Math. Res. Lett. **18** (2011), *no.* 01, 75–92

HÖLDER REGULARITY OF WEAK KAM SOLUTIONS IN A PRIORI UNSTABLE SYSTEMS

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Abstract. For *a priori* unstable Hamiltonian systems with two and half degrees of freedom, there is a continuous path in $H^1(\mathbb{T}^2,\mathbb{R})$ such that for each cohomology class c in this path, the c-minimal measure is supported on a normally hyperbolic cylinder. In this paper, we show that the weak KAM solutions for these classes can be parameterized by the area bounded by the graph of these solutions and obtain the $\frac{1}{4}$-Hölder regularity of these solutions in the parameter.

1. Introduction

A Hamiltonian system is usually called *a priori* unstable type if it is a perturbed coupling of a rotator and a pendulum:

$$H(x,y,t)=f(y_1)+g(x_2,y_2)+P(x,y,t),$$

where f, g and P stand for the rotator, the pendulum and the perturbation respectively. In this paper, we assume $x=(x_1,x_2)\in\mathbb{T}^2$, $y=(y_1,y_2)\in\mathbb{R}^2$, $H\in C^r(r\geq 2)$ and

1, $f+g$ is a convex function in y, i.e., the Hessian matrix $\partial_{yy}(f+g)$ is positive definite, finite everywhere, and has superlinear growth in y, $(f+g)/\|y\|\to\infty$ as $\|y\|\to\infty$.

2, g has a non-degenerate saddle critical point (x_2^*,y_2^*) and its stable manifold coincides with its unstable manifold. Without loss of generality, we assume $(x_2^*,y_2^*)=(0,0)$.

3, P is time-1-periodic and small in C^r-topology.

Under Legendre transformation $\mathcal{L}^*:H\to L$, we obtain the Lagrangian

$$L(x,\dot{x},t)=\max_y\{\langle y,\dot{x}\rangle-H(x,y,t)\}.$$

Here $\dot{x}=\dot{x}(x,y,t)$ is implicitly determined by $\dot{x}=\frac{\partial H}{\partial y}$. We use $\mathcal{L}:(x,y,t)\to(x,\dot{x},t)$ to denote the coordinate transformation determined by the Hamiltonian H.

As *a priori* unstable condition is assumed, there is a normally hyperbolic cylinder invariant for Φ_L, the time-1-map of the Lagrange flow. In this case, for each $c\in\mathcal{P}=$

Received by the editors August 31, 2010.

$\{(c_1, 0) : A \leq c_1 \leq B\}$, the c-minimal measure is supported on an Aubry-Mather set in this cylinder. Each of these Aubry-Mather sets determines a pair of weak-KAM solutions $u_c^{\pm}$ [5].

A natural question would be the regularity of these weak-KAM solutions with respect to the cohomology class c. However, it appears unclear whether the regularity exists. In this paper, we find another parameter "area", one-to-one corresponding to c, and obtain certain Hölder regularity of these weak-KAM solutions in the parameter.

Let $\mathcal{B}_{\epsilon,K}$ denote the ball in the function space $C^r(\{(x, y, t) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T} : \|y\| \leq K\} \to \mathbb{R})$, centered at the origin with radius of ϵ. We have the following result:

Theorem 1.1. *H is assumed a priori unstable. For each large $K > 0$ and small $a > 0$, a small $\epsilon > 0$ and a residual set $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ exist such that for each $P \in \mathcal{S}_{\epsilon,K}$, there exists a continuous and one-to-one correspondence $\sigma^- \in [A', B']$ (resp. $\sigma^+ \in [A'', B''])$ $\to \mathcal{P}$ such that $\{\bar{u}^-_{c(\sigma^-)}(x)|_{\{0\leq x_2\leq 1-a\}}\}$ (resp. $\{\bar{u}^+_{c(\sigma^+)}(x)|_{\{a\leq x_2\leq 1\}}\}$) is $\frac{1}{4}$-Hölder continuous in σ^- (resp. σ^+).*

This paper is organized as follows. In the section 2, an order-preserving property is obtained for backward and forward minimal configurations for monotone twist map. With such order-preserving property, we obtain the regularity of weak-KAM solutions for monotone twist maps in the section 3. The main result is obtained in the section 4 by using the normally hyperbolic structure of the invariant cylinder. In the last section, we apply the result to construct Arnold-type diffusion orbits shadowing a sequence of heteroclinic orbits, the shape of which is different from the shape of the diffusion orbits constructed in [3].

Let us briefly recall the standard notations and terminologies before our demonstration [9]. Let $\eta_c = \langle \eta_c(x), dx\rangle$ be a closed 1-form with de Rham cohomology class $c = [\eta_c]$. By abusing this notation, we call $\eta_c = \langle \eta_c(x), \dot{x}\rangle$ "closed 1-form" also. For each $c \in H^1(M, \mathbb{R})$, the c-action of an absolutely continuous curve $\gamma : [a, b] \to M$ is defined as

$$A_c(\gamma) = \int_a^b (L - \eta_c + \alpha(c))(\gamma(t), \dot{\gamma}(t), t)dt = \int_a^b L_c(\gamma(t), \dot{\gamma}(t), t)dt.$$

For $t \in \mathbb{R}$, let $[t] \in \mathbb{T}$ denote the decimal part of t. For any pair of points $(m_0, [s])$, $(m_1, [t])$ on $M \times \mathbb{T}$, let

$$\Phi_c^n((m_0, [s]), (m_1, [t])) = \min_{\gamma} \int_a^b (L - \eta_c + \alpha(c))(\gamma(\tau), \dot{\gamma}(\tau), \tau)d\tau,$$

where the minimum is taken over all absolutely continuous curve $\gamma : [a, b] \to M$ with $\gamma(a) = m_0$, $\gamma(b) = m_1$ such that $[a] = [s]$, $[b] = [t]$ and the integer part of $b - a$ is n.

Then the Mañé action functional is defined by

$$\Phi_c((m_0, [s]), (m_1, [t])) = \inf_n \Phi_c^n((m_0, [s]), (m_1, [t])),$$

and the barrier function is defined by ([10])

$$h_c((m_0,[s]),(m_1,[t])) = \liminf_{n\to\infty} \Phi_c^n((m_0,[s]),(m_1,[t])).$$

A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-semi-static if

$$A_c(\gamma|_{[s,t]}) = \Phi_c((\gamma(s),[s]),(\gamma(t),[t]))$$

for each $[s,t] \subset \mathbb{R}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-static if

$$A_c(\gamma|_{[s,t]}) + \Phi_c((\gamma(t),[t]),(\gamma(s),[s])) = 0$$

for each $[s,t] \subset \mathbb{R}$. An orbit $X(t) = (\gamma(t), \dot{\gamma}(t), t \mod 1)$ is called c-static (semi-static) if γ is c-static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of global c-semi-static orbits, and call the Aubry set $\tilde{\mathcal{A}}(c)$ the union of global c-static orbits. In the following we use the symbol $\tilde{\mathcal{N}}_s(c) = \tilde{\mathcal{N}}(c)|_{t=s}$ to denote the time s-section of a Mañé set, and so on.

The concept of c-semi-static curves can be extended to those only defined on $\mathbb{R}^{\pm}$ (cf. [2]). We call them backward or forward c-semi-static curves and use $\gamma_c^-(t,x) : (-\infty, 0] \to M$ to denote backward c-semi-static curve with $\gamma_c^-(0) = x$ and use $\gamma_c^+(t,x) : [0,+\infty) \to M$ to denote forward c-semi-static curve with $\gamma_c^+(0) = x$. Let Φ_L^t be the Lagrangian flow determined by L, let $\Phi_L^t(z,\tau)$ be the orbit of the Lagrangian flow with the initial value z at the time τ. Define

$$\mathcal{N}^-(c) = \{(z,\tau) \in TM \times \mathbb{T},\ \pi \circ \Phi_L^t(z,\tau)|_{(-\infty,0]} \text{ is } c\text{-semi-static}\},$$

$$\mathcal{N}^+(c) = \{(z,\tau) \in TM \times \mathbb{T},\ \pi \circ \Phi_L^t(z,\tau)|_{[0,+\infty)} \text{ is } c\text{-semi-static}\}.$$

Clearly, both $\mathcal{N}^-(c)$ and $\mathcal{N}^+(c)$ cover the configuration manifold in the sense that $\pi\mathcal{N}^{\pm}(c) = M$. The orbits in $\mathcal{N}^{\pm}(c)$ are called backward or forward c-semi-static orbits respectively.

2. order-preseving property of minimal orbits for monotone twist maps

Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ be an area-preserving and monotone twist diffeomorphism of cylinder, i.e., if we write $f(\theta,y) = (\theta', y')$, then $|\frac{\partial^2 f}{\partial\theta\partial y}| \equiv 1$ and $\frac{\partial\theta'}{\partial y} \geq d > 0$. Considering a lift of f to the universal cover $\mathbb{R}^2$, by abusing of terminology, we continue to denote $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x',y')$. This map has a global generating function $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$y = -\partial_1 h(x,x'),\ y' = \partial_2 h(x,x'),$$

where $\partial_1 h$ and $\partial_2 h$ denote the first partial derivatives of h with respect to x and x'. It was proved (see [1]) that h satisfies the following properties:

(1) $h(x,x') = h(x+1, x'+1)$, for all $x, x' \in \mathbb{R}$.

(2) $\lim_{|\xi|\to\infty} h(x, x+\xi) = \infty$, uniformly in x.

(3) $h(x,x') + h(\xi,\xi') < h(x,\xi') + h(\xi,x)$, if $x < \xi$, $x' < \xi'$.

(4) If $(\bar{x}, x, x') \neq (\bar{\xi}, \xi, \xi')$ are both minimal segments and $x = \xi$, then $(\bar{x} - \bar{\xi})(x' - \xi') < 0$.

Given a sequence $X = (\cdots, x_j, \cdots, x_k, \cdots) \in \mathbb{R}^{\mathbb{Z}}$, we call it minimal configuration for the generating function h, if for any $j < k$,

$$\begin{aligned} H(x_j, \cdots, x_k) &= \sum_{i=j}^{k-1} h(x_i, x_{i+1}) \\ &\leq H(x_j^*, \cdots, x_k^*) \end{aligned} \tag{2.1}$$

holds for all $(x_j^*, \cdots, x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$. If $X \in \mathbb{R}^{\mathbb{Z}}$ is a minimal configuration, then $\rho(X) = \lim_{|i| \to \infty} i^{-1} x_i$ exists. The number $\rho(X)$ is called the rotation number of X. Indeed, if the rotation number is rational p/q, there are three type of minimal configurations, they correspond to the rotation symbols $p/q-$, p/q, $p/q+$ (see [1]). For each rotation symbol, all Aubry graphs of the minimal configurations do not intersect each other.

For each rotation number ω, the Aubry-Mather set $\tilde{\mathcal{M}}_\omega$ is defined on the cylinder $(x, y) \in \mathbb{T} \times \mathbb{R}$. Let $\mathcal{M}_\omega = \pi \tilde{\mathcal{M}}_\omega$ where $\pi : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ is the standard projection. Without causing confusion, we abuse the symbol $\mathcal{M}_\omega$ to denote its lift to the universal covering of $\mathbb{T}$, call them projected Aubry-Mather set for ω. Obviously, $\mathcal{M}_\omega$ contains exactly those points through each of which passes a minimal configuration with the rotation symbol, i.e. if $\omega = p/q$, then the rotation symbol is exactly p/q.

This definition of minimal configuration can be applied to one-sided infinite sequence. A sequence $(x_0, x_1, \cdots) \in \mathbb{R}^{\mathbb{Z}^+}$ is called forward minimal configuration if (2.1) holds for any $0 \leq j < k$. A sequence $(\cdots, x_{-1}, x_0) \in \mathbb{R}^{\mathbb{Z}^-}$ is called backward minimal configuration if (2.1) holds for any $j < k \leq 0$. Therefore, we can also define the rotation number for each forward (resp. backward) minimal configuration. Each minimal configuration $(\cdots, x_j, \cdots, x_k, \cdots)$ determines an orbit $(\cdots, (x_j, y_j), \cdots, (x_k, y_k), \cdots)$, i.e, $f(x_i, y_i) = (x_{i+1}, y_{i+1})$ for any $i \in \mathbb{Z}$. We call it minimal orbit. Clearly, a forward minimal configuration determines a forward minimal orbit $((x_0, y_0), (x_1, y_1), \cdots)$, the ω-limit set of the forward minimal orbit is contained in certain Aubry-Mather set. Similarly, a backward minimal configuration determines a backward minimal orbit, the α-limit set of the backward minimal orbit is also contained in certain Aubry-Mather set.

Lemma 2.1. *Let $X = (x_0, x_1, \cdots)$, $X' = (x_0', x_1', \cdots)$ be two forward minimal configurations, $x_0 = x_0'$. Assume $\rho(X) > \rho(X')$, then $\partial_1 h(x_0, x_1) < \partial_1 h(x_0, x_1')$, i.e., $y_0 > y_0'$. The case for the backward minimal configurations is similar.*

Proof. As both X and X' are forward minimal configurations, they determine forward minimal orbits, denoted by

$$f^k(x_0, y_0) = (x_k, y_k), \quad k = 0, 1, 2 \cdots,$$

$$f^k(x_0', y_0') = (x_k', y_k'), \quad k = 0, 1, 2 \cdots.$$

Clearly, we have $y_0 \neq y_0'$ as $\rho(X) \neq \rho(X')$. We consider the Aubry graphs $\{(k, x_k)\}_{k=0}^{\infty}$ and $\{(k, x_k')\}_{k=0}^{\infty}$ of X and X' respectively. If $y_0 < y_0'$, we would have $x_1' > x_1$ as f is monotonically twisting and $x_0' = x_0$. On the other hand, we find that $x_i > x_i'$ for sufficiently large i because of $\rho(X) > \rho(X')$. Therefore, two Aubry graphs must cross at least once at some $t > 1$, $t \in \mathbb{R}^+$. There are two cases: $(t) \neq 0$ or $(t) = 0$. Here (t) is decimal part of t.

If $(t) \neq 0$, there is an integer $k \geq 2$ such that $t \in (k, k+1)$. We consider two segments $(x_0, x_1, \cdots\cdots, x_k, x_{k+1}')$, $(x_0', x_1', \cdots\cdots, x_k', x_{k+1})$, using the property (3) of h, we see

$$\begin{aligned}
& H(x_0, \cdots, x_k, x_{k+1}') + H(x_0', \cdots, x_k', x_{k+1}) \\
= \; & H(x_0, \cdots, x_k) + h(x_k, x_{k+1}') + H(x_0', \cdots, x_k') + h(x_k', x_{k+1}) \\
< \; & H(x_0, \cdots, x_k) + h(x_k, x_{k+1}) + H(x_0', \cdots, x_k') + h(x_k, x_{k+1}) \\
= \; & H(x_0, \cdots, x_k, x_{k+1}) + H(x_0', \cdots, x_k', x_{k+1}').
\end{aligned}$$

For the segments $(x_0, \cdots, x_k, x_{k+1})$ and $(x_0', \cdots, x_k', x_{k+1}')$, it implies that at least one of them is not minimal.

If $(t) = 0$, i.e., $t = k$ for some integer $k \geq 2$ and $x_k = x_k'$. Because of the property (4) of h and $(x_{k-1} - x_{k-1}')(x_{k+1} - x_{k+1}') < 0$, it is impossible that both the segments (x_{k-1}, x_k, x_{k+1}') and $(x_{k-1}', x_k', x_{k+1})$ are minimal. Hence, there exist $\bar{x}_k$ and $\bar{x}_k'$ such that

$$\begin{aligned}
& H(x_{k-1}, \bar{x}_k, x_{k+1}') + H(x_{k-1}', \bar{x}_k', x_{k+1}) \\
< & H(x_{k-1}, x_k, x_{k+1}') + H(x_{k-1}', x_k', x_{k+1}).
\end{aligned}$$

Using $x_k = x_k'$, we have

$$\begin{aligned}
& H(x_{k-1}, \bar{x}_k, x_{k+1}') + H(x_{k-1}', \bar{x}_k', x_{k+1}) \\
< & H(x_{k-1}, x_k, x_{k+1}) + H(x_{k-1}', x_k', x_{k+1}').
\end{aligned}$$

We consider the segments $(x_0, \cdots, x_{k-1}, \bar{x}_k, x_{k+1}')$, $(x_0', \cdots, x_{k-1}', \bar{x}_k', x_{k+1})$, similarly, we have

$$\begin{aligned}
& H(x_0, \cdots, x_{k-1}, \bar{x}_k, x_{k+1}') + H(x_0', \cdots, x_{k-1}', \bar{x}_k', x_{k+1}) \\
= & H(x_0, \cdots, x_{k-1}) + H(x_{k-1}, \bar{x}_k, x_{k+1}') \\
& + H(x_0', \cdots, x_{k-1}') + H(x_{k-1}', \bar{x}_k', x_{k+1}) \\
< & H(x_0, \cdots, x_{k-1}) + H(x_{k-1}, x_k, x_{k+1}) \\
& + H(x_0', \cdots, x_{k-1}') + H(x_{k-1}', x_k', x_{k+1}') \\
= & H(x_0, \cdots, x_{k-1}, x_k, x_{k+1}) + H(x_0', \cdots, x_{k-1}', x_k', x_{k+1}').
\end{aligned}$$

For the segments $(x_0, \cdots, x_{k-1}, x_k, x_{k+1})$ and $(x_0', \cdots, x_{k-1}', x_k', x_{k+1}')$, it also implies that at least one of them is not minimal.

The contradiction shows that $y_0 > y_0'$. $\square$

Let $X = (\cdots, x_i, \cdots)$ and $X' = (\cdots, x'_i, \cdots)$ be the minimal configurations with the rotation number p/q. Thus, we have $x_{i+q} = x_i + p$ and $x'_{i+q} = x'_i + p$. We assume that $X < X'$ are two adjacent minimal configurations, i.e. between them there is no other minimal configuration with the same rotation number, that is, there is no other minimal configuration X'' such that $x_i < x''_i < x'_i$. For each $m \in (x_0, x'_0)$ and each integer $k > 0$, there exists a sequence of $k+1$ numbers $X_{m,k} = (m_{-k,k}, m_{-k+1,k}, \cdots, m_{0,k})$ with $m_{-k,k} = x_{-k}$ and $m_{0,k} = m$ which minimizes the action

$$H(m_{-k,k}, \cdots, m_{0,k}) = \min_{\substack{\xi_{-k}=x_{-k} \\ \xi_0 = m}} \sum_{i=-k}^{-1} h(\xi_i, \xi_{i+1}).$$

By using the argument to prove Lemma 2.1, we see the monotonicity and the boundedness

$$x_{-i} < m_{-i,k} < m_{-i,k'} < x'_{-i}, \qquad \forall\ 0 < i < k < k'$$

and $m_{-i,k} > m_{-i,k'}$ holds for each $i \le k'$ provided $k > k'$. Let $k \to \infty$, there exists a sequence of infinitely many numbers $X_m = (\cdots, m_{-i}, \cdots, m_0)$ such that

$$m_{-i} = \lim_{k\to\infty} m_{-i,k}.$$

Because there is no minimal configuration between X and X', we see that

$$m_{-i} - x_{-i} \to 0, \qquad \text{as } \ i \to \infty,$$

and X_m is a backward minimal configuration. Similarly, we can show the existence of backward minimal configuration $X'_m = (\cdots, m'_{-1}, m'_0 = m)$ originating from m and approaching X' in the sense that

$$x'_{-i} - m'_{-i} \to 0, \qquad \text{as } \ i \to \infty.$$

Let us recall the minimal configuration for the rotation symbol $p/q-$ as well as $p/q+$ [1]. Thus, we call X_m backward minimal configuration with rotation symbol $p/q+$ and call X'_m backward minimal configuration with rotation symbol $p/q-$. These arguments lead to the following:

Lemma 2.2. *If the rotation number is rational $\omega = p/q$ and m is not in the projected Aubry-Mather set, then, originating from m there exists backward minimal configuration $X_m = (\cdots, m_{-1}, m_0 = m)$ with rotation symbol $p/q+$ as well as backward minimal configuration $X'_m = (\cdots, m'_{-1}, m'_0 = m)$ with rotation symbol $p/q-$, and*

$$\partial_2 h(m'_{-1}, m) < \partial_2 h(m_{-1}, m).$$

3. Regularity of weak KAM solutions for monotone twist maps

Given a cohomology class $c \in H^1(M, \mathbb{R})$, there is a backward (resp. forward) c-weak KAM solution ([5]), denoted by u_c^- (resp. u_c^+). It is also a viscosity solution of the Hamiltonian-Jacobi equation

$$H(t, x, \partial_x u_c^- + c) + \partial_t u_c^- = \alpha(c), \tag{3.2}$$

$$(\text{resp. } -H(t, x, \partial_x u_c^+ + c) - \partial_t u_c^+ = -\alpha(c)).$$

Such weak KAM solution u_c^- (resp. u_c^+) is also a fixed point of so called Lax-Oleinik operator on $C^0(M,\mathbb{R})$:

$$T^-_{t,t'}u(x,t') = \inf\Big\{u(\gamma(t),t) + \int_t^{t'} (L - \eta_c + \alpha(c))(d\gamma(\tau),\tau)d\tau\Big\}$$

$$(\text{resp. } T^+_{t,t'}u(x,t) = \sup\Big\{u(\gamma(t'),t') - \int_t^{t'} (L - \eta_c + \alpha(c))(d\gamma(\tau),\tau)d\tau\Big\})$$

where $t' > t$, the infimum is taken over all absolutely continuous curves such that $\gamma(t') = x$ (resp. $\gamma(t) = x$). The weak KAM solution u_c^- as well as u_c^+ is uniquely determined up to a constant provided the minimal measure is uniquely ergodic.

As u_c^- (resp. u_c^+) is a fixed point of the Lax-Oleinik operator, it satisfies the following properties [4]:

(1) $u_c^\pm$ are L_c-dominated, i.e.,

$$u_c^\pm(x',[t]) - u_c^\pm(x,[s]) \le \Phi_c((x,[s]),(x',[t])),$$

for any $(x,[s]),(x',[t]) \in M \times \mathbb{T}$. We use the notation $u_c^\pm \prec L_c$.

(2) For every $(x,s) \in M \times \mathbb{R}$, there exists a curve $\gamma_c^- : (-\infty,s) \to M$ (resp. $\gamma_c^+ : (s,\infty) \to M$) with $\gamma_c^-(s) = x$ (resp. $\gamma_c^+(s) = x$) such that

$$u_c^-(x,[s]) - u_c^-(\gamma_c^-(t),[t]) = \int_t^s (L - \eta_c + \alpha(c))(\gamma^-(\tau),\dot\gamma_c^-(\tau),\tau)d\tau$$

$$(\text{resp. } u_c^+(\gamma^+(t),[t]) - u_c^+(x,[s]) = \int_s^t (L - \eta_c + \alpha(c))(\gamma_c^+(\tau),\dot\gamma_c^+(\tau),\tau)d\tau)$$

for any $t < s$ (resp. $t > s$). The curve $\gamma_c^-(t)$ and $\gamma_c^+(t)$ are called the calibrated curves of backward c-weak KAM solution and forward c-weak KAM solution respectively.

Each weak KAM solution u_c^- is a Lipschitz function provided the Hamiltonian is assumed positive definite, thus it is differentiable almost everywhere. Starting from the point $(x,y) = (x,\partial_x u_c^-(x,0)+c)$, a backward c-minimal orbit $d\gamma_c^-(t) : (-\infty,0] \to TM$ will approach to the certain Mather set $\tilde{\mathcal{M}}(c)$ corresponding to the cohomology class c, provided u_c^- is differentiable at $(x,0)$. In this sense, we say that $\tilde{\mathcal{M}}(c)$ is associated to u_c^-. Obviously, $\partial_x u_c^-(x,0)+c$ and $\dot\gamma_c^-(0)$ are conjugate via the Legendre transformation. The similar result is valid for u_c^+ and $\dot\gamma_c^+$.

Let us consider the case that $M = \mathbb{T}$ in this section. By a result of Moser [12], for each monotone twist map there exists a positive definite periodic Hamiltonian system with one and half degrees of freedom such that the time-1-map of the Hamiltonian flow is exactly the twist map. We have

Lemma 3.1. *For each $x \in \mathbb{T}$, there is at least one backward (resp. forward) c-semi-static curve $\gamma_c^-(t,x)$ (resp. $\gamma_c^+(t,x)$) such that $\gamma_c^-(0,x) = x$ (resp. $\gamma_c^+(0,x) = x$).*

Proof. Obviously, every calibrated curve $\gamma_c^-(t)$ (resp. $\gamma_c^+(t)$) of backward (resp. forward) c-weak KAM solution with $\gamma_c^-(0) = x$ (resp. $\gamma_c^+(0) = x$) is corresponding backward (resp. forward) c-semi-static curve. $\square$

As both the α-limit set of $d\gamma_c^-$ and the ω-limit set of $d\gamma_c^+$ are in the Aubry set, the set

$$\bigcup_{x \in M} \left\{ x, \frac{d\gamma_c^{\pm}(0,x)}{dt} \right\}$$

are the unstable and stable set of the c-Mather set respectively. Clearly, if c-minimal measure is uniquely ergodic, we have

$$W_c^{\pm} = \mathcal{L}^{-1}\Big(\bigcup_{x \in M} \left\{ x, \frac{d\gamma_c^{\pm}(0,x)}{dt} \right\} \Big) = \mathrm{Graph}(d_x u_c^{\pm}(x,0) + c)$$

holds almost everywhere. Let $\bar{u}_c^{\pm}(x) = u_c^{\pm}(x,0) + cx$, $x \in [0,1]$, here $[0,1]$ is a basic region for the covering space of $\mathbb{T}$, then

$$W_c^{\pm} = \mathrm{Graph}(d_x \bar{u}_c^{\pm}(x)) \quad \text{almost everywhere.}$$

We call $\bar{u}_c^-(x)$, $\bar{u}_c^+(x)$ the backward generating function and the forward generating function of W_c^- and W_c^+ respectively.

Let $\mathcal{L}_\beta$ be the Fenchel-Legendre transformation : $H_1(M,\mathbb{R}) \to H^1(M,\mathbb{R})$ determined by

$$c \in \mathcal{L}_\beta(\omega) \text{ iff } \langle c, \omega \rangle = \beta_L(\omega) + \alpha_L(c) \text{ holds.}$$

In the case $M = \mathbb{T}$, we have canonical identification $H_1(\mathbb{T},\mathbb{R}) = \mathbb{R}$ and $H^1(\mathbb{T},\mathbb{R}) = \mathbb{R}$. The following facts are proved in [8] or [2]:

• If $\omega \in H_1(\mathbb{T},\mathbb{R})$ is irrational, then $\mathcal{L}_\beta(\omega) \in H^1(\mathbb{T},\mathbb{R})$ is one point, denoted by $c_\omega = \mathcal{L}_\beta(\omega)$.

• If $\omega = p/q$, then $\mathcal{L}_\beta(p/q)$ is reduced to one point if and only if $\pi(\mathcal{M}_{p,q}) = \mathbb{T} \times \mathbb{T}$. In the generic case, there is only one p/q-minimal periodic orbit and

$$\mathcal{L}_\beta(p/q) = [c_{p/q-}, c_{p/q+}], \quad -\infty < c_{p/q-} < c_{p/q+} < \infty.$$

When $\mathcal{L}_\beta(p/q)$ is an interval, the projected Aubry-Mather set in $\mathbb{R}$ can be expressed in the form

$$\mathcal{M}_{p/q} = \bigcup_{i=-\infty}^{\infty} M_i$$

where each $M_i = [x_i^-, x_i^+]$ is a point or an interval, $x_i^+ < x_{i+1}^-$. In generic case, each $x_i^- = x_i^+ = x_i$ and $\mathcal{M}_{p/q}|_{[0,1)}$ contains q points, i.e. there is only one minimal periodic orbit.

Let us consider the set of backward generating functions $\{\bar{u}_c^-(x)\}_{c \in \mathbb{R}}$. The study of the set $\{\bar{u}_c^+(x)\}_{c \in \mathbb{R}}$ is similar. By adding a constant, we can assume $\bar{u}_c^-(0) = 0$ for each $c \in \mathbb{R}$. First of all, we study the set of backward generating functions $\{\bar{u}_c^-(x)\}_{c \in [c_{p/q-}, c_{p/q+}]}$.

Recall Lemma 2.2. Let $\bar{u}^-_{p/q-}$, $\bar{u}^-_{p/q+}$ be the functions such that

$$\partial_x \bar{u}^-_{p/q-}(m) = \partial_2 h(m'_{-1}, m),$$
$$\partial_x \bar{u}^-_{p/q+}(m) = \partial_2 h(m_{-1}, m),$$

and $\bar{u}^-_{p/q\pm}(0) = 0$. We see that $\bar{u}^-_{p/q+} - \bar{u}^-_{p/q-}$ monotonously increases with respect to the variable by the fact that $\partial_2\partial_1 h < 0$. Indeed, these two functions generate all weak KAM solutions for each $c \in [c_{p/q-}, c_{p/q+}]$ in the following sense.

Lemma 3.2. *Assume there is only one minimal periodic configuration for the rotation number p/q. For each $c \in (c_{p/q-}, c_{p/q+})$, let $\bar{u}^-_c(x) = u^-_c(0,x) + cx$ where $u^-_c(t,x)$ is the weak KAM solution of the equation 3.2. Then,*

1, there exists a unique $m_i = m_i(c) \in (x_i, x_{i+1})$ as well as constants a^-_i, a^+_i for each i such that m_i monotonously increases respect to c, $m_i(c) \to x_i$ as $c \to c_{p/q-}$, $m_i(c) \to x_{i+1}$ as $c \to c_{p/q+}$ and

$$\bar{u}^-_c(x) = \bar{u}^-_{p/q+}(x) + a^+_i, \qquad \forall\ x \in [x_i, m_i]$$
$$\bar{u}^-_c(x) = \bar{u}^-_{p/q-}(x) + a^-_i, \qquad \forall\ x \in [m_i, x_{i+1}]$$

In particular, $\bar{u}^-_{c_{p/q\pm}} = \bar{u}^-_{p/q\pm} + \text{constant}$ holds for all $x \in \mathbb{R}$;

2, if we set $\bar{u}^-_c(0) = 0$ for each c, then $\bar{u}^-_c(x) \le \bar{u}^-_{c'}(x)$ for all $x \in (0,\infty)$ and $\bar{u}^-_c(x) \ge \bar{u}^-_{c'}(x)$ for all $x \in (-\infty, 0)$ provided $c < c' \in (c_{p/q-}, c_{p/q+})$.

Proof. Let X and X' be the minimal configuration with rotation symbol p/q passing through x_0 and x_1 respectively. Clearly, they are adjacent minimal configurations. Originating from each $m \in (x_0, x_1)$, there are two backward minimal configurations X_m and X'_m which approach to X and X' respectively. We define

$$A_c(X_m) = \lim_{k\to\infty}\Big(\sum_{-kq}^{-1} h(m_i, m_{i+1}) - (m - m_{-kq})c + kq\alpha(c)\Big),$$

for each $c \in [c_{p/q-}, c_{p/q+}]$. $A_c(X'_m)$ is defined in similar way. If we write $X = (\cdots, \zeta_i, \cdots, \zeta_0 = x_0, \cdots)$, $X' = (\cdots, \zeta'_i, \cdots, \zeta'_0 = x_1, \cdots)$, then $m_{-i} - \zeta_{-i} \to 0$ and $\zeta'_{-i} - m'_{-i} \to 0$ as $i \to \infty$. Thus we find that

$$A_c(X'_m) - A_c(X_m) = \sum_{-\infty}^{-1}(h(m'_i, m'_{i+1}) - h(m_i, m_{i+1})) + c(x_1 - x_0). \tag{3.3}$$

It is easy to see that for each $c \in (c_{p/q-}, c_{p/q+})$

$$A_c(X'_m) - A_c(X_m) > 0, \qquad \text{as } m \searrow x_0;$$
$$A_c(X'_m) - A_c(X_m) < 0, \qquad \text{as } m \nearrow x_1.$$

Indeed, the quantity $A_c(X'_m) - A_c(X_m)$ is monotonously decreasing in m in the interval (x_0, x_1). To see it, we note that the Aubry graph of X'_m intersects the

Aubry graph of X_ξ if $\xi > m$. If they intersect each other at $s \in (-i, -i+1)$, then we have

$$h(m'_{-i}, \xi_{-i+1}) + h(\xi_{-i}, m'_{-i+1}) < h(\xi_{-i}, \xi_{-i+1}) + h(m'_{-i}, m'_{-i+1}).$$

We define two configurations

$$X_{m\vee\xi} = (\cdots, m'_{-j}, \cdots, m'_{-i}, \xi_{-i+1}, \cdots, \xi_0 = \xi),$$

and

$$X_{m\wedge\xi} = (\cdots, \xi_{-j}, \cdots, \xi_{-i}, m'_{-i+1}, \cdots, m'_0 = m).$$

If they intersect each other at $-i$, i.e. $\xi_{-i} = m'_{-i}$, there exists $a < \xi_{-i} < a'$ such that

$$\begin{aligned} &H(\xi_{-i-1}, a, m'_{-i+1}) + H(m'_{-i-1}, a', \xi_{-i+1}) \\ <&H(\xi_{-i-1}, \xi_{-i}, \xi_{-i+1}) + H(m'_{-i-1}, m'_{-i}, m'_{-i+1}). \end{aligned}$$

In this case, we define the two configurations

$$X_{m\vee\xi} = (\cdots, m'_{-j}, \cdots, m'_{-i-1}, a', \xi_{-i+1}, \cdots, \xi_0 = \xi),$$

and

$$X_{m\wedge\xi} = (\cdots, \xi_{-j}, \cdots, \xi_{-i-1}, a, m'_{-i+1}, \cdots, m'_0 = m).$$

By the definition we then have

$$\begin{aligned} &(A_c(X'_m) - A_c(X_m)) - (A_c(X'_\xi) - A_c(X_\xi)) \\ >&A_c(X_{m\wedge\xi}) - A_c(X_m) + A_c(X_{m\vee\xi}) - A_c(X'_\xi) \\ \geq& 0. \end{aligned}$$

The second inequality comes from the observation that $X_{m\wedge\xi}$ is the configuration originating from m and approaching X which is a minimal configuration, and $X_{m\vee\xi}$ is the configuration originating from ξ and approaching X' which is again a minimal configuration. This verifies the decreasing monotonicity in m. Therefore, for each $c \in (c_{p/q-}, c_{p/q+})$ there is exactly one point $m \in (x_0, x_1)$ such that

$$A_c(X'_m) - A_c(X_m) = 0,$$

thus we can write $m_c = m(c)$ and see its increasing monotonicity in c if we note further that $A_c(X'_m) - A_c(X_m)$ monotonously increases with respect to c (see (3.3)).

Therefore, for $c \in (c_{p/q^-}, c_{p/q^+})$ we have $A_c(X_m) < A_c(X'_m)$ if $m \in (x_0, m_c)$ and $A_c(X_m) > A_c(X'_m)$ if $m \in (m_c, x_1)$. By the definition of weak KAM solution, $(m, \partial\bar{u}_c^-(m))$ uniquely determines a backward minimal configuration X along which the action $A_c(X)$ reaches the minimum provided u_c^- is a weak KAM solution and differentiable at m. Clearly, this minimal configuration is exactly X_m if $m \in (x_0, m_c)$, is X'_m if $m \in (m_c, x_1)$. This verifies the expression of $\bar{u}_c^-$ for $c \in (c_{p/q-}, c_{p/q+})$. Let $c \to c_{p/q\pm}$, it is obvious that $\bar{u}_c^- \to \bar{u}_{p/q\pm}^-$.

Indeed, we have

$$\begin{aligned} &A_{c_{p/q-}}(X'_m) - A_{c_{p/q-}}(X_m) \to 0, \qquad \text{as } m \searrow x_0; \\ &A_{c_{p/q+}}(X'_m) - A_{c_{p/q+}}(X_m) \to 0, \qquad \text{as } m \nearrow x_1. \end{aligned}$$

Indeed, by letting $\sigma_k X'_m = (\cdots, (\sigma_k m')_0 = m'_{-k}, \cdots, (\sigma_k m')_k = m'_0 = m)$ be the shift of X'_m, we see that $\sigma_k X'_m$ approaches to a minimal homoclinic configuration as

$m \to x_0$ and $k \to \infty$. This minimal configuration is in the Aubry set for $c = c_{p/q-}$. Similarly, $\sigma_k X_m$ approaches another minimal homoclinic configuration as $m \to x_1$ and $k \to -\infty$. It is in the Aubry set for $c = c_{p/q+}$.

To obtain the monotonicity of $\bar{u}_c^-$ in c, we observe two facts, the first one is that $\partial_x \bar{u}_{p/q+}^- > \partial_x \bar{u}_{p/q-}^-$ whenever both exist, the second one is that m_c monotonously increases in c. Therefore, we obtain that

$$\partial_x \bar{u}_c^- \leq \partial_x \bar{u}_{c'}^- \quad a.e. \qquad \text{whenever } c < c'.$$

This completes the proof of the second part. □

Let $C^0([0,1],\mathbb{R})$ be the space of continuous functions equipped with supremum norm

$$\|\bar{u}_c^- - \bar{u}_{c'}^-\| = \max_{x\in[0,1]} |\bar{u}_c^-(x) - \bar{u}_{c'}^-(x)|.$$

We have:

Theorem 3.1. *The set of functions $\{\bar{u}_c^- : c \in \mathbb{R}\}$ can be parameterized by some parameter σ, so that the map $\sigma \to \bar{u}_{c(\sigma)}^-$ is $\frac{1}{2}$-Hölder regular in σ.*

Proof. Note that each cohomology class c uniquely determines a rotation number $\omega_c = \mathcal{L}_\beta^{-1}(c)$ for monotone twist map and $c > c'$ if $\omega_c > \omega_{c'}$. According to Lemma 2.1, we have

$$\partial_x \bar{u}_c^-(x) > \partial_x \bar{u}_{c'}^-(x) \ \text{ a.e,} \qquad \text{if } \ \omega_c > \omega_{c'}.$$

If we set $\bar{u}_c^-(0) = 0$ for each c, we have $\bar{u}_c^- > \bar{u}_{c'}^-$ for $x > 0$ if $\omega_c > \omega_c'$ and if $c_{p/q+} \geq c > c' \geq c_{p/q-}$, in virtue of Lemma 3.2, we find that $\bar{u}_c^-(x) \geq \bar{u}_{c'}^-(x)$ for $x > 0$, moreover, there exists $x \in (0,1]$ such that $\bar{u}_c^-(x) > \bar{u}_{c'}^-(x)$.

We arbitrarily choose one function $\bar{u}_0^-$ corresponding to a cohomology class c_0 and parameterize another $\bar{u}_\sigma^- = \bar{u}_{c(\sigma)}^-$ by the algebraic area between graph$\bar{u}_\sigma^-$ and graph$\bar{u}_0^-$ in $[0,1]$,

$$\sigma = \int_0^1 (\bar{u}_\sigma^-(x) - \bar{u}_0^-(x))dx.$$

Obviously, there exists one-to-one and continuous correspondence between σ and c although we do not know whether some regularity exists if we think σ as a function of c, or vice versa. Anyway, we can think $\bar{u}_\sigma^-$ as a map to function space C^0 equipped with supremum norm $\bar{u}^- : \mathbb{R} \to C^0([0,1],\mathbb{R})$,

$$\|\bar{u}_{\sigma_1}^- - \bar{u}_{\sigma_2}^-\| = \max_{x\in[0,1]} |\bar{u}_{\sigma_1}^-(x) - \bar{u}_{\sigma_2}^-(x)|.$$

Straight forward calculation shows

$$\begin{aligned} |\sigma_1 - \sigma_2| &= \Big|\int_0^1 (\bar{u}_{\sigma_1}^-(x) - \bar{u}_{\sigma_2}^-(x))dx\Big| \\ &\geq \frac{1}{2|C_{\sigma_1} \pm C_{\sigma_2}|} \|\bar{u}_{\sigma_1}^- - \bar{u}_{\sigma_2}^-\|^2, \end{aligned}$$

i.e,

$$\|\bar{u}^-_{\sigma_1} - \bar{u}^-_{\sigma_2}\| \quad \leq \quad \sqrt{2|C_{\sigma_1} \pm C_{\sigma_2}}|\sigma_1 - \sigma_2|^{\frac{1}{2}}.$$

Here the constant C_σ is the Lipschitz constant for $\bar{u}^-_\sigma$. □

4. Regularity of weak KAM solutions in a priori unstable case

This section aims at the regularity of those weak KAM solutions for which the associated Mather sets are contained in a two dimensional cylinder.

Let Φ^t, Φ^t_{f+g} and Φ^t_g denote the Hamiltonian flow determined by H, $f+g$ and g respectively, let Φ, Φ_{f+g} and Φ_g be their time-1-maps accordingly. For the map Φ_g, the origin in the two dimensional space $(x_2, y_2) \in \mathbb{T} \times \mathbb{R}$ is a hyperbolic fixed point, its stable and unstable manifolds are smooth curves, denoted by Γ^s_0 and Γ^u_0 respectively. For the map of Φ_{f+g}, the cylinder $\Sigma_0 = \mathbb{T} \times \mathbb{R} \times \{(x_2, y_2) = (0, 0)\}$ is a normally hyperbolic invariant manifold, its stable (unstable) manifold has the form $W^{s,u}_{\Sigma_0} = \mathbb{T} \times \mathbb{R} \times \Gamma^{s,u}_0$, foliated into a family of invariant fibers $W^{s,u}_{\Sigma_0} = \cup_{z \in \Sigma_0} \Gamma^{s,u}_{0,z}$ in which $\Gamma^{s,u}_{0,z} = \{z\} \times \Gamma^{s,u}_0$.

As the *a priori* unstable condition is assumed, it follows from the fundamental theorem of normally hyperbolic invariant manifold [6] that

Theorem 4.1. *For any $K > 0$, there is $\epsilon_0 = \epsilon_0(K) > 0$ such that if $\|P\|_{C^r} \leq \epsilon_0$ in the region $\{|y_1| \leq K\}$ then the map Φ has an invariant C^{r-1}-manifold Σ with the properties:*

1, it is a small deformation of manifold $\Sigma_0|_{|y_1| \leq K}$

$$\Sigma = \{x_1, y_1, x_2(x_1, y_1), y_2(x_1, y_1) : x_1 \in \mathbb{T}^1, y_1 \in [-K, K]\},$$

and is also normally hyperbolic for Φ, its locally invariant stable and unstable manifold are denoted by W^s_Σ and W^u_Σ respectively.

2, $W^{s,u}_\Sigma$ has a foliation of stable (unstable) fibers $W^{s,u}_\Sigma = \cup_{z \in \Sigma} \Gamma^{s,u}_z$. The points of Γ^u_z are characterized by sharp backward asymptoticity towards z, the points of Γ^s_z are characterized by sharp forward asymptoticity towards z. The laminae $\cup_{z \in \Sigma} \Gamma^{s,u}_z$ for Φ are C^{r-1} near $\cup_{z \in \Sigma_0} \Gamma^{s,u}_{0,z}$ for Φ_{f+g}.

Let $\bar{M} = \mathbb{T} \times 2\mathbb{T}$ be the covering space of $M = \mathbb{T}^2$, i.e. $x_2 \in [0, 2)$ modulus 2. In $T^*\bar{M}$, the life of Σ has two connected components, one is in a small neighborhood of $\{x_2 = 0\}$, denoted by $\bar{\Sigma}$, another one is in a small neighborhood of $\{x_2 = 1\}$, denoted by $\bar{\Sigma}'$. Similarly, the life of each point $z \in \Sigma$ and the associated fiber $\Gamma^{s,u}_z$ have their two copies, denoted by $\bar{z}$, $\bar{z}'$, $\Gamma^{s,u}_{\bar{z}}$ and $\Gamma^{s,u}_{\bar{z}'}$ respectively. The unstable fibre $\Gamma^u_{\bar{z}}$ originates form $\bar{\Sigma}$ and extends to the right, the stable fiber $\Gamma^s_{\bar{z}'}$ originates form $\bar{\Sigma}'$ and extends to the left. For suitably small $a > 0$, there exists $0 < \epsilon < \epsilon_0$ such that if $\|P\|_{C^r} \leq \epsilon$ on the region $\{|y_1| \leq K\}$, $\Gamma^u_{\bar{z}}$ keep horizontal in the region $\{x_2(x_1, y_1) \leq x_2 \leq 1 - a\}$ and $\Gamma^s_{\bar{z}'}$ keep horizontal in the region $\{a \leq x_2 \leq x_2(x_1, y_1) + 1\}$.

As the manifold Σ can be considered as the image of a map $\psi : \Sigma_0 \to \mathbb{T}^2 \times \mathbb{R}^2$, $\Sigma = \{x_1, y_1, x_2(x_1, y_1), y_2(x_1, y_1)\}$, this map induces a 2-form $\psi^*\omega$ on Σ_0,

$$\psi^*\omega = (1 + \frac{\partial(y_2, x_2)}{\partial(y_1, x_1)})dy_1 \wedge dx_1,$$

here ω is canonical 2-form. Since the second de Rham co-homology group of Σ_0 is trivial, by using Moser's argument on the isotopy of symplectic form [11], we find that there exists a diffeomorphism ψ_1 on $\Sigma_0|_{\{|y_1| \le K\}}$ such that

$$(\psi \circ \psi_1)^*\omega = dy_1 \wedge dx_1.$$

Since Σ is invariant for Φ and $\Phi^*\omega = \omega$, we obtain

$$\Big((\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)\Big)^* dy_1 \wedge dx_1 = dy_1 \wedge dx_1.$$

Note that $\phi = \psi \circ \psi_1 : \Sigma_0 \to \Sigma$ is a small perturbation of identity. Consequently, $(\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)$ is monotonously twist.

As the second de Rham cohomology group of a suitably small neighborhood of Σ_0 is trivial also, both ψ and ψ_1 can be smoothly extended to the neighborhood of Σ_0 so that

$$\Big((\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)\Big)^* \omega = \omega.$$

It is shown in [3] that there is a channel $\mathcal{S} = \{(c_1, c_2) \in H^1(M, \mathbb{R}) : c_1 \in \mathbb{R}, A < c_1 < B, a(c_1) \le c_2 \le b(c_1), -\infty < a(c_1) < 0 < b(c_1) < \infty\}$, such that, if $c \in \text{int}\mathcal{S}$, the Mañé set is contained in $\mathcal{L}\Sigma$.

The following estimation is in the sense that we pull it back to the standard cylinder by $\phi \in \text{diff}(\Sigma_0, \Sigma)$, and denote the perturbed generating functions still by $\bar{u}_c^{\pm}$.

For each $c \in \text{Int}\mathcal{S}$, as it was studied in [3], the corresponding Aubry-Mather set lies in the cylinder. Thus, restricted on the universal covering of Σ_0, we have $y_1 = \partial_{x_1}\bar{u}_c^-$ and $y_2 = \partial_{x_2}\bar{u}_c^- = 0$ almost everywhere. Without loss of generality, we assume $\bar{u}_c^-(0,0) = 0$. We choose a path $\mathcal{P} = \{c = (c_1, 0) : A \le c_1 \le B\} \subset \mathcal{S}$. By choosing the parameter σ as in Theorem 3.1, there exists a one-to-one correspondence between c_1 and σ. Clearly, a constant $C_1 = C_1(K) > 0$ exists such that

$$\|\bar{u}_\sigma^-(x_1, 0) - \bar{u}_{\sigma'}^-(x_1, 0)\| \le C_1|\sigma - \sigma'|^{\frac{1}{2}},$$

where

$$\|\bar{u}_\sigma^-(x_1, 0) - \bar{u}_{\sigma'}^-(x_1, 0)\| = \max_{x_1 \in [0,1]} |\bar{u}_\sigma^-(x_1, 0) - \bar{u}_{\sigma'}^-(x_1, 0)|$$

and

$$\sigma - \sigma' = \int_0^1 (\bar{u}_\sigma^-(x_1, 0) - \bar{u}_{\sigma'}^-(x_1, 0))dx_1.$$

Theorem 4.2. *The set of functions $\{\bar{u}_c^-(x)\}_{c \in \mathcal{P}}$ can be parameterized by σ so that restricted in $\{0 \le x_2 \le 1 - a\}$, $\{\bar{u}_\sigma^-\}$ is $\frac{1}{4}$-Hölder continuous in σ.*

Proof. Recall the proof of Theorem 3.1, there exist upper and lower bound for σ: $A' \le \sigma \le B'$ so that each $c \in \mathcal{P}$ corresponds to some $\sigma \in [A', B']$. Let us establish the lemma first:

Lemma 4.1. *For $\sigma,\ \sigma' \in [A', B']$ with $|\sigma - \sigma'| < \frac{1}{16}$, let*

$$D = \left\{ x_1 \in [0,1] \middle| \; |\partial_{x_1} \bar{u}^-_\sigma(x_1, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(x_1, 0)| > |\sigma - \sigma'|^{\frac{1}{2}} \right\},$$

then the Lebesgue measure of D is bounded by $m(D) < 2|\sigma - \sigma'|^{\frac{1}{4}}$.

Proof. As we set $\bar{u}_\sigma(0,0) = 0$ for each σ, straightforward calculation shows

$$\begin{aligned} \sigma - \sigma' &= \int_0^1 (\bar{u}^-_\sigma(x_1, 0) - \bar{u}^-_{\sigma'}(x_1, 0)) dx_1 \\ &= \int_0^1 \int_0^{x_1} (\partial_{x_1} \bar{u}^-_\sigma(s, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(s, 0)) ds dx_1 \\ &= \int_0^1 (1 - s)(\partial_{x_1} \bar{u}^-_\sigma(s, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(s, 0)) ds. \end{aligned}$$

Let $D_1 = D \cap [0, 1 - |\sigma - \sigma'|^{\frac{1}{4}}]$, we have

$$m(D) \le m(D_1) + |\sigma - \sigma'|^{\frac{1}{4}}.$$

By the way of defining σ, we have

$$\partial_{x_1} \bar{u}^-_\sigma(x_1, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(x_1, 0) \ge 0 \quad a.e.$$

if $\sigma > \sigma'$, thus,

$$\begin{aligned} \sigma - \sigma' &> \int_{D_1} (1 - s)(\partial_{x_1} \bar{u}^-_\sigma(s, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(s, 0)) ds \\ &> m(D_1)|\sigma - \sigma'|^{\frac{3}{4}}. \end{aligned}$$

Thus, we have $m(D) < 2|\sigma - \sigma'|^{\frac{1}{4}}$. □

This lemma implies that the existence of a points x_1^* in each interval $[a, b] \subset [0, 1]$ with the property

$$|\partial_{x_1} \bar{u}^-_\sigma(x_1^*, 0) - \partial_{x_1} \bar{u}^-_{\sigma'}(x_1^*, 0)| \le |\sigma - \sigma'|^{\frac{1}{2}}.$$

provided $m([a, b]) \ge 2|\sigma - \sigma'|^{\frac{1}{4}}$.

Given $\sigma,\ \sigma' \in [A', B']$ with $|\sigma - \sigma'| < \frac{1}{16}$, we assume that both $\bar{u}^-_\sigma$ and $\bar{u}^-_{\sigma'}$ are differentiable at $x^* = (x_1^*, x_2^*) \in \{0 \le x_2 \le 1 - a\}$. Let $y = \partial_x \bar{u}^-_\sigma(x^*)$ and $y' = \partial_x \bar{u}^-_{\sigma'}(x^*)$. Thus, by Theorem 4.1, there exist exactly two points $\hat{z}, \hat{z}'$ in the cylinder such that $z = (x^*, y) \in \Gamma^u_{\hat{z}}$ and $z' = (x^*, y') \in \Gamma^u_{\hat{z}'}$. As they are in the cylinder, we can write $\hat{z} = (\hat{x}_1, 0, \hat{y}_1, 0)$ and $\hat{z}' = (\hat{x}'_1, 0, \hat{y}'_1, 0)$.

Thanks to Lemma 4.1, there exist two points $z_\xi = (\xi, 0, y_\xi, 0)$ and $z'_\xi = (\xi, 0, y'_\xi, 0)$ in the cylinder such that $|\xi - \hat{x}_1| \le 2|\sigma - \sigma'|^{\frac{1}{4}}$

$$y_\xi = \partial_{x_1} \bar{u}^-_\sigma(\xi, 0), \qquad y'_\xi = \partial_{x_1} \bar{u}^-_{\sigma'}(\xi, 0),$$

and

$$|y_\xi - y'_\xi| \le |\sigma - \sigma'|^{\frac{1}{2}}.$$

The unstable fiber $\Gamma^u_{z_\xi}$ (resp. $\Gamma^u_{z'_\xi}$) intersects the hyperplane $\{x_2 = x_2^*\}$ at $z_\zeta = (x_\zeta, y_\zeta)$ (resp. $z'_\zeta = (x'_\zeta, y'_\zeta)$), where $x_\zeta = (\zeta, x_2^*)$ and $x'_\zeta = (\zeta', x_2^*)$. Let $\gamma_1 = \Gamma^u_{z_\xi}|_{[z_\xi, z_\zeta]}$ and $\gamma_2 = \Gamma^u_{z'_\xi}|_{[z'_\xi, z'_\zeta]}$,

$$\begin{aligned}\bar{u}^-_\sigma(x_\zeta) - \bar{u}^-_{\sigma'}(x'_\zeta) =& \bar{u}^-_\sigma(\xi, 0) - \bar{u}^-_{\sigma'}(\xi, 0)\\ &+ \int_{\gamma_1} ydx - \int_{\gamma_2} ydx.\end{aligned}$$

By the theorem of normally hyperbolic manifold, the unstable (stable) fiber C^{r-1}-smoothly depends on the base point. Therefore, we have

$$\Big|\int_{\gamma_1} ydx - \int_{\gamma_2} ydx\Big| = O(|\sigma - \sigma'|^{\frac{1}{2}}).$$

In virtue of Theorem 3.1, we see that there is a positive constant C_3 such that

$$|\bar{u}^-_\sigma(x_\zeta) - \bar{u}^-_{\sigma'}(x'_\zeta)| \le C_3 |\sigma - \sigma'|^{\frac{1}{2}}.$$

To complete the proof, we claim that

$$\max\{\|x_\zeta - x^*\|, \|x'_\zeta - x^*\|\} = O(|\sigma - \sigma'|^{\frac{1}{4}}).$$

Towards this goal, we recall a fact that the unstable manifold of Σ is invariantly foliated by unstable fiber $W^u_\Sigma = \cup_{z\in\Sigma}\Gamma^u_z$. Since the 3-dimensional manifold W^u_Σ intersects the hyperplane $x_2 =$ constant transversally, we obtain a map $F_H : W^u_\Sigma|_{x_2=0} \to W^u_\Sigma|_{x_2=x_2^*}$ by defining $F(z) \in \Gamma^u_z|_{x_2=x_2^*}$. Clearly, for the map Φ_{f+g}, F_{f+g} is a translation and the coordinate x_1 keeps constant. By Theorem 4.1, the foliation into unstable fiber is permanent for the map Φ_H, the laminae $\cup_{z\in\Sigma}\Gamma^{s,u}_z$ for Φ_H are C^{r-1} near $\cup_{z\in\Sigma_0}\Gamma^{s,u}_{0,z}$ for Φ_{f+g}. It implies the tangent map dF_H is close to identity provided $r \ge 2$. This verifies our claim.

Therefore, we obtain

$$\|\bar{u}^-_\sigma(x_1, x_2) - \bar{u}^-_{\sigma'}(x_1, x_2)\|_{C^0(\{0\le x_2\le 1-a\},\mathbb{R})} \le C|\sigma - \sigma'|^{\frac{1}{4}}$$

for σ, $\sigma' \in [A', B']$ with $|\sigma - \sigma'|$ suitably small. □

From the proof, we see that the parametrization $\sigma \to c(\sigma)$ depends on the family of $\bar{u}^-_c$, we use the symbol σ^- to specify it for backward weak-KAM solutions. Similarly, area parameter $\sigma^+ \in [A'', B'']$ can be also introduced for $\bar{u}^+_c$, and the $\frac{1}{4}$-Hölder regularity exists in σ^+. It is not clear whether $\sigma^-(c) = \sigma^+(c)$. Anyway, we complete the proof of Theorem 1.1.

Remark 1: By using the modulus continuity of Peierl's barrier function obtained in [7], we find that, restricted on the invariant cylinder, the Hausdorff dimension of the barrier function set is not larger than 2. But we don't know, in this way, how to obtain the Hausdorff dimension estimate when they are treated as functions defined on two dimensional configuration space.

Remark 2: The result in this paper can be extended to the case that the Hamiltonian g has arbitrary k degrees of freedom provided $(x,y)=0$ is a hyperbolic fixed point supporting a minimal measure. There is a neighborhood of the origin where the stable and unstable manifolds keep horizontal. Under small perturbations to $f+g$, a two-dimensional cylinder still exists. Therefore, the regularity is obtained in the same way if we restrict ourselves in a suitable neighborhood of the cylinder. We can obtain the regularity by noticing the fact that a uniform upper bound T exists such that $\Phi_H^{\pm t}(z)$ stays in this neighborhood for any $t\geq T$ and any z provided it is a forward (backward) c-minimal orbit for $c\in\mathcal{P}$.

5. Application

By the regularity obtained above, one immediately obtains the genericity of Arnold diffusion in *a priori* unstable systems by following the way of [3]. Different from the diffusion orbit obtained in [3], the diffusion orbit obtained here shadows a sequence of heteroclinic orbits. It was claimed in [13].

Indeed, by the regularity of the weak-KAM solutions, we obtain immediately the regularity of the barrier functions $B_{c,e_2}(x)$ for the co-homology class $c\in\mathcal{P}$ with $\sigma^-(c)\in[A',B']$ and $\sigma^+(c)\in[A'',B'']$, it measures the limit infimum of the action along each curve passing through x and homoclinic to $\mathcal{M}(c)$ in the covering space $\bar{M}$. Here $B_{c,e_2}(x)$ is defined as well as in [3],

$$B_{c,e_2}(x)=\inf\{h_{c,e_2}(m_0,x,m_1)-h_{c,e_2}(m_0,m_1):m_0,m_1\in\mathcal{M}_0(c)\},$$

$$h_{c,e_2}(m_0,x,m_1)=\liminf_{\substack{k_1\to\infty\\ k_2\to\infty}}\inf_{\substack{\gamma(-k_1)=m_0\\ \gamma(0)=x\\ \gamma(k_2)=m_1\\ [\gamma]_2\neq 0}}\int_{-k_1}^{k_2}(L-\eta_c)(d\gamma(t),t)dt+(k_1+k_2)\alpha(c),$$

$$h_{c,e_2}(m_0,m_1)=\liminf_{k\to\infty}\inf_{\substack{\gamma(0)=m_0\\ \gamma(k)=m_1\\ [\gamma]_2\neq 0}}\int_0^k(L-\eta_c)(d\gamma(t),t)dt+k\alpha(c).$$

For the covering of the configuration manifold $\bar{M}=\mathbb{T}\times 2\mathbb{T}$ introduced in the last section, there are two lifts of the invariant cylinder Σ, denoted by $\bar{\Sigma}$, $\bar{\Sigma}'$ respectively. Each lift of orbits homoclinic to Σ turns out to be heteroclinic orbit connecting $\bar{\Sigma}$ to $\bar{\Sigma}'$ or vice versa. We consider those orbits originating from $\bar{\Sigma}$ and approaching to $\bar{\Sigma}'$. Among which the minimal heteroclinic orbit corresponds to the minimal point of the function

$$B_{c(\sigma^\pm),e_2}(x)=\bar{u}^-_{c(\sigma^-)}(x)-\bar{u}^+_{c(\sigma^+)}(x).$$

We call it barrier function, it is introduced in [3], somewhat different from the barrier function introduced by Mather [10]. Recall the parametrization $\sigma^-\in[A',B']$ (resp. $\sigma^+\in[A'',B''])\to c=(c_1(\sigma^\pm),0)\in\mathcal{P}$ with $c_1\in[A,B]$, we have

Theorem 5.1. *Given $A < B$, there exists $C_5 > 0$ such that*

$$|B_{c(\sigma^{\pm}),e_2}(x) - B_{c(\sigma'^{\pm}),e_2}(x)| \leq C_5(|\sigma^- - \sigma'^-|^{\frac{1}{4}} + |\sigma^+ - \sigma'^+|^{\frac{1}{4}})$$

holds for each $x \in M\backslash\{|x_2| \leq a\}$, provided $\sigma^-, \sigma'^- \in [A', B']$, $\sigma^+, \sigma'^+ \in [A'', B'']$ and $|\sigma^{\pm} - \sigma'^{\pm}|$ is suitably small.

Consequently, the Hausdorff dimension of the set

$$\mathscr{B}_\sigma = \{B_{c(\sigma^{\pm}),e_2}|_{x\in M\backslash\{|x_2|\leq a\}} : \ \sigma^- \in [A', B'], \ \sigma^+ \in [A'', B'']\}.$$

is finite:

$$D_H(\mathscr{B}_\sigma) \leq 8.$$

This finiteness of the Hausdorff dimension guarantees that genericity of transition chain (see [3]):

Let $c = (c_1, 0)$, $c' = (c_1', 0)$, transition chain $\Gamma : [0,1] \to \mathcal{P}$ is defined by $\Gamma(\tau) = ((1-\tau)c_1 + \tau c_1', 0)$. For each $\tau \in [0,1]$, $\Gamma(\tau)$ satisfies the following condition: there is a small number $\delta_\tau > 0$ such that $\pi_1(\mathcal{N}_0(\Gamma(\tau), \bar{M}))\backslash(\mathcal{A}_0(\Gamma(\tau)) + \delta_\tau)$ is non-empty and totally disconnected. Here $\mathcal{N}_0(\Gamma(\tau), \bar{M})$ denotes the Mañé set with respect to the covering space $\pi_1 : \bar{M} \to M$.

Indeed, $\tilde{\mathcal{A}}_0(\Gamma(\tau))$ is the image of the Aubry-Mather set in the cylinder under the Legendre transformation $\mathcal{L}$ and $(\mathcal{A}_0(\Gamma(\tau)) + \delta_\tau) \subset \{x : |x_2| \leq a\}$. Thus, $\pi_1(\mathcal{N}_0(\Gamma(\tau), \bar{M}))\backslash(\mathcal{A}_0(\Gamma(\tau)) + \delta_\tau)$ contains exactly the minimal points of the barrier function in the region $\{x : |x_2| > a\}$. Clearly, this set is totally disconnected in generic case, one obtains it immediately from the finiteness of the Hausdorff dimension (see Section 7 in [3]).

Therefore, the conditions required in the Theorem 5.1 in [3] are satisfied, (one needs not to consider c-equivalence here). By applying this theorem, we obtain:

Theorem 5.2. *Given two arbitrarily numbers $A < B$ and assume H satisfies the above conditions, then there exists a small number $\epsilon > 0$, a large number $K > 0$ and a residual set $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ such that for each $P \in \mathcal{S}_{\epsilon,K}$ there exists an Arnold-type orbit of the Hamiltonian flow which connects the region with $y_1 < A$ to the region with $y_1 > B$.*

Acknowledgements

The author thanks her advisor, Professor C.-Q. Cheng for his patience in the guidance. This work is partially supported by the National Basic Research Programme (973 Programme, 2007CB814800) of China, Basic Research Programme of Jiangsu Province (BK2008013) and the graduate student research fellowship of Jiangsu Province of China.

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