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ASYMPTOTIC LINEARITY OF REGULARITY AND a^*-INVARIANT OF POWERS OF IDEALS

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Abstract. Let $X = \operatorname{Proj} R$ be a projective scheme over a field k, and let $I \subseteq R$ be an ideal generated by forms of the same degree d. Let $\pi : \widetilde{X} \to X$ be the blowing up of X along the subscheme defined by I, and let $\phi : \widetilde{X} \to \bar{X}$ be the projection given by the divisor $dE_0 - E$, where E is the exceptional divisor of π and E_0 is the pullback of a general hyperplane in X. We investigate how the asymptotic linearity of the regularity and the a^*-invariant of I^q (for $q \gg 0$) is related to invariants of fibers of ϕ.

1. Introduction

Let k be a field and let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme over k. Let $I \subseteq R$ be a homogeneous ideal. It is well known (cf. [1, 4, 6, 7, 9, 12, 18, 20]) that $\operatorname{reg}(I^q) = aq + b$, a linear function in q, for $q \gg 0$. While the linear constant a is quite well understood from reduction theory (see [20]), the free constant b remains mysterious (see [10, 19] for partial results). Recently, Eisenbud and Harris [10] showed that when I is generated by general forms of the same degree, whose zeros set is empty in X, b can be related to a set of *local* data, namely, the regularity of fibers of the projection map defined by the generators of I. The aim of this paper is to exhibit a similar phenomenon in a more general situation, when I is generated by arbitrary forms of the same degree. In this case, the generators of I do not necessarily give a morphism. The projection map that we will examine is the map from the blowup of X along the subscheme defined by I, considered as a bi-projective scheme, to its second coordinate.

Let $I = (F_0, \ldots, F_m)$, where $F_0, \ldots, F_m$ are homogeneous elements of degree d in R. Let $\pi : \widetilde{X} \to X$ be the blowing up of X along the subscheme defined by I. Let $\mathcal{R} = R[It]$ be the Rees algebra of I. By letting $\deg F_i t = (d, 1)$, the Rees algebra $\mathcal{R}$ is naturally bi-graded with $\mathcal{R} = \bigoplus_{p,q \in \mathbb{Z}} \mathcal{R}_{(p,q)}$, where $\mathcal{R}_{(p,q)} = (I^q)_{p+qd} t^q$. Under this bi-gradation of $\mathcal{R}$, we can identify $\widetilde{X}$ with $\operatorname{Proj} \mathcal{R} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ (cf. [8, 15, 16]). Also, the projection $\phi : \operatorname{Proj} \mathcal{R} \to \mathbb{P}^m$ is in fact the morphism given by the divisor $D = dE_0 - E$, where E is the exceptional divisor of π and E_0 is the pullback of a general hyperplane in X. For a close point $\wp \in \bar{X} = \operatorname{image}(\phi)$, let $\widetilde{X}_\wp = \widetilde{X} \times_{\bar{X}} \operatorname{Spec} \mathcal{O}_{\bar{X},\wp}$ be the fiber of ϕ over the affine neighborhood $\operatorname{Spec} \mathcal{O}_{\bar{X},\wp}$ of $\wp$. Then $\widetilde{X}_\wp = \operatorname{Proj} \mathcal{R}_{(\wp)}$, where $\mathcal{R}_{(\wp)}$ is the homogeneous localization of $\mathcal{R}$ at $\wp$. We define the *regularity* of $\widetilde{X}_\wp$, denoted by $\operatorname{reg}(\widetilde{X}_\wp)$, to be that of $\mathcal{R}_{(\wp)}$. Inspired by the work of Eisenbud and Harris [10], we propose the following conjecture.

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Conjecture 1.1. Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree d. Let $\operatorname{reg}(\phi) = \max\{\operatorname{reg}(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Then for $q \gg 0$,

$$\operatorname{reg}(I^q) = qd + \operatorname{reg}(\phi).$$

We provide a strong evidence[1] for Conjecture 1.1. More precisely, we prove a similar statement to Conjecture 1.1 for the a^*-invariant, a closely related variant of the regularity. For a closed point $\wp \in \bar{X}$, we define the *a^*-invariant* of $\widetilde{X}_\wp$, denoted by $a^*(\widetilde{X}_\wp)$, to be the a^*-invariant of its homogeneous coordinate ring $\mathcal{R}_{(\wp)}$. Our first main result is stated as follows.

Theorem 1.2 (Theorems 2.6). *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree d. Let $a^*(\phi) = \max\{a^*(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Then for $q \gg 0$, we have*

$$a^*(I^q) = qd + a^*(\phi).$$

As a consequence of Theorem 1.2, we obtain in Theorem 3.1 an upper and a lower bounds for the asymptotic linear function $\operatorname{reg}(I^q)$. We prove that for $q \gg 0$,

$$qd + a^*(\phi) \leq \operatorname{reg}(I^q) \leq qd + a^*(\phi) + \dim R.$$

This, in particular, allows us to settle Conjecture 1.1 in an important case. A fiber $\widetilde{X}_\wp$ is said to be *arithmetically Cohen-Macaulay* if its homogeneous coordinate ring $\mathcal{R}_\wp$ is Cohen-Macaulay. Our next result shows that Conjecture 1.1 holds under the additional condition that each fiber $\widetilde{X}_\wp$ is arithmetically Cohen-Macaulay. This hypothesis is satisfied, for instance, when the Rees algebra $\mathcal{R}$ is a Cohen-Macaulay ring.

Theorem 1.3 (Theorem 3.2). *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be an irreducible projective scheme of dimension at least 1, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree d. Let $\operatorname{reg}(\phi) = \max\{\operatorname{reg}(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Assume that each fiber $\widetilde{X}_\wp$ is an arithmetically Cohen-Macaulay scheme. Then for $q \gg 0$, we have*

$$\operatorname{reg}(I^q) = qd + \operatorname{reg}(\phi).$$

Our method in proving Theorem 1.2, and subsequently Theorem 1.3, is based upon investigating different graded structures of the Rees algebra $\mathcal{R}$. More precisely, beside the natural bi-graded structure mentioned above, $\mathcal{R}$ possesses two other $\mathbb{N}$-graded structures; namely

$$\mathcal{R} = \bigoplus_{q \in \mathbb{Z}} \mathcal{R}^1_q, \text{ where } \mathcal{R}^1_q = \bigoplus_{p \in \mathbb{Z}} \mathcal{R}_{(p,q)}, \text{ and}$$
$$\mathcal{R} = \bigoplus_{p \in \mathbb{Z}} \mathcal{R}^2_p, \text{ where } \mathcal{R}^2_p = \bigoplus_{q \in \mathbb{Z}} \mathcal{R}_{(p,q)}.$$

Under these $\mathbb{N}$-graded structures, it can be seen that $R = \mathcal{R}^1_0 \hookrightarrow \mathcal{R}$, $\mathcal{R}^1_q$ is a graded R-modules for any $q \in \mathbb{Z}$, $S = \mathcal{R}^2_0 \hookrightarrow \mathcal{R}$, and $\mathcal{R}^2_p$ is a graded S-modules for any $p \in \mathbb{Z}$. Let $\widetilde{\mathcal{R}^1_q}$ be the coherent sheaf associated to $\mathcal{R}^1_q$ on X, and let $\widetilde{\mathcal{R}^2_p}$ be the coherent

[1]Marc Chardin in a recent preprint [5] has proved that Conjecture 1.1 holds in general.

sheaf associated to $\mathcal{R}^2_p$ on $\bar{X}$. Observe further that $\mathcal{R}^1_q = \bigoplus_{p\in\mathbb{Z}} \left[I^q\right]_{p+qd} = I^q(qd)$, the module I^q shifted by qd. As a consequence, for any $p, q \in \mathbb{Z}$ we have

$$\widetilde{\mathcal{R}^1_q}(p) = \widetilde{I^q}(p+qd).$$

Thus, to study the regularity of I^q, we examine sheaf cohomology groups of $\widetilde{\mathcal{R}^1_q}(p)$. Our results are obtained by investigating how these sheaf cohomology groups behave by pulling back via the blowup map π and pushing forward through the projection map ϕ.

Our paper is outlined as follows. In the next section, we consider $\widetilde{X}$ as a biprojective scheme and prove a similar statement to Conjecture 1.1 for the a^*-invariant. In the last section, we prove an important case of Conjectures 1.1.

2. Bi-projective schemes and a^*-invariants

The goal of this section is to give a similar statement to Conjecture 1.1 for the a^*-invariant of powers of an ideal. We first recall the definition of regularity and a^*-invariant.

Definition 2.1. For any $\mathbb{N}$-graded algebra T, let T_+ denote its irrelevant ideal. For $i \ge 0$, let $a^i(T) = \max\{l \mid [H^i_{T_+}(T)]_l \neq 0\}$ (if $H^i_{T_+}(T) = 0$ then take $a^i(T) = -\infty$). The *a^*-invariant* and the *regularity* of T are defined to be

$$a^*(T) = \max_{i\ge 0}\{a^i(T)\} \text{ and } \operatorname{reg}(T) = \max_{i\ge 0}\{a^i(T) + i\}.$$

Note that $H^i_{T_+}(T) = 0$ for $i > \dim T$, so $a^*(T)$ and $\operatorname{reg}(T)$ are well-defined and finite invariants.

Let S denote the homogeneous coordinate ring of $\bar{X} \subseteq \mathbb{P}^m$. For each closed point $\wp \in \bar{X}$, i.e., $\wp$ is a homogeneous prime ideal in S, let $\mathcal{R}_\wp$ be the localization of $\mathcal{R}$ at $\wp$; that is, $\mathcal{R}_\wp = \mathcal{R} \otimes_S S_\wp$. The *homogeneous localization* of $\mathcal{R}$ at $\wp$, denoted by $\mathcal{R}_{(\wp)}$, is the collection of all element of degree 0 (in t) of $\mathcal{R}_\wp$. Observe that homogeneous localization at $\wp$ annihilates the grading with respect to powers of t. Thus, inheriting from the bi-graded structure of $\mathcal{R}$, the homogeneous localization $\mathcal{R}_{(\wp)}$ is a $\mathbb{N}$-graded ring. The regularity and a^*-invariant of $\mathcal{R}_{(\wp)}$ are therefore defined as usual.

Associated to $\phi : \widetilde{X} \to \bar{X}$, let

$$\begin{aligned} a^i(\phi) &= \max\{a^i(\mathcal{R}_{(\wp)}) \mid \wp \in \bar{X}\} \text{ for } i \ge 0, \\ a^*(\phi) &= \max\{a^*(\mathcal{R}_{(\wp)}) \mid \wp \in \bar{X}\}, \text{ and} \\ \operatorname{reg}(\phi) &= \max\{\operatorname{reg}(\mathcal{R}_{(\wp)}) \mid \wp \in \bar{X}\}. \end{aligned}$$

Remark 2.2. By definition, $a^*(\phi) = \max_{i\ge 0}\{a^i(\phi)\}$ and $\operatorname{reg}(\phi) = \max_{i\ge 0}\{a^i(\phi)+i\}$. Note that $H^i_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)}) = \left[H^i_{\mathcal{R}_+}(\mathcal{R})\right]_{(\wp)}$, where on the right hand side we view $\mathcal{R}$ under its $\mathbb{N}$-graded structure $\mathcal{R} = \bigoplus_{p\in\mathbb{Z}} \mathcal{R}^2_p$, which induces the embedding $S \hookrightarrow \mathcal{R}$. Thus, $a^i(\phi)$ is a well-defined and finite invariant for any $i \ge 0$. As a consequence, $a^*(\phi)$ and $\operatorname{reg}(\phi)$ are well-defined and finite invariants. These invariants are defined in a similar fashion to the *projective a^*-invariant* that was introduced in [16]. We shall also let r_ϕ denote the smallest integer r such that

$$a^*(\phi) = a^r(\phi).$$

Recall further that the Rees algebra $\mathcal{R} = R[It]$ of I is naturally bi-graded with $\mathcal{R}_{(p,q)} = (I^q)_{p+qd}t^q$, and we identify $\widetilde{X}$ with $\operatorname{Proj}\mathcal{R} \subseteq \mathbb{P}^n \times \mathbb{P}^m$. It can also be seen that $\widetilde{X}$ and $\bar{X}$ can be realized as the (closure of the) graph and the (closure of the) image of the rational map $\varphi : X \dashrightarrow \mathbb{P}^m$ given by $P \mapsto [F_0(P) : \cdots : F_m(P)]$ (cf. [8, 15, 16]). Under this identification, π and ϕ are restrictions on $\widetilde{X}$ of natural projections $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$. We have the following diagram:

$$\begin{array}{ccccc} & & \widetilde{X} & \subseteq & \mathbb{P}^n \times \mathbb{P}^m \\ \pi & \swarrow & & \searrow & \phi \\ X & & \stackrel{\varphi}{\dashrightarrow} & & \bar{X} \end{array}$$

Let $\mathcal{I}$ be the ideal sheaf of I, and let $\mathcal{L} = \mathcal{I}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(0,1)$.

Lemma 2.3. *With notations as above.*

1. *The homogeneous coordinate ring of $\bar{X}$ is $S \simeq k[F_0t, \ldots, F_mt]$.*
2. $\phi^*\mathcal{O}_{\bar{X}}(q) = \mathcal{L}^q \otimes \pi^*\mathcal{O}_X(qd)\ \forall\ q \in \mathbb{Z}$.
3. $\mathcal{O}_{\widetilde{X}}(p,q) = \pi^*\mathcal{O}_X(p) \otimes \phi^*\mathcal{O}_{\bar{X}}(q) \simeq \mathcal{L}^q \otimes \pi^*\mathcal{O}_X(p+qd)\ \forall\ p,q \in \mathbb{Z}$.

Proof. (1) follows from the construction of φ. (2) and (3) follow from the graded structures of $R, \mathcal{R}$ and S. $\square$

The next few lemmas exhibit how the a^*-invariant of fibers of ϕ governs sheaf cohomology groups via a push forward along ϕ.

Lemma 2.4. *Let $p > a^*(\phi)$. Then*

1. $\phi_*\mathcal{O}_{\widetilde{X}}(p,q) = \widetilde{\mathcal{R}^2_p}(q)$ *and* $R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,q) = 0$ *for any $j > 0$ and any $q \in \mathbb{Z}$,*
2. $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q)) = 0$ *for $i > 0$ and $q \gg 0$.*

Proof. By Lemma 2.3 and the projection formula we have

$$\phi_*\mathcal{O}_{\widetilde{X}}(p,q) = \phi_*\mathcal{O}_{\widetilde{X}}(p,0) \otimes \mathcal{O}_{\bar{X}}(q) \text{ and } R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,q) = R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,0) \otimes \mathcal{O}_{\bar{X}}(q).$$

Thus, to show (1) it suffices to prove the assertion for $q = 0$.

Let $\wp$ be any closed point of $\bar{X}$, and consider the restriction $\phi_\wp : \widetilde{X}_\wp = \operatorname{Proj}\mathcal{R}_{(\wp)} \to \operatorname{Spec}\mathcal{O}_{\bar{X},\wp}$ of ϕ over an open affine neighborhood $\operatorname{Spec}\mathcal{O}_{\bar{X},\wp}$ of $\wp$. We have

$$R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,0)\Big|_{\operatorname{Spec}\mathcal{O}_{\bar{X},\wp}} = R^j\phi_*\Big(\widetilde{\mathcal{R}_{(\wp)}}(p)\Big) = H^j(\widetilde{X}_\wp, \widetilde{\mathcal{R}_{(\wp)}}(p))^{\sim} \quad \forall\ j \geq 0. \tag{2.1}$$

For any $j \geq 0$ and any $\wp \in \bar{X}$, we have $p > a^*(\phi) \geq a^j(\mathcal{R}_{(\wp)})$; and thus, $\big[H^j_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)})\big]_p = 0$. Moreover, the Serre-Grothendieck correspondence give us an exact sequence

$$\begin{aligned} 0 \to \big[H^0_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)})\big]_p \to \big[\mathcal{R}_{(\wp)}\big]_p = \big(\mathcal{R}^2_p\big)_{(\wp)} \\ \to H^0(\widetilde{X}_\wp, \widetilde{\mathcal{R}_{(\wp)}}(p)) \to \big[H^1_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)})\big]_p \to 0 \end{aligned}$$

and isomorphisms

$$H^i(\widetilde{X}_\wp, \widetilde{\mathcal{R}_{(\wp)}}(p)) \simeq \big[H^{i+1}_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)})\big]_p \text{ for } i > 0.$$

Therefore, for any $j \ge 0$ and any $\wp \in \bar{X}$,

$$R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,0)\Big|_{\operatorname{Spec}\mathcal{O}_{\bar{X},\wp}} = H^j(\widetilde{X}_\wp, \widetilde{\mathcal{R}_{(\wp)}(p)})^{\sim} = \begin{cases} \widetilde{(\mathcal{R}^2_p)_{(\wp)}} & \text{for } j = 0 \\ 0 & \text{for } j > 0. \end{cases}$$

This is true for any $\wp \in \bar{X}$, and so (1) is proved.

Now, it follows from (1) that the Leray spectral sequence $H^i(\bar{X}, R^j\phi_*\mathcal{O}_{\widetilde{X}}(p,q)) \Rightarrow H^{i+j}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q))$ degenerates. Thus, for any $j \ge 0$,

$$H^j(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q)) = H^j(\bar{X}, \widetilde{\mathcal{R}^2_p}(q)).$$

Moreover, since $\mathcal{O}_{\bar{X}}(1)$ is a very ample divisor, we have $H^j(\bar{X}, \widetilde{\mathcal{R}^2_p}(q)) = 0$ for all $q \gg 0$, and (2) is proved. □

Lemma 2.5. *Let r_ϕ be defined as above.*

(1) *If $r_\phi \le 1$ then $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a^*(\phi), q)) \ne \mathcal{R}_{(a^*(\phi),q)}$ for $q \gg 0$.*
(2) *If $r_\phi \ge 2$ then $H^{r_\phi-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a^*(\phi), q)) \ne 0$ for $q \gg 0$.*

Proof. For simplicity, let $a = a^*(\phi)$. By the definition of r_ϕ, we have

$$\begin{cases} \left[H^i_{\mathcal{R}_{(\wp)+}}(\mathcal{R}_{(\wp)})\right]_a = 0 & \text{for } i < r_\phi \text{ and any } \wp \in \bar{X} \\ \left[H^{r_\phi}_{\mathcal{R}_{(\mathfrak{q})+}}(\mathcal{R}_{(\mathfrak{q})})\right]_a \ne 0 & \text{for some } \mathfrak{q} \in \bar{X}. \end{cases} \tag{2.2}$$

(1) If $r_\phi \le 1$ then it follows from (2.2) and the Serre-Grothendieck correspondence that $H^0(\widetilde{X}_\mathfrak{q}, \widetilde{\mathcal{R}_{(\mathfrak{q})}}(a)) \ne \left[\mathcal{R}_{(\mathfrak{q})}\right]_a = \left(\mathcal{R}^2_a\right)_{(\mathfrak{q})}$. This and (2.1) imply that $\phi_*\mathcal{O}_{\widetilde{X}}(a,0) \ne \widetilde{\mathcal{R}^2_a}$, and so

$$\phi_*\mathcal{O}_{\widetilde{X}}(a,q) \ne \widetilde{\mathcal{R}^2_a}(q) \text{ for any } q \in \mathbb{Z}.$$

Since both $\phi_*\mathcal{O}_{\widetilde{X}}(a,q) = \phi_*\mathcal{O}_{\widetilde{X}}(a,0) \otimes \mathcal{O}_{\bar{X}}(q)$ (by Lemma 2.3 and the projection formula) and $\widetilde{\mathcal{R}^2_a}(q)$ are generated by global sections for $q \gg 0$, we must have

$$H^0(\bar{X}, \phi_*\mathcal{O}_{\widetilde{X}}(a,q)) \ne H^0(\bar{X}, \widetilde{\mathcal{R}^2_a}(q)) = \mathcal{R}_{(a,q)} \ \forall\ q \gg 0.$$

Moreover, $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a,q)) = H^0(\bar{X}, \phi_*\mathcal{O}_{\widetilde{X}}(a,q))$. Thus,

$$H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a,q)) \ne \mathcal{R}_{(a,q)} \text{ for } q \gg 0.$$

(2) If $r_\phi \ge 2$, then it follows from (2.2) and (2.1) that

$$\begin{cases} R^j\phi_*\mathcal{O}_{\widetilde{X}}(a,q) = 0 \text{ for } 0 < j < r_\phi - 1, \\ R^{r_\phi-1}\phi_*\mathcal{O}_{\widetilde{X}}(a,q) \ne 0. \end{cases} \tag{2.3}$$

By Lemma 2.3 and the projection formula, $\phi_*\mathcal{O}_{\widetilde{X}}(a,q) = \phi_*\mathcal{O}_{\widetilde{X}}(a,0) \otimes \mathcal{O}_{\bar{X}}(q)$. Thus, for $q \gg 0$ we have $H^{r_\phi-1}(\bar{X}, \phi_*\mathcal{O}_{\widetilde{X}}(a,q)) = 0$. From this, together with (2.3) and the Leray spectral sequence $H^i(\bar{X}, R^j\phi_*\mathcal{O}_{\widetilde{X}}(a,q)) \Rightarrow H^{i+j}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a,q))$, we can deduce that

$$H^{r_\phi-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a,q)) = H^0(\bar{X}, R^{r_\phi-1}\phi_*\mathcal{O}_{\widetilde{X}}(a,q)) \text{ for } q \gg 0.$$

It then follows, since $R^{r_\phi-1}\phi_*\mathcal{O}_{\widetilde{X}}(a,q) = R^{r_\phi-1}\phi_*\mathcal{O}_{\widetilde{X}}(a,0) \otimes \mathcal{O}_{\bar{X}}(q)$ is globally generated for $q \gg 0$, that

$$H^{r_\phi-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a,q)) \ne 0 \text{ for } q \gg 0.$$

□

Our first main result is a similar statement to Conjecture 1.1 for the a^*-invariant.

Theorem 2.6. *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree d. Let $a^*(\phi) = \max\{a^*(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Then for $q \gg 0$, we have*

$$a^*(I^q) = qd + a^*(\phi).$$

Proof. By a similar argument as in Lemma 2.4, considering π_* instead of ϕ_*, we can show that for $q \gg 0$,

$$\pi_* \mathcal{O}_{\widetilde{X}}(p,q) = \widetilde{\mathcal{R}^1_q}(p) = \widetilde{I^q}(p+qd) \text{ and } R^j \pi_* \mathcal{O}_{\widetilde{X}}(p,q) = 0 \ \forall\ j > 0. \tag{2.4}$$

This implies that for $q \gg 0$, the Leray spectral sequence $H^i(X, R^j\pi_*\mathcal{O}_{\widetilde{X}}(p,q)) \Rightarrow H^{i+j}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q))$ degenerates and we have

$$H^j(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q)) = H^j(X, \widetilde{I^q}(p+qd)) \ \forall\ j \geq 0.$$

Therefore, for $j > 0$, $q \gg 0$ and $p > a^*(\phi)$, it follows from Lemma 2.4 that $H^j(X, \widetilde{I^q}(p+qd)) = 0$. That is,

$$\big[H^{j+1}_{R_+}(I^q)\big]_{p+qd} = 0. \tag{2.5}$$

Furthermore, for $j = 0$ and $q \gg 0$, we have $H^0(\bar{X}, \widetilde{\mathcal{R}^2_p}(q)) = H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q)) = H^0(X, \widetilde{I^q}(p+qd))$, where the first equality follows from Lemma 2.4. On the other hand, for $q \gg 0$, $H^0(\bar{X}, \widetilde{\mathcal{R}^2_p}(q)) = (\mathcal{R}^2_p)_q = \mathcal{R}_{(p,q)} = [I^q]_{p+qd}$. Thus, for $q \gg 0$, $H^0(X, \widetilde{I^q}(p+qd)) = [I^q]_{p+qd}$. This and (2.5) imply that for $q \gg 0$,

$$a^*(I^q) \leq qd + a^*(\phi).$$

To prove the other inequality, let r_ϕ be defined as in Remark 2.2. We consider two cases: $r_\phi \leq 1$ and $r_\phi \geq 2$. If $r_\phi \leq 1$ then by Lemma 2.5, $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a^*(\phi),q)) \neq \mathcal{R}_{(a^*(\phi),q)}$ for all $q \gg 0$. This implies that $H^0(X, \pi_*\mathcal{O}_{\widetilde{X}}(a^*(\phi),q)) \neq \mathcal{R}_{(a^*(\phi),q)}$ for $q \gg 0$. That is,

$$H^0(X, \widetilde{I^q}(a^*(\phi)+qd)) \neq \big[I^q\big]_{a^*(\phi)+qd} \ \forall\ q \gg 0.$$

By the Serre-Grothendieck correspondence, for $q \gg 0$, we have either

$$\big[H^0_{R_+}(I^q)\big]_{(a^*(\phi)+qd,q)} \neq 0 \text{ or } \big[H^1_{R_+}(I^q)\big]_{(a^*(\phi)+qd,q)} \neq 0.$$

It then follows that $a^*(I^q) \geq qd + a^*(\phi)$ for $q \gg 0$.

If $r_\phi \geq 2$, then by Lemma 2.5, $H^{r_\phi - 1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a^*(\phi),q)) \neq 0$ for $q \gg 0$. Moreover, for $q \gg 0$, it follows from (2.4) that the Leray spectral sequence

$$H^i(X, R^j\pi_*\mathcal{O}_{\widetilde{X}}(p,q)) \Rightarrow H^{i+j}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p,q))$$

degenerates. Thus, for $q \gg 0$, we have $H^{r_\phi-1}(X, \widetilde{I^q}(a^*(\phi)+qd)) \neq 0$. By the Serre-Grothendieck correspondence, we have $\big[H^{r_\phi}_{R_+}(I^q)\big]_{a^*(\phi)+qd} \neq 0$ for $q \gg 0$. This implies that $a^*(I^q) \geq qd + a^*(\phi)$ for $q \gg 0$. $\square$

Example 2.7. Let $R = \bigoplus_{n\geq 0} R_n$ be a Cohen-Macaulay standard graded domain, and let $A = (a_{ij})_{1\leq i\leq r, 1\leq j\leq s}$ be an $r \times s$ matrix ($r \leq s$) of entries in R_1. Let $I_t(A)$ denote the ideal generated by $t \times t$ minors of A, and let $I = I_r(A)$. Assume that for any $1 \leq t \leq r$, $\operatorname{ht} I_t(A) \geq (r-t+1)(s-r)+1$. Let $\imath(\omega_R)$ be the least generating degree of ω_R, the canonical module of R. Then for $q \gg 0$,

$$a^*(I^q) = qr - \imath(\omega_R).$$

Indeed, let $S = k[It]$ denote the homogeneous coordinate ring of $\bar{X}$, let $\wp$ be any point in $\bar{X}$, and let $T = \mathcal{R}_{(\wp)}$. By [11, Theorem 3.5], the Rees algebra $\mathcal{R}$ is Cohen-Macaulay. Thus, $\mathcal{R}_{(\wp)}$ is Cohen-Macaulay. This implies that

$$a^*(T) = a^{\dim T}(T) = -\min\{s \mid [\omega_T]_s \neq 0\}.$$

Furthermore, by [17, Example 3.8],

$$\omega_{\mathcal{R}} = \omega_R(1,t)^{g-2} = \omega_R \oplus \omega_R t \oplus \cdots \oplus \omega_R t^{g-2} \oplus \omega_R I t^{g-1} \oplus \ldots,$$

where $g = \operatorname{ht} I$. Hence, by localizing at $\wp$, we obtain

$$\omega_T = \big(\omega_{\mathcal{R}}\big)_{(\wp)} = \big(\omega_R(1,t)^{g-2}\big)_{(\wp)}.$$

Observe that the homogeneous localization at $\wp$ annihilates the grading inherited from powers of t, so it follows that the degrees of ω_T arise from the degrees of ω_R. That is, $\imath(\omega_T) = \imath(\omega_R)$, and the conclusion follows from Theorem 2.6.

3. Regularity of powers of ideals

In this section, we investigate the asymptotic linearity of regularity and prove a special case of Conjecture 1.1.

We start by giving an upper and a lower bound for the free constant of $\operatorname{reg}(I^q)$ in terms of $a^*(\phi)$.

Theorem 3.1. *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subsetneq R$ be a homogeneous ideal generated by forms of degree d. Let $a^*(\phi) = \max\{a^*(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Then there exists an integer $0 \leq r \leq \dim R$ such that for $q \gg 0$, we have $\operatorname{reg}(I^q) = qd + a^*(\phi) + r$. In particular, for $q \gg 0$,*

$$qd + a^*(\phi) \leq \operatorname{reg}(I^q) \leq qd + a^*(\phi) + \dim R.$$

Proof. Suppose $\operatorname{reg}(I^q) = aq + b$ for $q \gg 0$. It can be easily seen from the definition of the regularity and a^*-invariant of graded R-modules that $a^*(I^q) \leq \operatorname{reg}(I^q) \leq a^*(I^q) + \dim R$ for any q. This and Theorem 2.6 imply that $a = d$; that is, $\operatorname{reg}(I^q) = qd + b$ for $q \gg 0$. Let $r = b - a^*(\phi)$. Then $\operatorname{reg}(I^q) = qd + a^*(\phi) + r$, and since $a^*(I^q) \leq \operatorname{reg}(I^q) \leq a^*(I^q) + \dim R$, we have $0 \leq r \leq \dim R$. □

Our next result shows that Conjecture 1.1 holds under an extra condition that each fiber $\widetilde{X}_\wp$ is an arithmetically Cohen-Macaulay scheme.

Theorem 3.2. *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be an irreducible projective scheme of dimension at least 1, and let $I \subsetneq R$ be a homogeneous ideal generated by forms of degree d. Let $\operatorname{reg}(\phi) = \max\{\operatorname{reg}(\widetilde{X}_\wp) \mid \wp \in \bar{X}\}$. Assume that each fiber $\widetilde{X}_\wp$ is an arithmetically Cohen-Macaulay scheme. Then for $q \gg 0$, we have*

$$\operatorname{reg}(I^q) = qd + \operatorname{reg}(\phi).$$

Proof. Let $l = \dim X \geq 1$. Since X is irreducible, $\widetilde{X}$ is also irreducible. Moreover, for any point $\wp \in \bar{X}$, $\operatorname{Spec} \mathcal{O}_{\bar{X},\wp}$ is an open neighborhood of $\wp$, and so $\widetilde{X}_\wp = \phi^{-1}(\operatorname{Spec} \mathcal{O}_{\bar{X},\wp})$ is an open subset in $\widetilde{X}$. Thus, $\dim \widetilde{X}_\wp = \dim \widetilde{X} = \dim X$.

By the hypothesis, for each $\wp \in \bar{X}$, $\mathcal{R}_{(\wp)}$ is a Cohen-Macaulay ring of dimension $\dim \widetilde{X}_\wp + 1 = l + 1$. This implies that $a^*(\mathcal{R}_{(\wp)}) = a^{l+1}(\mathcal{R}_{(\wp)})$ and $\operatorname{reg}(\mathcal{R}_{(\wp)}) = a^{l+1}(\mathcal{R}_{(\wp)}) + (l+1)$. Therefore,

$$a^*(\phi) = a^{l+1}(\phi), \tag{3.1}$$

$$\operatorname{reg}(\phi) = a^*(\phi) + l + 1. \tag{3.2}$$

It follows from (3.1) that $r_\phi = l + 1 \geq 2$. By the same arguments as the last part of the proof of Theorem 2.6, we have that for $q \gg 0$, $\operatorname{reg}(I^q) \geq qd + a^*(\phi) + r_\phi = qd + a^*(\phi) + \dim R$. This, together with Theorem 3.1, implies that for $q \gg 0$, $\operatorname{reg}(I^q) = qd + a^*(\phi) + \dim R$. The conclusion now follows from (3.2). □

Corollary 3.3. *Let $X = \operatorname{Proj} R \subseteq \mathbb{P}^n$ be an irreducible projective scheme of dimension at least 1, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree d. Assume that $\mathcal{R}$ is a Cohen-Macaulay ring. Then for $q \gg 0$,*

$$\operatorname{reg}(I^q) = qd + \operatorname{reg}(\phi).$$

Proof. Since $\mathcal{R}$ is Cohen-Macaulay, so is $\mathcal{R}_{(\wp)}$ for any $\wp \in \bar{X}$. Thus, each fiber $\widetilde{X}_\wp$ is arithmetically Cohen-Macaulay. The conclusion follows from Theorem 3.2. □

We shall end the paper with a number of examples in which the hypotheses of Corollary 3.3 are satisfied.

Example 3.4. Let R and I be as in Example 2.7. In this case, I is generated in degree r. As noted before, the Rees algebra $\mathcal{R}$ is Cohen-Macaulay. Notice further that $X = \operatorname{Proj} R$ is an irreducible projective scheme. Thus, by Corollary 3.3, we have

$$\operatorname{reg}(I^q) = qr + \operatorname{reg}(\phi) \ \forall\ q \gg 0.$$

Example 3.5. Let $R = k[x_{ij}]_{1 \leq i \leq r, 1 \leq j \leq s}$ and let I be the ideal generated by $t \times t$ minors of $M = (x_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ for some $1 \leq t \leq \min\{r, s\}$. By [11, Theorem 3.5] and [3, Corollary 3.3], the Rees algebra $\mathcal{R}$ of I is Cohen-Macaulay. Also, $X = \operatorname{Proj} R$ is an irreducible projective scheme. It follows from Corollary 3.3 that

$$\operatorname{reg}(I^q) = qt + \operatorname{reg}(\phi) \ \forall\ q \gg 0.$$

Example 3.6. Let R be a Cohen-Macaulay graded domain of dimension at least 2. Let I be either a complete intersection, or an almost complete intersection that is also generically a complete intersection. Assume that I is generated in degree d. Then the Rees algebra $\mathcal{R}$ of I is Cohen-Macaulay (cf. [2, 21]). By Corollary 3.3, we have

$$\operatorname{reg}(I^q) = qd + \operatorname{reg}(\phi) \ \forall\ q \gg 0.$$

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