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ON THE DERIVED DG FUNCTORS

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Abstract. Assume that abelian categories $\mathcal{A}$, $\mathcal{B}$ over a field admit countable direct limits and that these limits are exact. Let $\mathcal{F} : D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$ be a DG quasi-functor such that the functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ carries $D^{\geq 0}(\mathcal{A})$ to $D^{\geq 0}(\mathcal{B})$ and such that, for every $i > 0$, the functor $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is effaceable. We prove that $\mathcal{F}$ is canonically isomorphic to the right derived DG functor $RH^0(\mathcal{F})$. We also prove a similar result for bounded derived DG categories and a formula that expresses Hochschild cohomology of the categories $D^b_{dg}(\mathcal{A})$, $D^+_{dg}(\mathcal{A})$ as the *Ext* groups in the abelian category of left exact functors $\mathcal{A} \to Ind\mathcal{A}$. The proofs are based on a description of Drinfeld's category of quasi-functors as the derived category of a certain category of sheaves.

1. Main results

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let

$$RF_{tri} : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

be the right derived functor of some left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then, the corresponding cohomological δ-functor $R^*F = H^*RF_{tri} : \mathcal{A} \to \mathcal{B}$ has the following property: the functor H^iRF_{tri} is 0 for $i < 0$, effaceable for $i > 0$, and H^0RF_{tri} is isomorphic to F. Conversely, according to a result of Grothendieck ([G]), every cohomological δ-functor $T^* : \mathcal{A} \to \mathcal{B}$ satisfying the above property is canonically isomorphic to the right derived functor R^*F. The purpose of this paper is to extend this extremely useful characterization of R^*F to the derived category level. Unfortunately, Verdier's notion of triangulated functor seems too poor to allow such a simple characterization of the derived functors. In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of *DG quasi-functor* (or essentially equivalent notion of A_∞-functor). Indeed, works of Keller and Drinfeld ([K2], [Dri]) provide a canonical DG enhancement $D^+_{dg}(\mathcal{A})$ of Verdier's triangulated derived category. Roughly, a DG quasi-functor $\mathcal{F} : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$ is a diagram of the form

$$D^+_{dg}(\mathcal{A}) \xleftarrow{S} \mathcal{C} \xrightarrow{G} D^+_{dg}(\mathcal{B}). \tag{1.1}$$

where $\mathcal{C}$ is a DG category, S and G are DG functors, and, in addition, S is a homotopy equivalence. Every quasi-functor (1.1) yields a triangulated functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor RF can be canonically promoted to a DG

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quasi-functor ([Dri] §5). The main result of this paper asserts that under certain mild assumptions on abelian categories $\mathcal{A}$ and $\mathcal{B}$ the DG quasi-functors isomorphic to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck's condition above. To state the result we need to introduce a bit of notation.

Let k be a commutative ring. Denote by $Mod(k)$ the category of k-modules. We shall say that a k-linear category [1] is k-flat if, for every two objects X, Y, the k-module $Hom(X, Y)$ is flat. Given a k-linear exact category $\mathcal{A}$ we denote by $D^b_{dg}(\mathcal{A})$ the corresponding bounded derived DG category over k. This is the DG quotient ([Dri]) of the DG category $C^b_{dg}(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones ([N], §1). The homotopy category of $D^b_{dg}(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in ([N]). Let $\mathcal{B}$ be another k-linear abelian category, $D^b_{dg}(\mathcal{B})$ the corresponding bounded derived DG category, and let $\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F} : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$ ([Dri], §16.1). Given such $\mathcal{F}$ and an integer i we denote by $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ the composition

$$\mathcal{A} \to D^b_{dg}(\mathcal{A}) \xrightarrow{\mathcal{F}} D^b_{dg}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$

Theorem 1. *Let $\mathcal{A}$ be a small k-flat exact idempotent complete category [2] and $\mathcal{B}$ a small abelian k-linear category.*

(1) Assume that a DG quasi-functor

$$\mathcal{F} : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$$

has the following property:

(P) The functor $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is 0 for every $i < 0$ and effaceable [3] for every $i > 0$.

Then the functor $F := H^0\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is left exact, has a right derived DG quasi-functor ([Dri] *§5)*

$$RF : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals Id. Conversely, the right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \to \mathcal{B}$ satisfies property (P).

(2) For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ satisfying property (P) and every $i < 0$, we have

$$Hom_{\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$Hom_{\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{Fct(\mathcal{A},\mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G}).$$

Here $Fct(\mathcal{A}, \mathcal{B})$ denotes the category of all k-linear functors $\mathcal{A} \to \mathcal{B}$.

Remark 1.1. I do not know if the analogous statement holds for merely triangulated functors.

[1] *i.e.*, a category enriched over $Mod(k)$.

[2] An additive category is called idempotent complete if any its morphism $p : X \to X$ such that $p \circ p = p$ is the projection on a direct summand of a decomposition $X \simeq Y \oplus Z$.

[3] That is, for every object $X \in \mathcal{A}$, there exists an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^i\mathcal{F}(X) \to H^i\mathcal{F}(Y)$ is 0.

Remark 1.2. It is likely that the k-flatness assumption on $\mathcal{A}$ is unnecessary. However, I can not prove this.

We have a similar result for bounded from below derived DG categories. If $\mathcal{A}$ is a k-linear abelian category we will write $D^+_{dg}(\mathcal{A})$ for the bounded from below derived DG category of $\mathcal{A}$ and $D^+(\mathcal{A})$ for the corresponding triangulated category. Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D^+(\mathcal{A})$ that consists of complexes with trivial cohomology in degrees less then n. We say that a DG quasi-functor

$$\mathcal{F} : D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$$

has property (P') if

(P') The functor $Ho(\mathcal{F})$ takes every object of the category $D^{\geq 0}(\mathcal{A}))$ to an object of $D^{\geq 0}(\mathcal{B})$ and, for every $i > 0$, the functor $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is effaceable.

Theorem 2. *Let k be a field and let $\mathcal{A}$, $\mathcal{B}$ be small abelian k-linear categories. Assume that both categories are closed under countable direct limits and that these limits are exact.*

1. *Let $\mathcal{F} \in \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be a DG quasi-functor satisfying property (P') and $F := H^0\mathcal{F} : \mathcal{A} \to \mathcal{B}$. The functor F admits a right derived DG quasi-functor $RF : D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$ and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals Id. Conversely, a right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \to \mathcal{B}$ satisfies property (P').*
2. *For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ satisfying property (P') and every $i < 0$, we have*

$$Hom_{\mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$Hom_{\mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{Fct(\mathcal{A},\mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G}).$$

The main ingredient of the proof of Theorem 2 is the following construction. Let $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the category of k-linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \to Mod(k)$ that are left exact with respect to both arguments. Every k-linear left exact functor $F : \mathcal{A} \to \mathcal{B}$ yields $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B})$:

$$s(F)(X \otimes X') = Hom_{\mathcal{B}}(X', F(X)).$$

Let $\mathcal{T}^+ \subset \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors $\mathcal{F}$ such that $Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some n. Using key Lemma 2.1 we construct a fully faithful embedding

$$\mathcal{T}^+ \hookrightarrow D(Sh(\mathcal{A}^o \otimes_k \mathcal{B})) \tag{1.2}$$

that carries every DG quasi-functor $\mathcal{F}$ satisfying property (P') to $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$.

Remark 1.3. In ([T], Th. 8.9), Toën gave an analogous description of the category of quasi-functors between the derived DG categories of (quasi)-coherent sheaves.

As another application of (1.2) we compute the Hochschild cohomology of a derived DG category. Recall (see, e.g. [K1], §5.4, [T], §8.1) that the Hochschild cohomology of a DG category $\mathcal{C}$ can be interpreted as

$$HH^i(\mathcal{C},\mathcal{C}) = Hom_{\mathcal{T}(\mathcal{C},\mathcal{C})}(Id_{\mathcal{C}}, Id_{\mathcal{C}}[i]). \tag{1.3}$$

The composition in $\mathcal{C}$ makes $HH^*(\mathcal{C},\mathcal{C})$ a graded commutative algebra over k.

Theorem 3. *Let k be a field, and let $\mathcal{A}$ be a small abelian k-linear category. There is an isomorphism of algebras*

$$HH^*(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{A})) \simeq Ext^*_{Sh(\mathcal{A}^o\otimes_k\mathcal{A})}(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})). \tag{1.4}$$

If, in addition, $\mathcal{A}$ is closed under countable direct limits and that these limits are exact, we have

$$HH^*(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{A})) \simeq Ext^*_{Sh(\mathcal{A}^o\otimes_k\mathcal{A})}(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})). \tag{1.5}$$

Remark 1.4. This is a remarkable phenomenon the Hochschild cohomology does not change we "enlarge" the DG category. A similar result, that the Hochschild cohomology of a small DG category coincides with the Hochschild cohomology of its DG ind-completion, is due to Toën ([T], §8). An analogous statement for Grothendieck abelian categories was proved by Lowen and Van den Bergh ([LV]).

Remark 1.5. The category $Sh(\mathcal{A}^o \otimes_k \mathcal{A})$ has a tensor structure that extends the tensor structure on the category of left exact endofunctors $\mathcal{A} \to \mathcal{A}$ given by the composition. This can be used to promote (1.4), (1.5) to isomorphisms of *Gerstenhaber algebras* (see, e.g. [K1], §5.4).

Notation. Given a category $\mathcal{C}$ we denote by $\mathcal{C}^o$ the opposite category. If $\mathcal{C}$ is a DG category we will write $Ho\mathcal{C}$ for the corresponding homotopy category ([Dri], §2.7). For example, $HoC(Mod(k))$ denotes the homotopy category of complexes of k-modules. The derived category of right DG modules over a DG category $\mathcal{C}$ will be denoted by $\mathbb{D}(\mathcal{C})$ ([Dri], §2.3) [4]. We will write $\underrightarrow{\mathcal{C}}$ for the DG category of semi-free right DG modules over $\mathcal{C}$ ([BV], 1.6.1). We have a canonical equivalence of triangulated categories $Ho\underrightarrow{\mathcal{C}} \xrightarrow{\sim} \mathbb{D}(\mathcal{C})$ ([BV], 1.6.4). For DG categories $\mathcal{C}, \mathcal{C}'$ we denote by $\mathcal{T}(\mathcal{C},\mathcal{C}')$ the category of DG quasi-functors ([Dri], §16.1). If $\mathcal{C}'$ is a pretriangulated ([Dri], §2.4) $\mathcal{T}(\mathcal{C},\mathcal{C}')$ has a canonical structure of triangulated category. If $\mathcal{F} \in \mathcal{T}(\mathcal{C},\mathcal{C}')$ we will write $Ho(\mathcal{F})$ for the corresponding functor between the homotopy categories. The expression "direct limit" always means "filtrant direct limit" ([KS], §3).

2. Proofs

Proof of theorem 1. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors $\mathcal{F}$ such that $H^i\mathcal{F} = 0$ for sufficiently small i. To prove the Theorem, we shall construct (in Lemma 2.1 below) a fully faithful embedding of $\mathcal{T}^+$ into the derived category of a certain abelian category $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ that takes every functor $\mathcal{F} \in \mathcal{T}^+$ satisfying property (P) to an object of the heart $Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$.

[4] Drinfeld's notation for this category is $D(\mathcal{C})$. We use a different notation to avoid a possible confusion with Verdier's derived category of an abelian category $\mathcal{C}$ that is denoted by $D(\mathcal{C})$.

Under our flatness assumption on $\mathcal{A}$, the category $\mathcal{T}$ is a full subcategory of the derived category $\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ of right DG modules over $D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$ that consists of all $M \in \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ such that, for every X in $D^b_{dg}(\mathcal{A})^o$, the module $M(X) \in \mathbb{D}(D^b_{dg}(\mathcal{B}))$ belongs to the essential image of the Yoneda embedding $D^+_{dg}(\mathcal{B}) \to \mathbb{D}(D^b_{dg}(\mathcal{B}))$ ([Dri], §16.1).

Consider the restriction functor

$$\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} \mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$$

induced by the DG quasi-functor $\mathcal{A}^o \otimes_k \mathcal{B} \to D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$. By definition, the triangulated category $\mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$ is the derived category of the abelian category $PSh := PSh(\mathcal{A}^o \otimes_k \mathcal{B})$ of k-linear presheaves *i.e.*, the category of k-linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \to Mod(k)$. Consider a Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$ whose covers are maps of the form $f \otimes g : Y \otimes Y' \to X \otimes X'$, where $X, Y \in \mathcal{A}^o$, $X', Y' \in \mathcal{B}$, and $f : Y \to X$, $g : Y' \to X'$ are admissible epimorphisms [5] *i.e.,* a sieve $\mathcal{C}$ over $X \otimes X'$ is a covering sieve if there exist $f : Y \to X$, $g : Y' \to X'$ as above such that $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in \mathcal{C}$. The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X'$ and every morphism $Z \otimes Z' \xrightarrow{\phi} X \otimes X'$ there exists a cover $T \otimes T' \xrightarrow{p \otimes q} Z \otimes Z'$ and a morphism $T \otimes T' \xrightarrow{\psi} Y \otimes Y'$ such that $(f \otimes g) \circ \psi = \phi \circ (p \otimes q)$, which is a consequence of the base change axiom of exact category ([Q], §2). Let $Sh := Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the subcategory of PSh that consists of objects satisfying the sheaf property. Explicitely, objects of the category $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ are contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \to Mod(k)$ that are left exact with respect to both arguments. The embedding $Sh \to PSh$ has a left adjoint functor (sheafification)

$$\tilde{} : PSh \to Sh,$$

which is exact ([KS], §17.4). We denote by $\gamma : D(PSh) \to D(Sh)$ the induced functor between the derived categories. The composition

$$\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is *not* fully faithful in general, however, we have the following result.

Lemma 2.1. *(cf.* [T]*, Th. 8.9) Let $\mathbb{D}^+ \subset \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ be the full subcategory whose objects are DG modules M such that $\beta(M)$ is bounded from below. Then the functor*

$$S : \mathbb{D}^+ \xrightarrow{\beta} D^+(PSh) \xrightarrow{\gamma} D^+(Sh)$$

is an equivalence of categories.

Proof. The category $D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$ is the DG quotient of the category $C^b_{dg}(\mathcal{A})^o \otimes_k C^b_{dg}(\mathcal{B})$ by the full subcategory whose objects are of the form $X^\cdot \otimes X'^\cdot$, where either $X^\cdot$ or $X'^\cdot$ is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor

$$\beta : \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \to \mathbb{D}(C^b_{dg}(\mathcal{A})^o \otimes_k C^b_{dg}(\mathcal{B})) = D(PSh)$$

[5] By definition, admissible epimorphisms $Y \to X$ in $\mathcal{A}^o$ are admissible monomorphisms $X \to Y$ in $\mathcal{A}$.

is fully faithful and that its essential image consists of all DG-modules $M \in \mathbb{D}(C^b_{dg}(\mathcal{A})^o \otimes_k C^b_{dg}(\mathcal{B}))$ that carry every $X^{\cdot} \otimes X'^{\cdot}$ with the above property to an acyclic complex. Identifying the category $\mathbb{D}(C^b_{dg}(\mathcal{A})^o \otimes_k C^b_{dg}(\mathcal{B}))$ with $D(PSh)$ and observing that the subcategories of acyclic complexes in the homotopy categories $HoC^b_{dg}(\mathcal{A})$, $HoC^b_{dg}(\mathcal{B})$ are generated by short exact sequences ([N], §1) we exhibit $\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ as a full subcategory $\mathcal{R} \subset D(PSh)$ whose objects are complexes $F^{\cdot}$ of presheaves satisfying the following two conditions:

- For any exact sequence $0 \to Z \to Y \to X \to 0$ in $\mathcal{A}^o$ and any $X' \in \mathcal{B}$ the total complex of
$$F^{\cdot}(X \otimes X') \to F^{\cdot}(Y \otimes X') \to F^{\cdot}(Z \otimes X') \tag{2.1}$$
is acyclic.
- For any $X \in \mathcal{A}^o$ and any exact sequence $0 \to Z' \to Y' \to X' \to 0$ in $\mathcal{B}$ the total complex of
$$F^{\cdot}(X \otimes X') \to F^{\cdot}(X \otimes Y') \to F^{\cdot}(X \otimes Z')$$
is acyclic.

Observe that, for every $F^{\cdot} \in \mathcal{R}$ and an exact sequence $0 \to Z \to Y \to X \to 0$ in $\mathcal{A}^o$, we have a long exact sequence of k-modules

(2.2)
$$\to H^{m-1}(F^{\cdot}(Z\otimes X')) \to H^m(F^{\cdot}(X\otimes X')) \to H^m(F^{\cdot}(Y\otimes X')) \to H^m(F^{\cdot}(Z\otimes X')) \to$$

The equivalence of categories
$$\beta : \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\sim} \mathcal{R} \subset D(PSh)$$
carries $\mathbb{D}^+$ to the subcategory $\mathcal{R}^+$ of $\mathcal{R}$ that consists of bounded from below complexes.

The derived category of sheaves $D(Sh)$ is the quotient of the derived category of presheaves by the subcategory $\mathcal{I}_{lac} \subset D(PSh)$ of locally (for our Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$) acyclic complexes ([BV], §1.11). We shall prove that
$$\mathcal{R}^+ \subset \mathcal{I}^{\perp}_{lac}, \tag{2.3}$$
where $\mathcal{I}^{\perp}_{lac}$ denotes the right orthogonal complement to $\mathcal{I}_{lac}$ in $D(PSh)$ ([BV] §1.1); *i.e.*
$$Hom_{D(PSh)}(G^{\cdot}, F^{\cdot}) = 0. \tag{2.4}$$
for every $G^{\cdot} \in \mathcal{I}_{lac}$ and $F^{\cdot} \in \mathcal{R}^+$. Without loss of generality we may assume that $F^{\cdot}$ has trivial cohomology in negative degrees: $F^{\cdot} = F^0 \to F^1 \to \cdots$. Let $\tilde{F}^{\cdot} = \tilde{F}^0 \to \tilde{F}^1 \to \cdots$ be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. [KS], Th. 9.6.2, 18.1.6) there exists a complex $I^{\cdot} = I^0 \to I^1 \to \cdots$ of injective sheaves together with a morphism $\tilde{F}^{\cdot} \to I^{\cdot}$ which is an isomorphism in the derived category of sheaves. Let us show that the composition
$$\delta : F^{\cdot} \to \tilde{F}^{\cdot} \to I^{\cdot}$$
is an isomorphism in the derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of $\mathcal{R}$. Thus $I^{\cdot}$ and $cone(\delta)$ are in $\mathcal{R}^+$. Assuming that $cone(\delta) \neq 0$ choose the smallest integer m such that
$$0 \neq H^m(cone(\delta)) \in PSh.$$

Then, there exist an object $X \otimes X' \in \mathcal{A}^o \otimes_k \mathcal{B}$ and a nonzero element $a \in H^m(cone(\delta))(X \otimes X')$. Since the sheafification of $H^m(cone(\delta))$ is 0 there exists a cover $p : Y \otimes Y' \to X \otimes X'$ such that

$$0 = p^* a \in H^m(cone(\delta))(Y \otimes Y').$$

Writing p as a composition

$$Y \otimes Y' \xrightarrow{1 \otimes g} Y \otimes X' \xrightarrow{f \otimes 1} X \otimes X'$$

we may assume $(f \otimes 1)^* a = 0$ (otherwise, we replace $X \otimes X'$ by $Y \otimes X'$). Let us look at the following fragment of the long exact sequence (2.2) applied to $F = cone(\delta)$ and the exact sequence $0 \to Z \to Y \xrightarrow{f} X \to 0$:

$$H^{m-1}(cone(\delta))(Z \otimes X') \to H^m(cone(\delta))(X \otimes X') \to H^m(cone(\delta))(Y \otimes X').$$

Since, by our assumption, $H^{m-1}(cone(\delta)) = 0$, it follows that $(f \otimes 1)^*$ is injective and, hence, $a = 0$. This contradiction proves that $cone(\delta) = 0$ *i.e.*, δ is a quasi-isomorphism. Thus, to complete the proof of (2.4) it suffices to show that

$$Hom_{D(PSh)}(G^{\cdot}, I^{\cdot}) = 0,$$

for every $G^{\cdot} \in \mathcal{I}_{lac}$ and every bounded from below complex of injective sheaves $I^{\cdot}$. Indeed, every morphism $h : G^{\cdot} \to I^{\cdot}$ in the derived category is represented by a diagram in $C(PSh(\mathcal{A}^o \otimes_k \mathcal{B}))$

$$G^{\cdot} \leftarrow G'^{\cdot} \xrightarrow{h'} I^{\cdot},$$

where the first arrow is a quasi-isomorphism (and, in particular, $G'^{\cdot} \in \mathcal{I}_{lac}$). If h' is homotopic to 0 then h is 0 in the derived category. Thus, it is enough to show that

$$Hom_{K(PSh)}(G'^{\cdot}, I^{\cdot}) = 0,$$

where $K(PSh)$ denotes the homotopy category of complexes. We have

$$Hom_{K(PSh)}(G'^{\cdot}, I^{\cdot}) \xrightarrow{\sim} Hom_{K(Sh)}(\tilde{G}'^{\cdot}, I^{\cdot}) \xrightarrow{\sim} Hom_{D(Sh)}(\tilde{G}'^{\cdot}, I^{\cdot}).$$

The first arrow is an isomorphism because all terms of the complex $I^{\cdot}$ are sheaves; the second arrow is an isomorphism by ([KS], Lemma 13.2.4). Finally, the group $Hom_{D(Sh)}(\tilde{G}'^{\cdot}, I^{\cdot})$ is trivial because the sheafification $\tilde{G}'^{\cdot}$ is 0 in $D(Sh)$.

To finish the proof of the lemma, we observe that, for every triangulated category $\mathcal{C}$ and its full triangulated subcategory $\mathcal{I}$, the composition

$$\mathcal{I}^{\perp} \to \mathcal{C} \to \mathcal{C}/\mathcal{I}$$

is a fully faithful embedding: for every $X, Y \in \mathcal{C}$

$$Hom_{\mathcal{C}/\mathcal{I}}(X, Y) := \underset{f: X' \to X}{colim}\, Hom_{\mathcal{C}}(X', Y),$$

where the colimit is taken over the filtrant category of pairs $(X' \in \mathcal{C}, f : X' \to X)$ such that $cone\, f \in \mathcal{I}$. If $Y \in \mathcal{I}^{\perp}$, then

$$Hom_{\mathcal{C}}(X, Y) \xrightarrow{\sim} Hom_{\mathcal{C}}(X', Y),$$

and, hence,

$$Hom_{\mathcal{C}/\mathcal{I}}(X, Y) = Hom_{\mathcal{C}}(X, Y).$$

Applying this remark to $\mathcal{C} = D(PSh)$, $\mathcal{I} = \mathcal{I}_{lac}$ and using (2.4) we conclude that the functor $\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$ is fully faithful and, hence, so is the composition $\mathbb{D}^+ \xrightarrow{\sim}$

$\mathcal{R}^+ \stackrel{\gamma}{\longrightarrow} D(Sh)$. The essential image the functor $\mathcal{R}^+ \stackrel{\gamma}{\longrightarrow} D(Sh)$ coincides with $D^+(Sh)$ because because every complex of injective sheaves viewed as a complex of presheaves is an object of $\mathcal{R}^+$. □

Remark 2.2. Applying Lemma 2.1 to $k = \mathbb{Z}$ and $\mathcal{A}$ being the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category $\mathcal{B}$

$$\mathbb{D}^+(D^b_{dg}(\mathcal{B})) \stackrel{\sim}{\longrightarrow} D^+(PSh(\mathcal{B})) = D^+(Ind(\mathcal{B})),$$

where $\mathbb{D}^+(D^b_{dg}(\mathcal{B}))$ is the full subcategory of $\mathbb{D}(D^b_{dg}(\mathcal{B}))$ that maps to $D^+(PSh(\mathcal{B}))$ under the restriction functor (and the ind-completion $Ind(\mathcal{B})$ is just another name for $PSh(\mathcal{B})$ ([KS], §8.6)). Note the functor

$$\mathbb{D}(D^b_{dg}(\mathcal{B})) \to D(Ind(\mathcal{B})) \tag{2.5}$$

is not an equivalence of categories in general. In fact, the functor (2.5) factors as

$$\mathbb{D}(D^b_{dg}(\mathcal{B})) \xrightarrow[\sim]{\phi} HoC(Ind(\mathcal{B}))/Ho\overline{C^b_{ac}(\mathcal{B})} \stackrel{p}{\longrightarrow} D(Ind(\mathcal{B})), \tag{2.6}$$

where $Ho\overline{C^b_{ac}(\mathcal{B})}$ is the smallest triangulated subcategory of the homotopy category of acyclic complexes $HoC_{ac}(Ind(\mathcal{B}))$ that contains *finite* acyclic complexes $HoC^b_{ac}(\mathcal{B})$ and closed under arbitrary direct sums; the functor p is the projection

$$HoC(Ind(\mathcal{B}))/Ho\overline{C^b_{ac}(\mathcal{B})} \to HoC(Ind(\mathcal{B}))/HoC_{ac}(Ind(\mathcal{B})).$$

The equivalence ϕ can be constructed as follows. Let $\overline{C^b_{ac}(\mathcal{B})}$ be the full subcategory of the DG category $C(Ind(\mathcal{B}))$ whose objects are those of $Ho\overline{C^b_{ac}(\mathcal{B})}$. The DG quasi-functor $D^b_{dg}(\mathcal{B}) \to C(Ind(\mathcal{B}))/\overline{C^b_{ac}(\mathcal{B})}$ extends uniquely to a quasi-functor

$$\phi_{dg} : \underrightarrow{D^b_{dg}(\mathcal{B})} \to C(Ind(\mathcal{B}))/\overline{C^b_{ac}(\mathcal{B})}$$

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

$$\phi := Ho\phi_{dg}.$$

Let us show that ϕ is an equivalence of categories. The subcategory $Ho\overline{C^b_{ac}(\mathcal{B})} \subset HoC(Ind(\mathcal{B}))$ is generated by compact objects (e.g., objects of $HoC^b_{ac}(\mathcal{B})$); it follows that the projection $HoC(Ind(\mathcal{B})) \to HoC(Ind(\mathcal{B}))/Ho\overline{C^b_{ac}(\mathcal{B})}$ carries compact objects of $HoC(Ind(\mathcal{B}))$ to compact objects of the quotient category ([BV], §1.4.2). In particular, in the following commutative diagram

$$\begin{array}{ccc} D^b_{dg}(\mathcal{B}) & = & D^b_{dg}(\mathcal{B}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathbb{D}(D^b_{dg}(\mathcal{B})) & \stackrel{\phi}{\longrightarrow} & HoC(Ind(\mathcal{B}))/Ho\overline{C^b_{ac}(\mathcal{B})} \end{array}$$

the image of j consists of compact objects. The same is true for the image of i ([BV], §1.7). The functors i,j are fully faithful and their images generate the categories $\mathbb{D}(D^b_{dg}(\mathcal{B}))$, $HoC(Ind(\mathcal{B}))/Ho\overline{C^b_{ac}(\mathcal{B})}$ respectfully. It follows that ϕ is an equivalence of categories.

In general, (e.g., if $\mathcal{B}$ is the category of finitely generated modules over a finite group) the projection p is not conservative. However, if the category $\mathcal{B}$ has *finite* *homological dimension* the objects of $D^b_{dg}(\mathcal{B})$ are compact in $D^b_{dg}(Ind(\mathcal{B}))$ [6] and the above argument proves that (2.5) is an equivalence of categories.

Corollary 2.3. *The composition*

$$S : \mathcal{T}^+ \xrightarrow{\alpha} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh) \tag{2.7}$$

is a fully faithful embedding.

Consider the Yoneda embedding

$$s : Fun(\mathcal{A}, \mathcal{B}) \to PSh$$

that takes a functor $F \in Fun(\mathcal{A}, \mathcal{B})$ to the presheaf

$$s(F)(X \times X') = Hom_{\mathcal{B}}(X', F(X)).$$

If F is left exact then $s(F)$ is actually a sheaf.

Let $\mathcal{F} \in \mathcal{T}$ be a DG quasi-functor satisfying property (P). It follows from the definition of $\mathcal{T}^+$ given at the beginning of this section that $\mathcal{F} \in \mathcal{T}^+$. We shall prove that $S(\mathcal{F}) \xrightarrow{\sim} s(H^0\mathcal{F})$. Having in mind applications to Theorem 2 we will actually show a slightly more general statement. Namely, let us extend the functor (2.7) to a larger category:

$$S' : \mathcal{T}(D^b_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B})) \xrightarrow{\alpha'} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B})) \xrightarrow{\beta'} D(PSh) \xrightarrow{\gamma} D(Sh).$$

Lemma 2.4. *Let $\mathcal{F} \in \mathcal{T}(D^b_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be a DG quasi-functor such that $H^i\mathcal{F}$ is zero for $i < 0$ and effaceable for $i > 0$. Set $s(F) = s(H^0\mathcal{F}) \subset Sh \subset D(Sh)$* [7]*. Then the complex $S'(\mathcal{F}) \in D(Sh)$ is canonically quasi-isomorphic to $s(F)$.*

Proof. By definition, the cohomology presheaves of the complex $\beta'\alpha'(\mathcal{F}) \in D(PSh)$ are given by the formula

$$H^i(\beta'\alpha'\mathcal{F})(X \otimes X') = Hom_{D^+(\mathcal{B})}(X', Ho(\mathcal{F})(X)[i]).$$

Since the negative cohomology of the complex $Ho(\mathcal{F})(X) \in D^+(\mathcal{B})$ vanishes the same is true for $\beta'\alpha'\mathcal{F}$ and, thus, we have

$$H^0(\beta'\alpha'\mathcal{F})(X \otimes X') = Hom_{D^+(\mathcal{B})}(X', H^0\mathcal{F}(X)) = s(F).$$

It remains to prove that for every $i > 0$ the sheafification of the presheaf $H^i(\beta'\alpha'\mathcal{F})$ equals zero. Given an integer j define presheaves $G^{i,j}$ to be

$$G^{i,j}(X \otimes X') = Hom_{D^+(\mathcal{B})}(X', \tau_{\leq j}(Ho(\mathcal{F})(X))[i]).$$

We shall show by induction on j that for every $i > 0$ and every j the sheafification of $G^{i,j}$ is 0. This would complete the proof since $G^{i,j}$ is isomorphic to $H^i(\beta'\alpha'\mathcal{F})(X\otimes X')$ for $j \geq i$. For every $i > 0$ and every element v of the group

$$G^{i,0}(X \otimes X') = Ext^i_{\mathcal{B}}(X', H^0\mathcal{F}(X))$$

there exists an epimorphism $Y' \to X'$ such that v is annihilated by the map

$$Ext^i_{\mathcal{B}}(X', H^0\mathcal{F}(X)) \to Ext^i_{\mathcal{B}}(Y', H^0\mathcal{F}(X))$$

---

[6]Indeed, under our finiteness assumption every complex in $D^b_{dg}(\mathcal{B})$ is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of $\mathcal{B}$ is compact in $D(Ind(\mathcal{B}))$. This is clear because every such object is projective and compact in $Ind(\mathcal{B})$.

[7]The vanishing of $H^i\mathcal{F}$ implies that F is left exact and, hence, $s(F)$ is a sheaf.

([KS], Exercise 13.17). This proves that the sheafification of $G^{i,0}$ is 0. For the induction step, consider the distinguished triangle

$$\tau_{\leq j}(Ho(\mathcal{F})(X)) \to \tau_{\leq j+1}(Ho(\mathcal{F})(X)) \to H^{j+1}\mathcal{F}(X)[-j-1]$$

and the corresponding long exact sequence

$$\to G^{i,j}(X \times X') \to G^{i,j+1}(X \times X') \to Hom_{D^b(\mathcal{B})}(X', H^{j+1}\mathcal{F}(X)[-j-1+i]) \to .$$

It follows that $G^{i,j+1}$ fits in a long exact sequence

$$\to G^{i,j} \to G^{i,j+1} \to Ext^{i-j-1}_{\mathcal{B}}(\cdot, H^{j+1}\mathcal{F}(\cdot)) \to .$$

The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $Ext^{i-j-1}_{\mathcal{B}}(\cdot, H^{j+1}\mathcal{F}(\cdot))$ is 0 because the functor $H^{j+1}\mathcal{F}$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. □

Now we are ready to prove the second part of the theorem. Given quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.1, 2.4

$$Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i]) \xrightarrow{\sim} Hom_{D(Sh)}(S(\mathcal{F}), S(\mathcal{G})[i]) \xrightarrow{\sim} Ext^i_{Sh}(s(H^0\mathcal{F}), s(H^0\mathcal{G})). \tag{2.8}$$

In particular, $Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i])$ is isomorphic to $Hom_{Fun(\mathcal{A},\mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G})$ for $i = 0$ (since the functor $s : Fun(\mathcal{A}, \mathcal{B}) \to PSh$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a DG functor between small DG categories. Then the restriction functor $f_* : \mathbb{D}(\mathcal{C}_2) \to \mathbb{D}(\mathcal{C}_1)$ admits a left and a right adjoint functors (the derived induction and co-induction functors)

$$f^*, f^! : \mathbb{D}(\mathcal{C}_1) \to \mathbb{D}(\mathcal{C}_2) \tag{2.9}$$

([Dri], §14.12). In particular, we have the canonical morphisms

$$Id \to f_*f^*, \quad f_*f^! \to Id \tag{2.10}$$
$$Id \to f^!f_*, \quad f^*f_* \to Id.$$

It also follows from the adjunction property that f^* commutes with arbitrary direct sums and that $f^!$ commutes with arbitrary direct products. If the the functor $Ho(f) : Ho(\mathcal{C}_1) \to Ho(\mathcal{C}_2)$ is fully faithful so is f_* and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor RF of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ from ([Dri], §16). Consider the functor

$$\mathcal{T}(\mathcal{A}, D^b_{dg}(\mathcal{B})) \hookrightarrow \mathbb{D}(C^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{f^*} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$$

induced by the projection

$$f : C^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}) \to D^b_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}).$$

Given a k-linear functor $F \in Fun(\mathcal{A}, \mathcal{B}) \to \mathcal{T}(\mathcal{A}, D^b_{dg}(\mathcal{B}))$ we define the "derived functor"

$$\text{“}RF\text{”} = f^*(F) \in \mathbb{D}(D^b_{dg}(\mathcal{A})^{op} \otimes_k D^b_{dg}(\mathcal{B})). \tag{2.11}$$

The right derived DG quasi-functor $RF : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$, if it exists, is an object of $\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ whose image in $\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \supset \mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ is "RF".

Lemma 2.5. *Assume that F is left exact. Then "RF" $\in \mathbb{D}^+ \subset \mathbb{D}(D^b_{dg}(\mathcal{A})^{op} \otimes_k D^b_{dg}(\mathcal{B}))$ and the functor $S : \mathbb{D}^+ \hookrightarrow D(Sh)$ takes "RF" to $s(F)$.*

Proof. Let $\beta : \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \to D(PSh)$ be the restriction functor, and let $\gamma : D(PSh) \to D(Sh)$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^i(\beta(\text{“}RF\text{”}))$ can be computed as follows:

$$H^i(\beta(\text{“}RF\text{”}))(X \otimes X') = colim_Q \, Hom_{D^b(\mathcal{B})}(X', F(Y^{\cdot})[i]), \tag{2.12}$$

where the colimit is taken over the filtrant category Q of pairs $(Y^{\cdot} \in HoC^b_{dg}(\mathcal{A}), f \in Hom_{HoC^b_{dg}(\mathcal{A})}(X, Y^{\cdot}))$ such that $cone(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs $(Y^{\cdot}, f)$ with $Y^j = 0$ for $j < 0$ is cofinal in Q, the category Q in the equation (2.12) can be replaced by Q'. This proves that "RF" $\in \mathbb{D}^+$. Let us show that $\gamma \circ \beta(\text{“}RF\text{”}) \simeq s(F)$. We have

$$H^0(\beta(\text{“}RF\text{”}))(X \otimes X') = colim_{Q'} \, Hom_{D^b(\mathcal{B})}(X', F(Y^{\cdot})) \simeq$$

$$colim_{Q'} \, Hom_{D^b(\mathcal{B})}(X', \tau_{\leq 0} F(Y^{\cdot})) \simeq colim_{Q'} \, Hom_{D^b(\mathcal{B})}(X', F(X)) = s(F)(X \otimes X').$$

It remains to prove that, for every $i > 0$, the sheafification of $H^i(\beta(\text{“}RF\text{”}))$ is 0. Let s be the section of $H^i(\beta(\text{“}RF\text{”}))(X \otimes X')$ represented by an element

$$\tilde{s} \in Hom_{D^b(\mathcal{B})}(X', F(Y^{\cdot})[i]),$$

where $X \xrightarrow{f} Y^0 \to Y^1 \to \cdots$ is an object of Q'. Looking at the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y^0 & \to & Y^1 & \to \cdots \\ \downarrow f & & \downarrow Id & & \downarrow & \\ Y^0 & \xrightarrow{Id} & Y^0 & \to & 0 & \to \cdots \end{array}$$

we see that the pullback $(f \otimes Id)^* s \in H^i(\beta(\text{“}RF\text{”}))(Y^0 \otimes X')$ is represented by an element of the group $Hom_{D^+(\mathcal{B})}(X', F(Y^0)[i]) = Ext^i_{\mathcal{B}}(X', F(Y^0))$. For any positive i every element of this group is annihilated by the map $Ext^i_{\mathcal{B}}(X', F(Y^0)) \to Ext^i_{\mathcal{B}}(Y', F(Y^0))$ for some epimorphism $Y' \to X'$.

□

Let us prove the first part of the theorem. Let $\mathcal{F} \in \mathcal{T} \subset \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ be a DG quasi-functor satisfying property (P) together with an isomorphism $F \simeq H^0\mathcal{F}$. We need to construct an isomorphism $\mathcal{F} \simeq$ "RF". By Lemmas 2.4, 2.5 $\mathcal{F}$, "RF" are objects of $\mathbb{D}^+$. By Lemma 2.1 the functor $S : \mathbb{D}^+ \to D(Sh)$ is fully faithful. Thus, constructing an isomorphism $\mathcal{F} \simeq$ "RF" is equivalent to producing an isomorphism $S(\mathcal{F}) \simeq S(\text{“}RF\text{”})$ in $D(Sh)$ which was done in Lemmas 2.4, 2.5. Theorem 1 is proved.

Proof of theorem 2. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors $\mathcal{F}$ such that, for some integer n, we have

$$Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B}).$$

We shall prove that the composition

$$\mathcal{T}^+ \hookrightarrow \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B})) \xrightarrow{Res} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \to D(Sh)$$

is a fully faithful embedding. Here Res denotes the restriction functor induced by the embedding

$$D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}) \to D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}). \tag{2.13}$$

To show this we need to introduce a bit of notation. If $\mathcal{C}$ is an abelian category closed under countable direct sums and

$$X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \cdots$$

is a diagram of complexes $X^i \in C(\mathcal{C})$, we set

$$hocolim\, X^i = cone(\bigoplus_i X^i \xrightarrow{v} \bigoplus_i X^i) \in C(\mathcal{C}),$$

where $v_{|X^i} := Id_{X^i} - \phi_i : X^i \to \oplus_i X^i$. There is a canonical morphism

$$hocolim\, X^i \to colim\, X^i,$$

which is a quasi-isomorphism if countable direct limits in $\mathcal{C}$ are exact. If this is the case, every morphism $X^\cdot \to X'^\cdot$ of diagrams that is a term-wise quasi-isomorphism induces a quasi-isomorphism of the homotopy colimits [8]. Dually, for a category $\mathcal{C}$ closed under countable products and a diagram

$$\cdots \to X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0,$$

we set

$$holim\, X_i = cone(\prod_i X_i \xrightarrow{v} \prod_i X_i)[-1],$$

where $v_i := p_i - \phi_i p_{i+1} : \prod X_i \to X_i$ and $p_i : \prod X_i \to X_i$ are the projections.

Let $\mathbb{D}^f \subset \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}))$ be the full subcategory whose objects are the covariant DG functors $M : D^+_{dg}(\mathcal{A}) \otimes_k D^+_{dg}(\mathcal{B})^o \to C(Mod(k))$ such that, for every $X \in D^+_{dg}(\mathcal{A})$ and $X' \in D^+_{dg}(\mathcal{B})$, the canonical morphism

$$M(X \otimes X') \to holim\, M(X \otimes \tau_{<i} X'), \tag{2.14}$$

is a quasi-isomorphism, and, for every $X \in D^+_{dg}(\mathcal{A})$ and every *bounded* $X' \in D^b_{dg}(\mathcal{B})$, the canonical morphism

$$hocolim\, M(\tau_{<i} X \otimes X') \to M(X \otimes X'), \tag{2.15}$$

is a quasi-isomorphism.

Remark 2.6. Since countable direct limits are exact in $\mathcal{B}$, the morphism $hocolim\, \tau_{<i} X' \to X'$ is a quasi-isomorphism. Thus, property (2.14) is implied by the following: for every integer n and a countable collection $X'^i \in D^{\geq n}_{dg}(\mathcal{B})$, the morphism

$$M(X \otimes \oplus_i X'^i) \to \prod_i M(X \otimes X'^i)$$

is a quasi-isomorphism.

Remark 2.7. Since directed limits are exact in $Mod(k)$ property (2.15) is equivalent to the following: for every $X \in D^+_{dg}(\mathcal{A})$ and $X' \in \mathcal{B}$, we have

$$colim\, H^0(M(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(M(X \otimes X')). \tag{2.16}$$

[8] For the last property, it suffices to assume that countable direct sums are exact in $\mathcal{C}$.

Lemma 2.8. *The restriction functor*

$$\mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$$

is an equivalence of categories.

Proof. We shall first consider the restriction

$$f_* : \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B})) \to \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$$

and prove that $f^!$ and f_* define mutually inverse equivalences of categories

$$\mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \simeq \mathbb{D}', \tag{2.17}$$

where $\mathbb{D}'$ is the full subcategory of $\mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}))$ whose objects are DG functors M satisfying the property (2.14). Let us check that

$$f^!(\mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))) \subset \mathbb{D}'. \tag{2.18}$$

For every DG functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ between DG categories over a field, the functor $f^! : \mathbb{D}(\mathcal{C}_1) \to \mathbb{D}(\mathcal{C}_2)$ admits the following concrete description: if $M : \mathcal{C}_1 \to C(Mod(k))$ is a contravariant DG functor and X is an object of $\mathcal{C}_2$, we have

$$f^!(M)(X) = Hom_{\mathbb{D}_{dg}(\mathcal{C}_1)}(f^{dg}_* Hom_{\mathcal{C}_2}(\cdot, X), M). \tag{2.19}$$

Here $\mathbb{D}_{dg}(\mathcal{C}_i)$ denotes the DG derived category of right $\mathcal{C}_i$-modules, f^{dg}_* the derived restriction functor, and $Hom_{\mathcal{C}_2}(\cdot, X)$ is the image of X under the Yoneda embedding $\mathcal{C}_2 \to \mathbb{D}_{dg}(\mathcal{C}_2)$.

We shall prove that

$$hocolim\, Hom(\cdot\,, X \otimes \tau_{<i} X') \to f_* Hom(\cdot\,, X \otimes X')$$

is an isomorphism in $\mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$. Together with (2.19) it will imply (2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$,

$$Hom(Y \otimes Y', X \otimes X') = Hom(Y, X) \otimes_k Hom(Y', X').$$

Hence, it is enough to check that the morphism

$$hocolim\, Hom_{D^+_{dg}(\mathcal{B})}(Y', \tau_{<i} Y) \to Hom_{D^+_{dg}(\mathcal{B})}(Y', Y)$$

is a quasi-isomorphism, for every $Y' \in D^b_{dg}(\mathcal{B})$. Using the exactness of direct limits in $Mod(k)$ the last assertion is reduced to the formula

$$colim\, Hom_{D^b(\mathcal{B})^o}(Y', \tau_{<i} Y) \simeq Hom_{D^+(\mathcal{B})^o}(Y', Y),$$

which holds because the group $Hom_{D^+(\mathcal{B})^o}(Y', \tau_{>i} Y)$ is trivial for large i. This proves the assertion (2.18).

Since the functor $Ho(f)$ is fully faithful, we have

$$f_* f^! \xrightarrow{\sim} Id.$$

Let us check that for every $M \in \mathbb{D}'$ the canonical morphism $M \to f^! f_* M$ is an isomorphism. Set $G = cone(M \to f^! f_* M)$. As we have just proved G belongs to $\mathbb{D}'(D^b_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}))$. On the other hand, the isomorphism $f_* f^! f_* \simeq f_*$ shows that $f_* G$ is 0. Hence, G is 0 by (2.14).

Next, consider the DG functor

$$g : D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}) \to D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$$

and show that g^* and g_* define mutually inverse equivalences of categories

$$\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \simeq \mathbb{D}'', \tag{2.20}$$

where $\mathbb{D}''$ is a full subcategory of $\mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ whose objects are DG functors F satisfying property (2.15). Let us check that

$$g^*(\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))) \subset \mathbb{D}''. \tag{2.21}$$

If $M \in \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ is a functor representable by

$$Y \otimes Y' \in D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$$

then g^*M is represented by the same object $Y \otimes Y'$ (viewed as an object of $D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})$). Hence (2.16) is implied by the formula

$$hocolim\, Hom_{D^+_{dg}(\mathcal{A})}(Y, \tau_{<i}X) \simeq Hom_{D^+_{dg}(\mathcal{A})}(Y, X), \quad Y \in D^b_{dg}(\mathcal{A})$$

proved above (with $\mathcal{A}$ replaced by $\mathcal{B}$). Since g^* commutes with arbitrary direct sums and since $\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ is the smallest triangulated subcategory that contains representable functors and closed under direct sums, $g^*(M)$ is an object of $\mathbb{D}''(D^+_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ for every M. By (2.15) the functor g_* is conservative when restricted to $\mathbb{D}''$ and the adjoint functor g^* is fully faithful (because $Ho(g)$ is fully faithful). Hence, we have

$$Id \xrightarrow{\sim} g_*g^*, \quad (g^*g_*)_{|\mathbb{D}''} \xrightarrow{\sim} Id.$$

Combining equations (2.17) and (2.20) we see that the functors Res and $f^!g^*$ define mutually inverse equivalences between the category $\mathbb{D}^f$ and the category $\mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$. □

Consider the composition

$$\mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} D(PSh) \to D(Sh). \tag{2.22}$$

Combining Lemmas 2.1 and 2.8 we get the following.

Corollary 2.9. *Let $\mathbb{D}^{f+} \subset \mathbb{D}^f$ be the full subcategory whose objects are DG modules M such that $\beta \circ \mathrm{Res}(M)$ is bounded from below. Then (2.22) induces an equivalence of categories*

$$S : \mathbb{D}^{f+} \xrightarrow{\sim} D^+(Sh).$$

Lemma 2.10. *The functor $\mathcal{T} \hookrightarrow \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B}))$ carries $\mathcal{T}^+$ into $\mathbb{D}^{f+}$.*

Proof. Let us show that every $\mathcal{F} \in \mathcal{T}$ satisfies property (2.6). By definition of $\mathcal{T}$, for every $X \in D^+_{dg}(\mathcal{A})$, there exists $Y \in D^+_{dg}(\mathcal{B})$ and an isomorphism

$$\mathcal{F}(X \times ?) \simeq Hom_{D^+_{dg}(\mathcal{B})}(?, Y)$$

in the derived category of right $D^+_{dg}(\mathcal{B})$-modules. Property (2.6) follows because the morphism

$$Hom_{D^+_{dg}(\mathcal{B})}(\oplus_i X'^i, Y) \to \prod_i Hom_{D^+_{dg}(\mathcal{B})}(X'^i, Y).$$

is a quasi-isomorphism.

Let us show that every $\mathcal{F} \in \mathcal{T}^+$ satisfies the property (2.7). Denote by $Ho(\mathcal{F})$: $D^+_{dg}(\mathcal{A}) \to D^+(\mathcal{B})$ the triangulated functor associated with $\mathcal{F}$. By definition of $Ho(\mathcal{F})$ there is a functorial isomorphism

$$H^0(\mathcal{F}(X \otimes X')) \simeq Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X)) \tag{2.23}$$

In order to check (2.7) we will prove a stronger statement: for every $X' \in \mathcal{B}$ the morphism

$$Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(\tau_{<n}X)) \to Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X)) \tag{2.24}$$

is an isomorphism for sufficiently large n. By definition of $\mathcal{T}^+$ we can find an integer N such that the functor $Ho\mathcal{F}$ carries every object of $D^{>N}(\mathcal{A})$ to an object $D^{>0}(\mathcal{B})$. In particular, for every $n > N$, the complex $Ho\mathcal{F}(cone(\tau_{<n}X \to X))$ has trivial cohomology in non-positive degrees. Hence, we have

$$Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(cone(\tau_{<n}X \to X))) = 0.$$

□

Combining Lemma 2.10 and Corollary 2.9 we get a fully faithful embedding

$$S : \mathcal{T}^+ \hookrightarrow D(Sh). \tag{2.25}$$

By Lemma 2.4 S carries every quasi-functor $\mathcal{F}$ satisfying property (P') to $s(H^0\mathcal{F}) \in Sh$. This proves the second part of Theorem 2. For the first part, let $F \in Fun(\mathcal{A}, \mathcal{B})$ be a k-linear functor, and let

$$\text{“}RF\text{”} \in \mathbb{D}(D^+_{dg}(\mathcal{A})^o \otimes_k D^+_{dg}(\mathcal{B})) \tag{2.26}$$

be the “derived functor” (see (2.11)). To complete the proof of Theorem it suffices to show the following.

Lemma 2.11. *Assume that F is left exact. Then “RF” is an object of $\mathbb{D}^{f+}$ and S(“RF”) is isomorphic to $s(F)$.*

Proof. Let us show that “RF” satisfies property (2.14). According Remark 2.6 it will suffice to show that, for every integer n, $Y^i \in D^{\geq n}_{dg}(\mathcal{B})$ and $X \in HoC^+(\mathcal{A})$

$$H^0(\text{“}RF\text{”}(X \otimes \oplus_i X'^i)) \xrightarrow{\sim} \prod_i H^0(\text{“}RF\text{”}(X \otimes X'^i)).$$

We have ([Dri], §5)

$$H^0(\text{“}RF\text{”}(X \otimes X')) \simeq colim_{Q_X} Hom_{D^+(\mathcal{B})}(X', F(Y)), \tag{2.27}$$

where Q_X is the filtrant category of pairs

$$(Y \in HoC^+_{dg}(\mathcal{A}), f \in Hom_{HoC^+_{dg}(\mathcal{A})}(X, Y))$$

such that $cone(f)$ is acyclic. If X is in $HoC^{\geq n}(\mathcal{A})$ the subcategory $Q'_X \subset Q_X$ formed by pairs (Y, f) with $Y \in HoC^{\geq n}(\mathcal{A})$ is cofinal in Q_X and, hence, Q_X in equation (2.27) can be replaced by Q'_X. Thus, it is enough to prove that the category Q_X has the following property: for every countable collection $w_i = (Y_i, f_i) \in Q'_X$, $(i = 1, 2, \cdots)$, there exists $v \in Q_X$ such that, for every i, the set $Mor_{Q_X}(w_i, v)$ is not empty. In fact, the object

$$v = (cone(\bigoplus_i X \xrightarrow{\phi} \bigoplus_i Y_i), g),$$

where $\phi_j : X \to \bigoplus_i Y_i$ equals $f_j - f_{j-1}$ and g is induced by the morphisms $X \xrightarrow{f_1} Y_1 \hookrightarrow \bigoplus_i Y_i$, does the job.

Let us show that "RF" satisfies property (2.15). As we explained in Remark 2.7 it suffices to show that

$$colim\, H^0(\text{“}RF\text{”}(\tau_{<i}X \otimes X')) \xrightarrow{\sim} H^0(\text{“}RF\text{”}(X \otimes X')),$$

for every $X' \in \mathcal{B}$. In fact, formula (2.27) with $Q_{\tau_{\geq i}X}$ replaced by $Q'_{\tau_{\geq i}X}$ shows that $H^0(\text{“}RF\text{”}(\tau_{\geq i}X \otimes X'))$ is trivial for $i > 0$. Hence, the morphism $H^0(\text{“}RF\text{”}(\tau_{<i}X \otimes X')) \to H^0(\text{“}RF\text{”}(X \otimes X'))$ is an isomorphism for $i > 1$. This proves that "RF" belongs to $\mathbb{D}^{f+}$.

For the second claim, observe that the restriction $Res(\text{“}RF\text{”}) \in \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^b_{dg}(\mathcal{B}))$ is the bounded "derived functor" (2.11). Thus, we are done by Lemma 2.5. □

Proof of theorem 3. Apply Corollary 2.3 and equation (2.25).

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