Math. Res. Lett. **17** (2010), *no.* 05, 905–916

FREE INFINITE DIVISIBILITY FOR q-GAUSSIANS

M. Anshelevich, S. T. Belinschi, M. Bożejko, F. Lehner

Abstract. We prove that the q-Gaussian distribution is freely infinitely divisible for all $q \in [0, 1]$.

1. Introduction

In this note we prove that the q-Gaussian distribution introduced by Bożejko and Speicher in [10] (see also the paper [9] of Bożejko, Kümmerer and Speicher) is freely infinitely divisible when $q \in [0, 1]$.

We shall give a short outline for the context of this problem. In probability theory, the class of infinitely divisible distributions plays a crucial role, in the study of limit theorems, Lévy processes etc. So it was a remarkable discovery of Bercovici and Voiculescu [6] that there exists a corresponding class of *freely* infinitely divisible distributions in free probability. These distributions are typically quite different from the classically infinitely divisible ones; for example, many of them are compactly supported. Nevertheless, work of numerous authors culminating in the paper by [5] Bercovici and Pata showed that free ID distributions are in a precise bijection with the classical ones, this bijection moreover having numerous strong properties. For example, the semicircular law is the free analog of the normal distribution. From this bijection, one might get the intuition that, perhaps with very rare exceptions such as the Cauchy and Dirac distributions, some measures belong to the "classical" world and some to the "free" world. However, [4, Corollary 3.9] indicates that this intuition may be misleading: the normal distribution, perhaps the most important among the classical ones, is also freely infinitely divisible.

One approach towards understanding the relationship between the classical and free probability theories, and in particular the Bercovici-Pata bijection, has been constructions of interpolations between these two theories. The oldest such construction, due to Bożejko and Speicher, is the construction of the q-Brownian motion. In particular, it provides a probabilistic interpretation for a (very classical) family of q-Gaussian distributions, which interpolate between the normal ($q = 1$) and the semicircle ($q = 0$) laws. Probably the best known description of the q-Gaussian distributions is in terms of their orthogonal polynomials $H_n(x|q)$ - called the q-Hermite polynomials - defined by the 3-term recurrence relation $xH_n(x|q) = H_{n+1}(x|q) + \frac{1-q^n}{1-q} H_{n-1}(x|q)$,

Received by the editors March 10, 2010.

Work of M. Anshelevich was supported in part by NSF grant DMS-0900935.

Work of S. Belinschi was supported by a Discovery Grant from NSERC and a University of Saskatchewan start-up grant.

Work of M. Bożejko was partially supported by Polish Ministry of Science and Higher Education by grant no. N N201 364436.

with initial conditions $H_0(x|q) = 1$, $H_1(x|q) = x$. These polynomials have been studied for a long time: probably their first appearance under this guise occurred in L. J. Rogers' 1893 paper [22]. Bożejko and Speicher construct their example of q-Brownian motion using creation-annihilation operators on a "twisted" Fock space: given a separable Hilbert space H and $f \in H$, we let $c^*(f)$ be the left creation operator on the tensor algebra $\mathcal{F}(H)$ over H. The authors provide in [10] a scalar product $(\cdot \mid \cdot)_q$ on (a quotient space of) $\mathcal{F}(H)$ so that when considering $c(f)$ as the adjoint of $c^*(f)$ with respect to $(\cdot \mid \cdot)_q$, the so-called q-canonical commutation relation holds:

$$c(f)c^*(g) - qc^*(g)c(f) = (f|g)_q \mathbf{1}, \quad f, g \in H,$$

where $\mathbf{1}$ is the identity operator on $\mathcal{F}(H)$. It is a fundamental result that, when $(f|f)_q = 1$, the distribution of the self-adjoint operator $c(f) + c^*(f)$ with respect to the vacuum state on $\mathcal{F}(H)$ is the (centered) q-Gaussian distribution of variance one, having the q-Hermite polynomials as orthogonal polynomials (see [9, Theorem 1.10]). In this case, as mentioned before, when $q = 1$, $c(f) + c^*(f)$ is distributed according to the classical normal distribution $(2\pi)^{-1/2} \exp(t^2/2)dt$, and when $q = 0$ according to the free central limit - the Wigner distribution with density $(2\pi)^{-1}\sqrt{4 - t^2}\chi_{[-2,2]}(t)$. A formula for the density f_q of the q-Gaussian distribution is provided in [26]: $f_q(x) = \pi^{-1}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty}(1-q)^n|1 - q^n e^{2i\theta}|^2$, where $\cos\theta = \frac{x}{2}\sqrt{1-q}$. The interested reader might want to note [26, Section III] that, up to a multiplicative constant depending only on q, f_q is a theta function. For numerous details on properties and applications of q-Gaussian processes we refer to [9] and references therein.

As the construction described above suggests, q-Gaussians provide useful examples in operator algebras. The von Neumann algebras generated by families of q-Gaussians are known to exhibit several interesting properties, and we shall list a few below; however, the structure of these algebras still remains largely mysterious. It is shown in [19] that algebras generated by such families when f runs through H and $\dim(H) \geq 2$ are non-injective; the paper [21] proves that algebras generated by at least two q-Gaussians corresponding to orthogonal fs are always factors when $|q| < 1$ (see also [15, 9, 24]), and in [23] Shlyakhtenko provides estimates for the non-microstates free entropy of n-tuples of such q-Gaussians, estimates which guarantee that the algebra they generate is solid in the sense of Ozawa whenever $q < \sqrt{2} - 1$. Moreover, recently Bożejko [8] proved that in von Neumann algebras generated by two q-Gaussians, the Bessis-Moussa-Villani conjecture holds.

There are several strictly probabilistic approaches to q-Gaussians: we would like to mention the stochastic integration with respect to q-Brownian motion of Donati-Martin [14], the q-deformed cumulants, a q-convolution defined on a restricted class of probability measures and q-Poisson processes studied in [2] and a random matrix model provided in [18] (see also [7]).

However, classical or free probability aspects of q-Gaussians have been less studied. In this paper, we show that all of these distributions, for $0 \leq q \leq 1$, are freely infinitely divisible. This may be an indication that the class of freely infinitely divisible distributions, despite the Bercovici-Pata bijection, is quite different from the classical one, and is yet to be understood completely. The conjecture that q-Gaussian distributions are freely infinitely divisible when $q \in [0, 1]$, formulated initially by the two last named authors and R. Speicher, was motivated among others by the recently proved free infinite divisibility of the classical Gaussian (corresponding to $q = 1$).

It has been shown in [4] that the Askey-Wimp-Kerov distributions with parameter $c \in [-1, 0]$, and in particular the classical normal distribution (corresponding to $c = 0$) are $\boxplus$-infinitely divisible. This provided free infinite divisibility for distributions of several noncommutative Brownian motions (see for example [11, 12, 13]), interpolating between the classical central limit (the normal distribution) and the free central limit (the Wigner semicircle law). However, until now it remained a mystery whether this first (and most famous) such example of interpolation consists also of $\boxplus$-infinitely divisible distributions. Several numerical verifications performed by one of the present authors seemed to indicate this is indeed the case. Here we shall give an answer to this question:

Theorem 1. *The q-Gaussian distribution $f_q(x)dx$ is freely infinitely divisible for all $q \in [0, 1]$.*

Our method to prove this result will be the same as in [4], namely we will construct an inverse to the Cauchy transform G_{f_q} of the q-Gaussian defined on the whole lower half-plane. Then the Voiculescu transform $\phi_{f_q}(1/z) = G_{f_q}^{-1}(z) - (1/z)$ clearly has an extension to the whole complex upper half-plane $\mathbb{C}^+$. An application of the following theorem of Bercovici and Voiculescu from [6] yields the desired result:

Theorem 2. *A Borel probability measure μ on the real line is $\boxplus$-infinitely divisible if and only if its Voiculescu transform $\phi_\mu(z)$ extends to an analytic function $\phi_\mu \colon \mathbb{C}^+ \to \mathbb{C}^-$.*

The paper is organized as follows: in the second section, we introduce several notions and preliminary results used in our proof, and in the third section we give the proof of Theorem 1.

2. Preliminary results: the importance of an entire function

It is shown in the paper [25] of Pawel Szabłowski (see also [17]) that the density of the q-Gaussian distribution of mean zero and variance one, with respect to the Lebesgue measure is given by the formula

$$f_q(x) = \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\sum_{k=1}^{\infty}(-1)^{k-1}q^{\frac{k(k-1)}{2}}\mathcal{U}_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right),$$

for $|x| \leq 2/\sqrt{1-q}$, where $\mathcal{U}_k$ is the k^{th} Chebyshev polynomial of the second kind - defined by the relation $\mathcal{U}_k(\cos\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}$. In our present paper we will consider this as being the definition of the q-Gaussian distribution.For simplicity of notation, we will re-normalize this density to being supported on $[-2, 2]$, by replacing x with $x/\sqrt{1-q}$:

$$f_q(x) = \frac{1}{2\pi}\sqrt{4-x^2}\sum_{k=1}^{\infty}(-1)^{k-1}q^{\frac{k(k-1)}{2}}\mathcal{U}_{2k-2}\left(\frac{x}{2}\right), \quad -2 \leq x \leq 2.$$

Recall that the Cauchy (or Cauchy-Stieltjes) transform of a Borel probability measure μ on $\mathbb{R}$ is by definition

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

This function maps the upper half-plane $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$, satisfies $G_\mu(\overline{z}) = \overline{G_\mu(z)}$, and extends analytically through the complement of the support of μ. Also, of some importance for us will be the map $F_\mu(z) = \frac{1}{G_\mu(z)}$. This function satisfies the inequality $\Im F_\mu(z) > \Im z$, $z \in \mathbb{C}^+$, whenever μ is not a point mass. For more details we refer the reader to [1, Chapter III].

Integrating $(z-x)^{-1} f_q(x)$ with respect to the Lebesgue measure on $[-2, 2]$, we obtain for any $z \in \mathbb{C}^+$ that

$$\begin{aligned}
G_{f_q}(z) &= \int_{-2}^{2} \frac{1}{z-x} f_q(x)dx \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} q^{\frac{k(k-1)}{2}} \left(\frac{1}{2\pi} \int_{-2}^{2} \frac{1}{z-x} \mathcal{U}_{2k-2}\left(\frac{x}{2}\right) \sqrt{4-x^2}\, dx \right) \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} q^{\frac{k(k-1)}{2}} G_s(z)^{2k-1} \\
&= \sum_{k=0}^{\infty} (-1)^{k} q^{\frac{k(k+1)}{2}} G_s(z)^{2k+1},
\end{aligned}$$

where G_s is the Cauchy transform of the semicircular law:

$$G_s(z) = \frac{1}{2\pi} \int_{-2}^{2} \frac{1}{z-x} \sqrt{4-x^2}\, dx = \frac{z - \sqrt{z^2-4}}{2}, \quad z \in \mathbb{C}^+.$$

This function has an analytic extension to the lower half-plane $\mathbb{C}^-$ through the interval $(-2, 2)$ that does *not* coincide with $(2\pi)^{-1} \int_{-2}^{2} (z-x)^{-1} \sqrt{4-x^2} dx$ when $z \in \mathbb{C}^-$: when we consider the same branch of the square root as above, it is of the form $G_s(z) = \frac{z+\sqrt{z^2-4}}{2}$ (meaning, the asymptotics of this extension at $-i\infty$ is $O(z)$). An analysis of these two formulas guarantee us that G_s maps $\mathbb{C}^+ \cup (-2, 2) \cup \mathbb{C}^-$ bijectively onto $\mathbb{C}^-$ by mapping $\mathbb{C}^+$ into the lower half of the unit disc $\mathbb{D}$ (the piece of $\partial G_s(\mathbb{C}^+)$ that forms the interval $[-1, 1]$ is $G_s([-\infty, -2] \cup [2, +\infty])$, with the two infinities identified and $G_s(\infty) = 0$), while $G_s(\mathbb{C}^-)$ is the complement of $\overline{G_s(\mathbb{C}^+)}$ in $\mathbb{C}^-$. In addition, $G_s(i\mathbb{R}_+) = i[-1, 0)$, $G_s(i\mathbb{R}_-) = i(-\infty, -1]$ and, when discarding the i, G_s is monotonic increasing on the imaginary axis. These simple observations will be essential in our proof.

Now observe that the above remarks translate into $G_{f_q} = g_q \circ G_s$, where

$$g_q(w) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k+1)}{2}} w^{2k+1}, \quad w \in \mathbb{C}, \tag{1}$$

is an entire function for any $q \in [0, 1)$ (in fact for any $|q| < 1$). (The reader will note that in terms of basic hypergeometric functions g_q can be written as $g_q(w) = w \cdot {}_1\phi_0(q^{\frac{1}{2}} | q^{\frac{1}{2}}, (iq^{\frac{1}{2}} w)^2)$; however, we shall not use this fact directly in our proof.) We list below a few properties of g_q which will be used in our proof:

1. $\lim_{q\to 0} g_q(w) = w, w \in \mathbb{C}$, and the limit is uniform on compacts in $\mathbb{C}$;
2. $\lim_{q\to 1} g_q(w) = \frac{w}{w^2+1}, w \neq \pm i$;
3. $g_q(-\overline{w}) = -\overline{g_q(w)}$;
4. $g_q(\overline{w}) = \overline{g_q(w)}$ - in particular $g_q(\mathbb{R}) \subseteq \mathbb{R}$;

(5) $g_q(ic) = i\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} c^{2k+1}$ for any real c, so that $g_q(i(-\infty,0]) = i(-\infty,0]$, $g_q(i[0,+\infty)) = i[0,+\infty)$, with $g_q(i(\pm\infty)) = i(\pm\infty)$, $g_q(0) = 0$, and g_q is monotonic on the imaginary axis (in the same sense as G_s is);

(6) $g_q'(0) = 1$.

Since the limit in item (1) above is uniform on compact subsets of $\mathbb{C}$, we can conclude that, given $M > 0$, for $q > 0$ sufficiently small, depending on M, g_q is injective on the ball of radius M centered at the origin. In addition, Equation (4) below guarantees that g_q has no limit at infinity along the real axis; items (3) and (4) above together with this remark guarantee that g_q' will have at least two zeros on the real line, symmetric with respect to zero, if $q > 0$.

It is time to state a few results to be used in our proof:

Theorem 3. *A necessary and sufficient condition that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

should be an entire function of order ϱ is that

$$\liminf_{n\to\infty}\left(\frac{-\log|a_n|}{n\log n}\right) = \frac{1}{\varrho}.$$

(This is [16, Theorem 4.12.1].) Thus, for our g_q, we can take

$$\infty = \lim_{n\to\infty}\left(\frac{-n(n+1)\log q}{2(2n+1)\log(2n+1)}\right) = \frac{1}{\varrho}$$

to conclude that the order ϱ of g_q is zero.

Let us also state the Denjoy-Carleman-Ahlfors Theorem:

Theorem 4. (i) *An entire function of order ϱ has at most 2ϱ finite asymptotic values.*

(ii) *For an entire function of order ϱ the sets $\{z \in \mathbb{C} : |f(z)| > c\}$, $c \geq 0$, have at most* $\max\{2\varrho, 1\}$ *components.*

In particular, the function g_q has no finite asymptotic value and the sets $\{z : |g_q(z)| > c\}$ have exactly one connected component.

Two very famous theorems, available in any complex analysis text, are:

Theorem 5. *Any nonconstant analytic function is open, i.e. maps open sets into open sets. In particular, if f is analytic and nonconstant, $\partial f(D) \subseteq f(\partial D)$ for a given open D.*

Theorem 6. *Assume that f, g are analytic on the simply connected domain D, and γ is a rectifiable simple closed curve in D. If $|f(z) + g(z)| < |f(z)| + |g(z)|$ for all z in the range of γ, then f and g have the same number of zeros in the subdomain delimited by γ inside D.*

The first is the open mapping theorem, the second a weakened version of Rouché's theorem - the variation of argument.

Finally, let us denote following [3], page 81,

(2)
$$\theta_1(z) = -iq^{1/4}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+n} e^{i(2n+1)z} = -iqG\prod_{n=1}^{\infty}(1-q^{2n-2}e^{-2iz})(1-q^{2n}e^{2iz})e^{iz}.$$

We do the obvious substitution $z = -i \log w$ to get the new function

$$\begin{aligned} \Theta(w) &= -iq^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} w^{2n+1} \\ &= -iq^{1/4} \left(g_{q^2}(w) - g_{q^2}\left(\frac{1}{w}\right)\right). \end{aligned} \tag{3}$$

(The letter G in formula (2) denotes a complex constant, not a Cauchy transform; we preserve this notation just in order to follow [3].) While θ_1 is entire (and one can see that from either of the two formulas - the infinite product or the bi-infinite sum - since, for example, $|(e^{iz})^{(2\pm\frac{1}{n})}| < |q^{-|n|+1}|$ for $|n|$ large enough), this is not the case, of course, with Θ. (We will generally ignore reparametrizations of q, since they won't be important.) This new function is not analytic at zero. Here it is important however to observe the product formula for our function: the above provides us with

$$\Theta(w) = -iq^{1/4}\left(g_{q^2}(w) - g_{q^2}\left(\frac{1}{w}\right)\right) = -iqG\prod_{n=1}^{\infty}(1-q^{2n-2}w^{-2})(1-q^{2n}w^2)w. \tag{4}$$

Thus, all zeros of $\Theta : w \mapsto -iq^{1/4}\left(g_{q^2}(w) - g_{q^2}\left(\frac{1}{w}\right)\right)$ are real (namely $w = \pm q^{n-1}$ and $w = \pm q^{-n}$, $n \in \mathbb{N}$; $w = 0$ is not a zero of this function, since zero is an accumulation point of other zeros).

Let us give a few elementary lemmas from complex analysis:

Lemma 7. *Assume that an entire analytic function h has order zero. Then the preimage $h^{-1}(l)$ of any piecewise smooth curve l with both ends at infinity has all its ends tend to infinity and for any component γ of $h^{-1}(l)$, we have $h(\gamma) = l$.*

Proof. It is clear from Theorem 5 and the definition of an entire function that the preimage of any curve with both ends at infinity cannot have an end in the complex plane.

On the other hand, by the Denjoy-Carleman-Ahlfors Theorem (Theorem 4), we know that there is no path γ going to infinity so that the limit of h along γ is finite. If we assume that there is a branch γ of $h^{-1}(l)$ so that $h(\gamma) \neq l$, then there exists a complex number $c \in l$ which is an asymptotic value for h at infinity, contradicting Theorem 4. □

Lemma 8. (a) *If γ is the boundary of a simply connected domain D in $\mathbb{C}$, γ has both ends at infinity, and $h(\gamma) = \mathbb{R}$, then either $h(D)$ is one of the domains $\mathbb{C}^+, \mathbb{C}^-$, or $\overline{h(D)} = \mathbb{C}$.*

(b) *If $h(\partial D)$ is a half-line s in $\mathbb{C}$, then $h(D) \supseteq \mathbb{C} \setminus s$.*

Proof. As seen in Theorem 5, $\partial h(D) \subseteq h(\gamma) = \mathbb{R}$. It is clear that if $h(D)$ is not a half-plane, then its closure must be all of $\mathbb{C}$. This proves (a). The proof of (b) is identical: $\partial h(D) \subseteq h(\partial D) = s$, so $h(D) \supseteq \mathbb{C} \setminus s$. □

Lemma 9. *With the notation from the previous lemma, if γ is a rectifiable path, $h(D) = \mathbb{C}^+$, and h is injective on γ, then h maps D conformally onto $\mathbb{C}^+$.*

Proof. This lemma is a direct consequence of [20], Chapters 1 and 2. □

3. Proof of free infinite divisibility for $f_q(x)dx$

We are now ready to prove the main result. For the comfort of the reader, we restate our main Theorem 1 below.

Theorem 1. *The q-Gaussian distribution $f_q(x)dx$ is freely infinitely divisible for all $q \in [0,1]$.*

Proof. In our proof, we will follow the outline described in the introduction. Namely, we will find a domain X_q in the lower half-plane containing the lower half of the unit disc $\mathbb{D}$ with the property that $g_q(X_q) = \mathbb{C}^-$ and g_q is injective on X_q. Since we have shown that G_s maps $\mathbb{C}^+ \cup \mathbb{R}$ injectively into the closure of the lower half of the unit disc and (when considering the correct extension through $(-2,2)$) $\mathbb{C}^-$ injectively into $\mathbb{C}^- \setminus \overline{\mathbb{D}}$, it will follow that $G_{f_q}^{-1} = G_s^{-1} \circ g_q^{-1}$ extends to $\mathbb{C}^-$. Thus,

$$F_{f_q}(\cdot) = \frac{1}{g_q(G_s(\cdot))} \colon G_s^{-1}(X_q) \mapsto \mathbb{C}^+$$

is a bijective correspondence. (The reader should keep in mind that G_s^{-1} is the inverse of the extension of $G_s|_{\mathbb{C}^+}$ through $(-2,2)$, so $G_s^{-1}(X_q) \supset \mathbb{C}^+$.) Since X_q is included in the lower half-plane, the choice of the extension of G_s guarantee that $G_s^{-1}(X_q) \cap \mathbb{R} = (-2,2)$. This implies the existence of $\phi_{f_q}(z) = F_{f_q}^{-1}(z) - z$ for all $z \in \mathbb{C}^+$. Recalling now that $\Im F_{f_q}(w) > \Im w$ for all $w \in \mathbb{C}^+$, we obtain that $\Im F_{f_q}(w) > \Im w$ for all $w \in G_s^{-1}(X_q)$, and hence $\Im \phi_{f_q}(z) = \Im F_{f_q}^{-1}(z) - \Im z < 0$ for all $z \in \mathbb{C}^+$. An application of Theorem 2 yields the desired conclusion.

First, remark that, since $g_q'(0) = 1$, for any fixed $q \in (0,1)$ there exist two constants $K_q, M_q > 0$ so that g_q is injective on $K_q\mathbb{D}$ and $g_q(K_q\mathbb{D}) \supset M_q\mathbb{D}$.

Next, we shall construct the domain X_q described in the first paragraph of the proof. Since $g_q(z) = -g_q(-z)$ and $g_q(\overline{z}) = \overline{g_q(z)}$, $z \in \mathbb{C}$, it will clearly be enough to find the right side of X_q, as X_q must be symmetric with respect to the imaginary axis. As suggested by the last lemma, we shall first find a continuous simple path inside $\overline{\mathbb{C}^-}$ whose image via g_q is the real line. We observe in addition that, due to the symmetry noted in items (3), (4) and (5) in the above list of properties of g_q, it will be enough to determine the right half of such a path (see Figure 1.)

We start our path from zero. Let us now "walk" along the real axis, in the positive direction (by item (3) above, it is enough if we cover the right half) until we encounter a zero of g_q', call it d_q. Clearly, $g_q([0,d_q]) = [0, g_q(d_q)]$ is a bijective identification. Around d_q, g_q will be an n-to-one cover (where n is the order of the zero of $g_q(\cdot) - g_q(d_q)$), and we will choose the path "first to the right" which continues $g_q^{-1}([0, g_q(d_q)])$ beyond d_q, to $g_q^{-1}([0,+\infty))$. Observe that this path escapes now in the lower half-plane and, since $g_q(\mathbb{R}) \subseteq \mathbb{R}$, it remains in the lower half-plane. Now whenever we meet another such critical point for g_q on this path $g_q^{-1}([0,+\infty))$, we turn again "first to the right", so that immediately to the right of this branch of $g_q^{-1}([0,+\infty))$ of ours, g_q is injective. Call this branch - which is a rectifiable, piecewise analytic path - γ_q. Observe moreover that, by item (5), γ_q is confined to the lower right quadrant of the complex plane (it cannot cut the imaginary axis, as the imaginary axis is mapped into itself).

Apriori it is not clear whether this branch of $g_q^{-1}([0,+\infty))$ in fact exists, i.e. $g_q(\gamma_q) = [0,+\infty)$. However, as we have seen above, g_q has order zero, so by Lemma

7 (the application of the Denjoy-Carleman-Ahlfors Theorem), g_q can have no finite asymptotic values at infinity, so indeed γ_q as above (i.e. $g_q(\gamma_q) = [0, +\infty)$) exists. Thus $\gamma_q \cup (-\overline{\gamma_q})$ (here $\overline{\gamma}$ means complex conjugate, not closure) forms the boundary of a unique simply connected domain $X_q \subseteq \mathbb{C}^-$ which contains the lower half of the imaginary axis and has a piecewise analytic, everywhere continuous, boundary. Clearly the lower half of the unit disc is included in X_q. Indeed, if $\gamma_q \cap \mathbb{D} \neq \varnothing$, then there exists $z_0 \in \mathbb{C}^+$ so that $G_s(z_0) \in \gamma_q \cap \mathbb{D}$, and thus $G_{f_q}(z_0) = g_q(G_s(z_0)) \in \mathbb{R}$, an obvious contradiction.

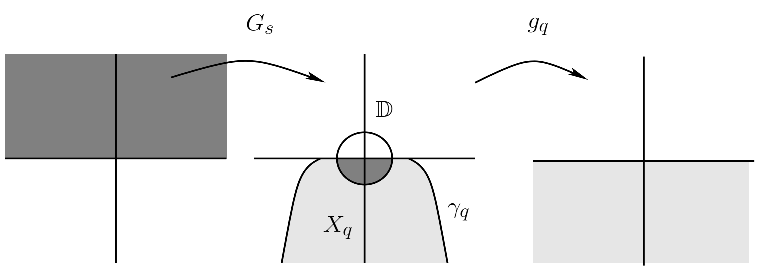

FIGURE 1. The curve γ_q is mapped by g_q onto the real line. The darker shadings correspond to each other via G_s, while g_q maps the entire shaded area (dark and light) onto the lower half-plane.

Fix now $q \in (0, 1)$. It is trivial that g_q has a unique inverse, call it Φ, defined around zero (more precise, at least on $M_q\mathbb{D}$), which fixes zero. We would like to extend this inverse to the whole lower half-plane; then it would easily follow that $\Phi(\mathbb{C}^-) = X_q$. The lack of finite asymptotic values for g_q guarantees that the only impediment to such an analytic extension would be a finite critical value of g_q. Thus, assume towards a contradiction that there exists a $c \in X_q$ (for precision, assume $\Re c > 0$) so that $g_q'(c) = 0$. Without loss of generality we may assume that $g_q(c) \in \mathbb{C}^-$ and $g_q(c)$ is the point closest to the origin in the lower half-plane to which Φ does not extend analytically. We consider a half-line r starting at $g_q(c)$ and going to infinity inside $\mathbb{C}^- \cap i\mathbb{C}^-$ without cutting $g_q(c)\mathbb{D}$ and so that no $z \in r \setminus \{g_q(c)\}$ is a critical value for g_q (possible since the set of critical values of g_q is at most countable). By the same Lemma 7, we conclude that r has a preimage p_q via g_q inside X_q, right of the imaginary axis, with both ends at infinity, which determines a domain $D \subset X_q$ that does not contain $i(-\infty, 0)$ and having boundary $\partial D = p_q$. The choice for p_q is made so that there is no other preimage in $X_q \setminus D$ of r which is in the same connected component of $g_q^{-1}(r)$ as p_q.

For the sake of clarity, we group most of the rest of the proof in the following lemma. The reader will probably find Figure 2 helpful in following it.

Lemma 10. (1) *There exists a point $b \in D$, $|b| > 1$, so that $g_q(b) = 0$.*

(2) *For any $m > |c| + |g_q(c)| + 1 + |b|$, there exists $M \geq m$ and a path ϖ in $\overline{D}$ connecting the two points of $p_q \cap g_q^{-1}(\partial(M\mathbb{D}))$ so that $g_q(\varpi) \cap m\mathbb{D} = \varnothing$.*

(3) *Let Π_m be the path obtained by concatenating ϖ, the bounded part(s) of $p_q \setminus (\varpi \cup \mathbb{D})$ and, if existing, the segments $\partial\mathbb{D} \cap D$. Then Π_m is a closed curve*

containing b *inside it and there exists* $m > 0$ *so that* $\{g_q(1/w)\colon w \in \Pi_m\} \subset \mathbb{C}^+$ *and* $g_q(\Pi_m) \cap \{g_q(1/w)\colon w \in \Pi_m\} = \varnothing$.

(4) *In particular,* $w \mapsto g_q(w)$ *and* $w \mapsto g_q(w) - g_q(1/w)$ *have the same number of zeros in the simply connected domain determined by the simple closed curve* Π_m.

Proof. Clearly, Lemma 8 guarantees that $g_q(D) \supseteq \mathbb{C} \setminus r$, which contains zero. Thus, there exists a point $b \in D$ so that $g_q(b) = 0$. Moreover, let us recall that $G_{f_q} = g_q \circ G_s$ is the Cauchy transform of a probability measure, and, as such, it maps the upper half-plane into the lower one. Thus, as $G_s(\mathbb{C}^+ \cup [-2, 2]) = \overline{\mathbb{D}} \cap \mathbb{C}^-$, it follows that $g_q(w) \neq 0$ for any $w \in \overline{\mathbb{D}} \cap \mathbb{C}^-$, so $b \in \mathbb{C}^- \setminus \overline{\mathbb{D}}$, so in particular $|b| > 1$. This proves (1).

To prove (2), recall that by Theorem 4 the set $\{z \in \mathbb{C}\colon |g_q(z)| > m\}$ is connected for any $m \geq 0$ and in particular for m as in the statement of the lemma. Also, from the construction of r and p_q, it is clear that $p_q \cap g_q^{-1}(\partial(M\mathbb{D}))$ contains exactly two points for any $M \geq m$. Assume now towards contradiction that there is no path uniting those two points inside the set $D \setminus g_q^{-1}(m\mathbb{D})$, and in particular, of course, in $D \setminus g_q^{-1}(m\overline{\mathbb{D}})$. Thus the open set $g_q^{-1}(m\mathbb{D}) \cap D$ must contain an unbounded smooth path. Choose such a path, and call it Γ. Choose $0 < T = 1 + 2\inf\{|z|\colon z \in \Gamma\}$ (so that $\Gamma \cap T\mathbb{D} \neq \varnothing$) and let $M > 2\max\{|g_q(z)|\colon |z| \leq T\} + m$. Since $g_q(\overline{z}) = \overline{g_q(z)}$, it follows that $\{z \in \mathbb{C}\colon |g_q(z)| > M\}$ does not intersect the set $T\mathbb{D} \cup \Gamma \cup \overline{\Gamma}$ (here again $\overline{\Gamma}$ denotes complex conjugate, not closure). But this set disconnects $\mathbb{C}$: for example, the set $p_q \cap \{z \in \mathbb{C}\colon |g_q(z)| > M\}$ contains two nonempty connected components separated by $T\mathbb{D} \cup \Gamma \cup \overline{\Gamma}$. This contradicts Theorem 4. Thus a path $\varpi = \varpi_m$ as described in our lemma must exist.

The fact that Π_m exists and surrounds b exactly once is a trivial consequence of (1), (2) and the entireness of g_q. Moreover, from Π_m's construction, the set $\{1/w\colon w \in \Pi_m\} \subset \mathbb{C}^+ \cap \overline{\mathbb{D}}$. Thus, as noted in the proofs of (1) and (2), $\{g_q(1/w)\colon w \in \Pi_m\} \subset g_q(\mathbb{D} \cap \mathbb{C}^+) \subset \mathbb{C}^+$ (recall that $g_q(\overline{z}) = \overline{g_q(z)}$) is a bounded set for any m (one can choose the bound $\max g_q(\overline{\mathbb{D}})$, which is obviously independent of m and depends only on q). To finish the proof of (3) we only need to argue that for m large enough, the set $g_q(\Pi_m)$ does not intersect $\mathbb{C}^+ \cap (\max g_q(\overline{\mathbb{D}})\mathbb{D})$. Indeed, if we have a point $z \in p_q$, then $g_q(z) \in r \subset \mathbb{C}^-$, while if $z \in \partial\mathbb{D} \cap \mathbb{C}^-$, then again $g_q(z) \in \mathbb{C}^-$. We only need to show that for m large enough $\varpi = \varpi_m$ is mapped in the complement of $\max g_q(\overline{\mathbb{D}})\mathbb{D}$. However, this follows from part (2) by simply choosing $m \geq \max g_q(\overline{\mathbb{D}})$. This proves (3).

In order to prove (4), we will apply Theorem 6 to $f(z) = -g_q(z) + g_q(1/z)$ and $g(z) = g_q(z)$. The relation $|g_q(z) + (-g_q(z) + g_q(1/z))| < |g_q(z) - g_q(1/z)| + |g_q(z)|$ is equivalent to $\left|\frac{g_q(1/z)}{g_q(z)}\right| < \left|\frac{g_q(z) - g_q(1/z)}{g_q(z)}\right| + 1 = \left|1 - \frac{g_q(1/z)}{g_q(z)}\right| + 1$, for $z \in \Pi_m$. Consider two cases: first, if $z \in \varpi$, then $|g_q(z)| > m > |g_q(1/z)|$, so that $\left|\frac{g_q(1/z)}{g_q(z)}\right| - 1 < 0 < \left|1 - \frac{g_q(1/z)}{g_q(z)}\right|$. Second case, if $z \in \Pi_m \setminus \varpi$, then $g_q(z) \in \mathbb{C}^-$ and $g_q(1/z) \in \mathbb{C}^+$. Thus, $g_q(z)$ and $g_q(1/z)$ cannot be positive multiples of each other, i.e. $\frac{g_q(1/z)}{g_q(z)} \notin [0, +\infty)$. Generally, for the relation $|a| - 1 = |a - 1|$ to hold it is necessary that $a \geq 1$. Applying

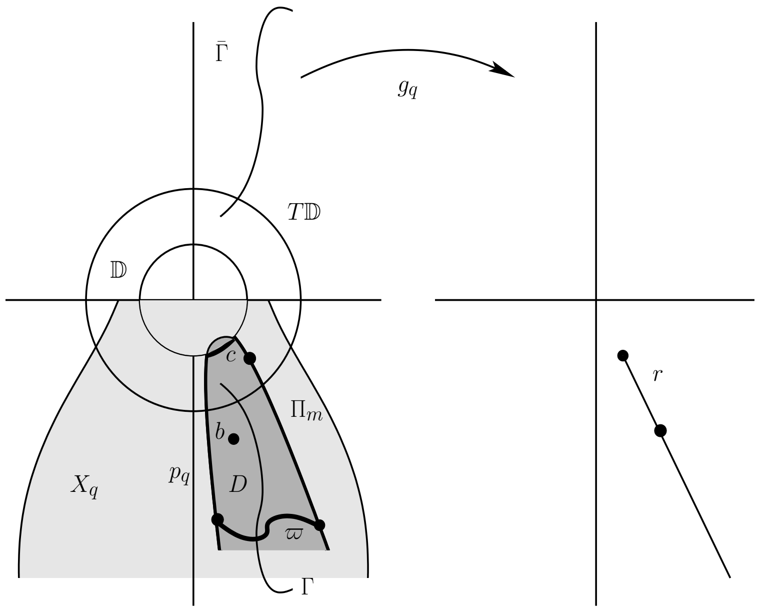

FIGURE 2. The region shaded in dark grey is mapped by g_q surjectively onto $\mathbb{C} \setminus r$.

this observation to $a = \frac{g_q(1/z)}{g_q(z)}$, we conclude that the inequality must be strict also for $z \in \Pi_m \setminus \varpi$.

Thus, we conclude that $g_q(z)$ and $g_q(z) - g_q(1/z)$ have the same number of zeros in the domain delimited by Π_m. This completes the proof of (4) and of our lemma. □

The proof of our main theorem is now almost complete. We will use equation (4) to obtain a contradiction. Our assumption (stated in the paragraph preceding the above Lemma) that g_q has a critical point c in X_q has led us to conclude by part (1) of the previous lemma that the equation $g_q(z) = 0$ has a solution $b \in D \subset X_q$. By part (4) of the same lemma, the map $D \ni w \mapsto k\Theta(w) = g_q(w) - g_q(1/w)$ must then have a zero in D. But $D \cap \mathbb{R} = \varnothing$, and we have seen in Equation (4) that the zeros of Θ are real. This is a contradiction.

We conclude that g_q has no critical points in X_q, and so Φ has an analytic continuation to the whole lower half-plane. As noted at the beginning of the proof, this implies free infinite divisibility for $f_q(x)dx$. □

Acknowledgements

We would like to thank Roland Speicher for many useful discussions regarding this paper and to the anonymous reviewer for numerous useful suggestions regarding the presentation of the paper.

References

[1] N. I. Akhiezer, The classical moment problem and some related questions in analysis, Translated by N. Kemmer, Hafner Publishing Co., New York (1965).

[2] M. Anshelevich, *Partition-dependent stochastic measures and q-deformed cumulants*, Doc. Math. **6** (2001) 343–384 (electronic).

[3] J. V. Armitage and W. F. Eberlein, Elliptic functions, Vol. 67 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge (2006), ISBN 978-0-521-78078-0; 978-0-521-78563-1; 0-521-78563-4.

[4] S. T. Belinschi, M. Bożejko, F. Lehner, and R. Speicher, *The classical normal distribution is* $\boxplus$*-infinitely divisible*, Preprint arXiv:0910.4263v2 [math.OA] (2009) 1–26.

[5] H. Bercovici and V. Pata, *Stable laws and domains of attraction in free probability theory*, Ann. of Math. (2) **149** (1999), no. 3, 1023–1060. With an appendix by Philippe Biane.

[6] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. **42** (1993), no. 3, 733–773.

[7] P. Biane, *Free hypercontractivity*, Commun. Math. Phys. **184** (1997) 457–474.

[8] M. Bożejko, *Bessis-Moussa-Villani conjecture and generalized Gaussian random variables*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **11** (2008), no. 3, 313–321.

[9] M. Bożejko, B. Kümmerer, and R. Speicher, *q-Gaussian processes: non-commutative and classical aspects*, Comm. Math. Phys. **185** (1997), no. 1, 129–154.

[10] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Comm. Math. Phys. **137** (1991), no. 3, 519–531.

[11] ———, *Interpolations between bosonic and fermionic relations given by generalized Brownian motions*, Math. Z. **222** (1996), no. 1, 135–159.

[12] W. Bryc, A. Dembo, and T. Jiang, *Spectral measure of large random Hankel, Markov and Toeplitz matrices*, Ann. Probab. **34** (2006), no. 1, 1–38.

[13] A. Buchholz, *New interpolation between classical and free Gaussian processes*, Infin. Dimens. Anal. Quantum Probab. Relat. Top.. To appear.

[14] C. Donati-Martin, *Stochastic integration with respect to q Brownian motion*, Probab. Theory Related Fields **125** (2003), no. 1, 77–95.

[15] F. Hiai, *q-deformed Araki-Woods algebras*, in Operator algebras and mathematical physics (Constanţa, 2001), 169–202, Theta, Bucharest (2003).

[16] A. S. B. Holland, Introduction to the theory of entire functions, Academic Press [a subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London (1973). Pure and Applied Mathematics, Vol. 56.

[17] M. E. H. Ismail, D. Stanton, and G. Viennot, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, European J. Combin. **8** (1987), no. 4, 379–392.

[18] T. Kemp, *Hypercontractivity in non-commutative holomorphic spaces*, Comm. Math. Phys. **259** (2005), no. 3, 615–637.

[19] A. Nou, *Non injectivity of the q-deformed von Neumann algebra*, Math. Ann. **330** (2004), no. 1, 17–38.

[20] C. Pommerenke, Boundary behaviour of conformal maps, Vol. 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin (1992), ISBN 3-540-54751-7.

[21] É. Ricard, *Factoriality of q-Gaussian von Neumann algebras*, Comm. Math. Phys. **257** (2005), no. 3, 659–665.

[22] L. G. Rogers, *On a Three-fold Symmetry in the Elements of Heine's Series*, Proc. London Math. Soc. (1893) 171–181.

[23] D. Shlyakhtenko, *Some estimates for non-microstates free entropy dimension with applications to q-semicircular families*, Int. Math. Res. Not. (2004), no. 51, 2757–2772.

[24] P. Śniady, *Factoriality of Bożejko-Speicher von Neumann algebras*, Comm. Math. Phys. **246** (2004), no. 3, 561–567.

[25] P. J. Szabłowski, *q-Gaussian Distributions: Simplifications and Simulations*, Journal of Probability and Statistics **2009** (2009) 1–18.

[26] H. van Leeuwen and H. Maassen, *A q deformation of the Gauss distribution*, J. Math. Phys. **36** (1995), no. 9, 4743–4756.

Department of Mathematics, Texas A&M University, College Station TX 77843-3368, U.S.A.
E-mail address: manshel@math.tamu.edu

Department of Mathematics & Statistics, University of Saskatchewan, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 106 Wiggins Road Saskatoon, SK S7N 5E6, CANADA
E-mail address: belinschi@math.usask.ca

Instytut Matematyczny, Uniwersytet Wrocławski, Pl. Grunwaldzki 2/4,50-384, Wrocław, POLAND
E-mail address: bozejko@math.uni.wroc.pl

Institute for Mathematical Structure Theory, Graz Technical University, Steyrergasse 30, 8010 Graz, AUSTRIA
E-mail address: lehner@finanz.math.tu-graz.ac.at