MODULI OF FLAT SU(3)-BUNDLES OVER A KLEIN BOTTLE

Thomas John Baird

ABSTRACT. In this short note, we compute the Betti numbers of the moduli stack of flat $SU(3)$-bundles over a Klein bottle. We also handle the general compact group case over $\mathbb{R}P^2$. In all cases the cohomology is found to be equivariantly formal, supporting a conjecture from the author's doctoral thesis. Our results also verify conjectural formulas obtained by Ho-Liu using Yang-Mills Morse theory.

1. Introduction

Let M be a smooth, compact nonorientable 2-manifold. For a choice of Lie group G, one may study the set of group homomorphisms $Hom(\pi_1(M), G)$, topologized using the compact open topology. The group G acts continuously on $Hom(\pi_1(M), G)$ by conjugation, and we deem two homomorphisms equivalent if they lie in the same orbit. The quotient stack $[Hom(\pi_1(M), G)/G]$ is well known to be isomorphic to the topological moduli stack of flat G-bundles over M (see [4] for example).

In previous work [1], the author computed the stack cohomology

$$H^*_{stack}([Hom(\pi_1(M), SU(2))/SU(2)]; \mathbb{Q}) = H^*_{SU(2)}(Hom(\pi_1(M), SU(2)); \mathbb{Q})$$

in the rank 2 case, where the conjugation action was found to be equivariantly formal (Recall that a G-space X is called equivariantly formal if $H^*_G(X) \cong H^*(X) \otimes H^*(BG)$ as graded vector spaces). This motivated the conjecture that equivariant formality remains true when $SU(2)$ is replaced by any compact connected Lie group G.

In this paper we compute rational Betti numbers and establish the conjecture in the case that $G = SU(3)$ and M is the Klein bottle. We also establish the conjecture for $M = \mathbb{R}P^2$ and arbitrary compact, connected G.

The case of $G = SU(3)$ has already been studied by Ho-Liu [6], using Yang-Mills Morse theory. The Yang-Mills functional determines a Morse stratification of the space $\mathcal{A}$ of connections, whose semistable stratum deformation retracts onto the space $\mathcal{A}_{flat}$ of flat connections. Ho and Liu prove that the normal bundles of the Morse strata have vanishing equivariant Euler classes and introduce the term *locally antiperfect* to describe this property. They conjecture that the stratification is also globally *antiperfect* and produce under this assumption candidate Poincaré series for the moduli stacks. For a G-space X, the Poincaré series $P_t^G(X)$ is defined by

$$P_t^G(X) = \sum_{i=0}^{\infty} \dim(H_G^i(X; \mathbb{Q})) t^i.$$

Let Σ_n denoted the connected sum of $n + 1$ copies of $\mathbb{R}P^2$. Ho-Liu propose that

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(1) $P_t^{SU(3)}(Hom(\pi_1(\Sigma_n), SU(3))) \overset{!}{=} \dfrac{P_t(SU(3))^n + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^n}{(1 - t^4)(1 - t^6)}$

In this paper we confirm (1) in the cases $n = 0, 1$.

Notational convention: Unless otherwise stated, cohomology is singular with $\mathbb{Q}$ coefficients.

2. Equivariant formality

We review the notion of equivariant formality.

Let G be a compact, connected Lie group with universal G-bundle $G \to EG \to BG$ and let X denote a compact, Hausdorff G-space. The equivariant cohomology of a G-space X is defined to be the cohomology of the associated fibre bundle $X \to EG \times_G X \to BG$,

$$H_G^*(X; \mathbb{Q}) = H^*(EG \times_G X; \mathbb{Q}).$$

The Serre spectral sequence $(E_*^{p,q})$ of the fibration $EG \times_G X$ satisfies $E_2^{p,q} \cong H^p(X) \otimes H^q(BG)$ and converges to $H_G^*(X)$. If the spectral sequence collapses at page 2, then $H_G^*(X) \cong H^*(X) \otimes H^*(BG)$ as graded vector spaces and we say the G-space X is equivariantly formal. The cohomology ring $H^*(BG)$ is a polynomial ring with even degree generators, so $H^{odd}(BG) = 0$. Consequently, if $H^{odd}(X) = 0$ then X must be equivariantly formal.

Let $T \subset G$ denote a maximal torus in G. The G-space X is G-equivariantly formal if and only if it is T-equivariantly formal under the restricted T-action. If X^T denotes the T-fixed point set, by [3] IV 5.5 we have

(2) $\dim H^*(X) \geq \dim H^*(X^T)$

with equality if and only if X is equivariantly formal.

3. The projective plane

The projective plane has fundamental group $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$. It follows that $Hom(\pi_1(\mathbb{R}P^2), G) \cong \{g \in G | g^2 = \mathbb{1}\}$ is exactly the set of square roots of the identity element.

Proposition 3.1. *Let G denote a compact, connected Lie group. The G-space $Hom(\pi_1(\mathbb{R}P^2), G)$ is equivariantly formal.*

Proof. The set of square roots of $\mathbb{1} \in G$ is clearly invariant under conjugation. Every conjugacy class intersects a maximal torus $T \subset G$ and the number of roots of $\mathbb{1}$ in T is finite, equal to $2^{\mathrm{rank}(T)}$. So $Hom(\pi_1(\mathbb{R}P^2), G)$ is a finite union of homogeneous spaces of the form G/H where $\mathrm{rank}(H) = \mathrm{rank}(G)$. Such G-spaces are equivariantly formal because $H^{odd}(G/H) = 0$ (see [5] pg. 467).   □

Example 1. The space $Hom(\pi_1(\mathbb{R}P^2), U(n))$ is isomorphic to $\{A \in U(n) | A^2 = \mathbb{1}\}$. Since $A^2 = \mathbb{1}$ implies that the eigenvalues of A are ± 1, by choosing (say) the $+1$ eigenspace, we obtain an isomorphism to a union of Grassmanians:

$$Hom(\pi_1(\mathbb{R}P^2), U(n)) \cong \coprod_{k=0}^{n} Gr_k(\mathbb{C}^n)$$

similarly

$$Hom(\pi_1(\mathbb{R}P^2), SU(n)) \cong \coprod_{k \text{ even}} Gr_k(\mathbb{C}^n)$$

In particular,

$$P_t(Hom(\pi_1(\mathbb{R}P^2), SU(3))) = 1 + (1 + t^2 + t^4).$$

confirming (1) when $n = 0$.

4. The Klein bottle

In this section we consider the moduli stack of rank 3 bundles over a Klein bottle, Kl. It will be useful to pass between structure groups $U(3)$ and $SU(3)$ using the following Lemma:

Proposition 4.1. *We have an equation of Poincaré polynomials*

$$P_t(Hom_{id}(\pi_1(Kl), U(3))) = (1 + t)P_t(Hom(\pi_1(Kl), SU(3))),$$

where $Hom_{id}(\pi_1(Kl), U(3))$ denotes the connected component of $Hom(\pi_1(Kl), U(3))$ containing the trivial homomorphism.

We postpone the proof of 4.1 until the end of the section.

The fundamental group of the Klein bottle Kl has a presentation

$$\pi_1(Kl) \cong \{a_0, a_1 | a_0^2 = a_1^2\}.$$

It follows that

$$Hom(\pi_1(Kl), G) \cong \{(g_0, g_1) \in G^2 | g_0^2 = g_1^2\}.$$

For the rest of this section, set $G := SU(3)$ and $Z := Hom(\pi_1(Kl), G)$. The main result of this paper is

Theorem 4.2. *The space $Hom(\pi_1(Kl), SU(3))$ has Poincaré polynomial*

$$P_t(Hom(\pi_1(Kl), SU(3))) = (1 + t^3)(1 + t^5) + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5).$$

Corollary 4.3. *The conjugation action of $SU(3)$ on $Hom(\pi_1(Kl), SU(3))$ is equivariantly formal. Consequently*

$$(3) \quad P_t^{SU(3)}(Hom(\pi_1(Kl), SU(3))) = \frac{(1 + t^3)(1 + t^5) + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)}{(1 - t^4)(1 - t^6)}$$

in accordance with (1).

Proof. Let $T \subset G = SU(3)$ be a maximal torus. The T-fixed point set is

$$Hom(\pi_1(Kl), G)^T = Hom(\pi_1(Kl), T).$$

Any such homomorphism must pass through the abelianization

$$Ab(\pi_1(Kl)) \cong H_1(Kl; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

, so

$$Hom(\pi_1(Kl), G)^T \cong T \times T_2,$$

where $T_2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the 2-torsion subgroup of T. In particular

$$\dim(Hom(\pi_1(Kl), G)^T) = 16 = \dim(Hom(\pi_1(Kl), G))$$

which by (2) proves equivariant formality. Thus $P_t^{SU(3)}(Hom(\pi_1(Kl), SU(3))) = P_t(Hom(\pi_1(Kl), SU(3)))P_t(BSU(3))$, giving rise to (3). □

We will compute the Poincaré polynomial for Z using the long exact sequence of a pair. Let $A \subset Z$ denote the subset of pairs (k_0, k_1) such that k_0 and k_1 lie in some common maximal torus. The subspace A is preserved by the G-action, and the stabilizers of points in A all contain maximal tori of G. Applying ([2], Theorem 3.3) we produce a G-equivariant cohomology isomorphism

$$\phi : G/T \times_W Z^T \to A. \tag{4}$$

Here $T \subset G$ is a maximal torus, G/T is a coset space, Z^T is the set of T-fixed points, $W = N(T)/T$ is the Weyl group and $G/T \times_W Z^T$ is the orbit space of the product $G/T \times Z^T$ under the natural diagonal W action. The map ϕ sends $\phi(gT, z) = g \cdot z$.

Lemma 4.4. *Let $W = N(T)/T$ act on $T_2 := \{t \in T | t^2 = \mathbb{1}\}$, T and G/T in the usual way. There is a cohomology isomorphism*

$$H^*(A; \mathbb{Q}) \cong H^*(G/T \times_W (T \times T_2); \mathbb{Q}) \tag{5}$$

with Poincaré polynomial

$$P_t(A) = (1 + t^3)(1 + t^5) + (1 + t^2 + t^4)(1 + t)(1 + t^3) \tag{6}$$

Proof. The T-fixed point set Z^T consists of pairs $(g_0, g_1) \in T^2$ such that $t_0^2 = t_1^2$ which is equivalent to $t_0 t_1 \in T_2$, so $Z^T \cong T \times T_2$. Equation (5) then follows from (4).

Choose the maximal torus to be the diagonal matrices in $SU(3)$. Then T_2 consists of the four matrices with diagonal entries $\{(1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$. The Weyl group $W \cong S_3$ acts on T_2 by fixing the identity and permuting the remaining three element. It follows that $G/T \times_W (T \times T_2)$ has two components. The first component $G/T \times_W T$ has cohomology isomorphic to $SU(3)$ and contributes the first term of (6). The second component is $G/T \times_{\mathbb{Z}_2} T$ where the $\mathbb{Z}_2$ action is generated by any of the order two elements in W. The Poincaré polynomials we need to know are $P_t(G/T) = (1 + 2t^2 + 2t^4 + t^6)$, $P_t((G/T)/\mathbb{Z}_2) = P_t(\mathbb{C}P^2) = 1 + t^2 + t^4$, $P_t(T) = (1 + t)^2$ and $P_t(T/\mathbb{Z}_2) = 1 + t$. The result follows by a simple calculation. $\square$

Equation (5) provides a description of A that can be generalized to any compact Lie group G. The more complicated part is describing the complement $Z - A$.

Lemma 4.5. *The Poincaré polynomial of the pair (Z, A) satisfies*

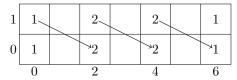$$P_t(Z, A) \leq (1 + t)(t^1 + t^3 + t^6 + t^8), \tag{7}$$

where the notation $p(t) \leq q(t)$ means that $q(t) - p(t)$ has no negative coefficients.

Proof. We will show that $Z - A$ is a 9-dimensional orientable manifold, so by Poincaré duality $H^*(Z, A) \cong H^{(9-*)}(Z - A)$ and we can focus instead on $Z - A$.

A pair $(g, h) \in Z - A$ is characterized by $g, h \in SU(3)$, $gh \neq hg$, $g^2 = h^2$. Since g and h both commute with g^2 but not with each other, we deduce that g^2 must have an eigenvalue λ^2 of multiplicity greater than one, and that g and h have eigenvalues $\lambda, -\lambda, \lambda^{-2}$. It is easily verified that g, h must share an eigenvector $\bar{x}$ such that $g\bar{x} = h\bar{x} = \lambda^{-2}\bar{x}$ and that up to scalar multiplication $\bar{x}$ is the only common eigenvector of g and h.

Now let $P(\mathbb{C}^3)$ denote the set of Hermitian, rank 1 projection operators on $\mathbb{C}^3$. We define the flag variety $Fl_3 := \{p = (p_1, p_2, p_3) \in P(\mathbb{C}^3) | p_i p_j = \delta_{i,j} p_i\}$ where $\delta_{i,j}$ denotes the Dirac delta. Define $\psi : Fl_3 \times U(1) \to SU(3)$ by $\psi(p, \lambda) = \lambda p_1 - \lambda p_2 + \lambda^2 p_3$.

Define $Y := \{(p,q) \in Fl_3 | p_3 = q_3 \text{ and } p_1 q_1 \neq q_1 p_1\}$. Then by the above discussion, the map $\rho : Y \times U(1) \to Z - A$ defined by $\rho((p,q),\lambda) = (\psi(p,\lambda), \psi(q,\lambda))$ is a two sheeted cover. It follows that the induced map $H^*(Z - A) \hookrightarrow H^*(Y \times U(1))$ is injective. We can therefore obtain upperbounds of the Betti numbers of $Z - A$ by computing $H^*(Y)$.

Let $\pi : Y \to Fl_3$ denote the projection $\pi(p,q) = p$. An element q in the fibre $\pi^{-1}(p)$ is uniquely determined by a choice $q_1 \in P(\mathbb{C}^3)$ with $q_1 p_3 = 0$ and $q_1 \neq p_1, p_2$. Thus π has fibres isomorphic to $\mathbb{C}P^1 - 2pts \cong \mathbb{C}^*$. In fact, if L_1, L_2, L_3 denote the tautological line bundles over Fl_3, it follows that Y is isomorphic to $Hom(L_1, L_2)$ minus the zero section. We see that $Y \times U(1)$ is a 9-dimensional orientable manifold. The Serre spectral sequence to the $\mathbb{C}^*$-bundle $Y \to Fl_3$ has E_2 page:

	0		2		4		6
1	1		2		2		1
0	1		2		2		1

The boundary map from $E_2^{p,1} \cong H^p(Fl_3)$ to $E_2^{p+2,0} \cong H^{p+2}(Fl_3)$ is simply cup product by the first Chern class of $Hom(L_1, L_2)$. Here

$$H^*(Fl_3) \cong Q[x_1, x_2, x_3]/(\text{symmetric polynomials}),$$

$$c_1(Hom(L_1, L_2)) = x_2 - x_1$$

and the E_3 page is:

	0		2		4		6
1	0		0		1		1
0	1		1		0		0

So $P_t(Y) = 1 + t^2 + t^5 + t^7$ so $P_t(Z - A) \leq (1 + t^2 + t^5 + t^7)(1 + t)$ by the Kunneth theorem, and Poincaré duality completes the proof. $\square$

In fact we will show in the course of proving Theorem 4.2 that (7) is an equality. We need one more Lemma for our calculation.

Lemma 4.6. *Modulo terms of order 4 or higher,*

$$P_t(Z) = 1 + 2t^3 \pmod{t^4}.$$

Proof. Let $\mathcal{A}$ denote the affine space of $U(3)$-connections on the trivial $\mathbb{C}^3$ bundle over Kl, $\mathcal{A}_{flat}$ the subspace of flat connections and let $\mathcal{G}_b$ denote the group of gauge transformations based at one point. According to ([1] proof of Theorem 1.1) we have an injection $H^*_{\mathcal{G}_b}(\mathcal{A}) \to H^*_{\mathcal{G}_b}(\mathcal{A}_{flat})$ and an isomorphism $H^*_{\mathcal{G}_b}(\mathcal{A}) \cong H^*(U(3))$, so

$$P_t^{\mathcal{G}_b}(\mathcal{A}_{flat}) = (1+t)(1+t^3)(1+t^5) + t^{-1} P_t^{\mathcal{G}}(\mathcal{A}, \mathcal{A}_{flat})$$
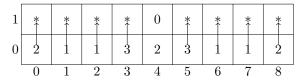
It was proven in [7] and [8] that the Yang-Mills functional determines a Morse stratification, $\mathcal{A} = \bigcup_{\mu=0}^{\infty} \mathcal{A}_\mu$, where $\mathcal{A}_\mu$ is a connected submanifold with codimension 4μ and orientable normal bundle and for which the open stratum $\mathcal{A}_0$ contains $\mathcal{A}_{flat}$

as a deformation retract. We deduce that $P_t^{\mathcal{G}_b}(\mathcal{A}, \mathcal{A}_{flat}) = P_t^{\mathcal{G}_b}(\mathcal{A}, \mathcal{A}_0) = t^4 \pmod{t^5}$, so

$$P_t^{\mathcal{G}_b}(\mathcal{A}_{flat}) = (1+t)(1+t^3)(1+t^5) + t^3 = 1 + t + 2t^3 \pmod{t^4}$$

Finally, holonomy determines a homeomorphism $\mathcal{A}_{flat}/\mathcal{G}_0 \cong Hom_{id}(\pi_1(Kl), U(3))$, so the lemma follows by applying Lemma 4.1. $\qquad\square$

Proof of Theorem 4.2. We consider the long exact sequence of the pair (Z, A) as the spectral sequence for the filtration, we get:

1	*	*	*	*	0	*	*	*	*
0	2	1	1	3	2	3	1	1	2
	0	1	2	3	4	5	6	7	8

where each of the star entries is either 0 or 1 by Lemma 4.5. Lemma 4.6 implies that the first four stars are 1 and are killed by the boundary map. Inequality (2) then implies that the remaining four stars are also 1, but are not killed by the boundary map. Thus the E_2 page looks like

1	0	0	0	0	0	1	1	1	1
0	1	0	0	2	2	3	1	1	2
	0	1	2	3	4	5	6	7	8

completing the proof.

$\qquad\square$

Proof of Proposition 4.1. The identity component is identified with

$$Hom_{id}(\pi_1(Kl), U(2)) \cong X := \{(g_0, g_1) \in U(2)^2 | g_0^2 = g_1^2, det(g_0) = det(g_1)\}.$$

It follows that the map $\rho : X \to U(1)$, $\rho((g_0, g_1)) = det(g_0)$ is a fibre bundle with fibre $Z = Hom(\pi_1(Kl), SU(3))$. We deduce that $\kappa : Z \times U(1) \to X$ sending $\kappa((g_0, g_1), \lambda) = (\lambda g_0. \lambda g_1)$ is a Galois cover. The deck transformation group $\mathbb{Z}_3 = \{\gamma \in U(1) | \gamma^3 = \mathbb{1}\}$ acts by $\gamma \cdot ((g_0, g_1), \lambda) = ((\gamma g_0, \gamma g_1), \gamma^2 \lambda)$. Thus $H^*(X) = H^*(Z \times U(1))^{\mathbb{Z}_3}$.

To prove the proposition, we must show that $\mathbb{Z}_3$ acts trivially on the cohomology of $Z \times U(1)$. The action on the $U(1)$ factor is by translation, which necessarily is cohomologically trivial. To understand the effect on $H^*(Z)$, notice that the action preserves the subspace $A \subset Z$ from (4) Thus the long exact sequence of the pair :

$$... \to H^*(Z, A) \to H^*(Z) \to H^*(A) \to ...$$

is acted on equivariantly by $\mathbb{Z}_3$. By Lemma 4.5, the cohomology groups $H^*(Z, A)$ have dimension 1 or 0 in all degrees, so $\mathbb{Z}_3$ must act trivially on $H^*(Z, A)$. The action on A lifts to the model space $(G/T \times T \times T_2)^W$ where it acts by translation of the T factor, which is an isotopy and hence trivial cohomologically. Since $\mathbb{Z}_3$ acts trivially on both $H^*(Z, A)$ and $H^*(A)$, we infer from the long exact sequence that it acts trivially also on $H^*(Z)$, completing the proof. $\qquad\square$

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Mathematical Institute, University of Oxford, 24-29 St. Giles' St., Oxford UK, OX1 3LB

E-mail address: `baird@maths.ox.ac.uk`