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INVARIANT HYPERSURFACES IN HOLOMORPHIC DYNAMICS

SERGE CANTAT

To Thierry Vust for his 65th birthday

ABSTRACT. We prove the following result, which is analogous to two theorems, one due to Kodaira and Krasnov and another one due to Jouanolou and Ghys. Let M be a compact complex manifold and f a dominant endomorphism of M. If there exist k totally invariant irreducible hypersurfaces $W_i \subset M$, with $k > \dim(M) + h^{1,1}(M)$ then f preserves a nontrivial meromorphic fibration. We then study the case where f is a meromorphic map.

1. Introduction

1.1. Let M be a connected compact complex manifold of dimension n, and $f : M \to M$ be a surjective endomorphism of M. Let $W \subset M$ be a hypersurface. By definition, W is *totally invariant* if $f^{-1}(W) = W$. This property implies, and is stronger than, the *forward invariance* $f(W) = W$. These notions coincide when f is an automorphism.

Locally, in a small open set $\mathcal{U}$, the hypersurface W is defined by an equation $\varphi = 0$; when W is irreducible, the hypersurface is totally invariant if there is a positive integer m such that $\varphi \circ f$ is locally the m-th power of an equation of W. More precisely, if $\mathcal{V}$ is any small open subset of M such that $f(\mathcal{V}) \subset \mathcal{U}$, and if ψ is an equation for W in $\mathcal{V}$, then there exists a non vanishing holomorphic function $\xi \in O^*(\mathcal{V})$ such that

$$\varphi \circ f = \xi \psi^m, \quad \text{on } \mathcal{V}.$$

The topological degree of f is then equal to the product of the multiplicity m and the topological degree of $f_{|W} : W \to W$; the divisor defined by the jacobian determinant of f contains W with multiplicity $m-1$.

Theorem A. *Let M be a compact complex manifold, and f be a dominant endomorphism of M. If there are k totally invariant hypersurfaces $W_i \subset M$, with*

$$(*) \qquad k > \dim(M) + \dim(H^1(M, \Omega^1_M)),$$

then there is a non constant meromorphic function Φ and a non zero complex number α such that $\Phi \circ f = \alpha \Phi$.

This result has already been used, and proved, by the author and by Kawaguchi when M is a surface (see [6, 4, 14]), and by Zhang when f is an automorphism of a compact Kähler manifold with positive entropy (see [18]). We shall extend (a weak form of) Theorem A to the case of meromorphic transformations in section 3, Theorem B.

1.2. Remarks.

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1.2.1. The sheaf Ω^1_M is the sheaf of holomorphic 1-forms, and $H^i(M,\Omega^1_M)$ denotes the i-th Cech cohomology group of this sheaf. By Hodge theory, these cohomology groups are finite dimensional complex vector spaces (see [12], chapter 0, section 6, page 100).

1.2.2. The proof of Theorem A provides a slightly stronger statement: The number k in inequality $(*)$ can be replaced by the total number of irreducible components of the W_i.

1.2.3. If g is an endomorphism of the complex projective space $\mathbb{P}^n(\mathbf{C})$ with two totally invariant hypersurfaces, then there is a constant $\alpha \neq 0$ and a non constant meromorphic function Ψ such that $\psi \circ g = \alpha \Psi^{\deg(g)}$. This conclusion is different from the invariance property $\Phi \circ f = \alpha\Phi$.

The coordinate axis of the plane $\mathbb{P}^2(\mathbf{C})$ are totally invariant under the action of the endomorphism $g[x:y:z] = [x^2:y^2:z^2]$. In this case,

$$\dim(\mathbb{P}^2(\mathbf{C})) + \dim(H^1(\mathbb{P}^2(\mathbf{C}),\Omega^1_{\mathbb{P}^2(\mathbf{C})})) = 2+1 = 3$$

is equal to the number of totally invariant lines, and all meromorphic functions Φ such that $\Phi \circ g = \alpha\Phi$ for some $\alpha \in \mathbf{C}$ are indeed constant. The same example, but in dimension n, shows that *Theorem A is sharp* for $M = \mathbb{P}^n(\mathbf{C})$.

1.2.4. Theorem A is analogous, in a dynamical setting, to the following statement: *If a compact complex manifold M has k irreducible hypersurfaces, with*

$$k > \dim(M) + \dim(H^1(M,\Omega^1_M)),$$

then there is a non constant meromorphic function on M (see [1], section IV.6, page 129, or [10, 15] and [3, 16]).

Another similar statement has been obtained by Jouanolou and Ghys for foliated manifolds: *If a codimension one (singular) holomorphic foliation $\mathcal{F}$ of a compact complex manifold M has an infinite number of compact leaves, then $\mathcal{F}$ has a meromorphic first integral* (Ghys-Jouanolou theorem, see [11, 13, 7]). The "infinite number" of compact leaves can be replaced by a large enough number of such leaves, $k > k(M,\mathcal{F})$, where $k(M,\mathcal{F})$ depends on M and on the degree of the meromorphic forms defining $\mathcal{F}$.

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2. Endomorphisms

This section is devoted to the proof of Theorem A; the results of this section will also be used in section 3 when we study dominant meromorphic transformations.

2.1. Meromorphic forms. Let M be a compact complex manifold. Let Ω^1_M (resp. $\mathcal{M}^1_M$) be the sheaf of holomorphic (resp. meromorphic) 1-forms on M; we have a short exact sequence of sheaves

$$0 \to \Omega^1_M \to \mathcal{M}^1_M \to Q^1_M \to 0,$$

where Q^1_M denotes the quotient sheaf; this sequence gives rise to a long exact sequence of Cech cohomology groups (see [12], chapter 0, and [1], §IV.6))

$$0 \to \Gamma(\Omega^1_M) \to \Gamma(\mathcal{M}^1_M) \to \Gamma(Q^1_M) \to H^1(M,\Omega^1_M) \to ...$$

where $\Gamma(\cdot)$ stands for $H^0(M,\cdot)$, i.e. for the complex vector space of global sections. By Dolbeault isomorphism, the first cohomology group $H^1(M,\Omega^1_M)$ is isomorphic to $H^{1,1}_{\overline{\partial}}(M)$. Both $\Gamma(\Omega^1_M)$ and $H^1(M,\Omega^1_M)$ have finite dimension, while $\Gamma(\mathcal{M}^1_M)$ and $\Gamma(Q^1_M)$ can be infinite dimensional.

2.2. Hypersurfaces and logarithmic forms. Let W be a (reduced) hypersurface of a compact complex manifold. Let $\mathcal{U}$ be an open set on which W is defined by an equation $\varphi \in O(\mathcal{U})$. The meromorphic form

$$\sigma_W = \mathrm{d}\log(\varphi) = \frac{\mathrm{d}\varphi}{\varphi}$$

depends on the choice of an equation, but does not depend on this choice modulo holomorphic 1-forms: If φ is replaced by $\xi\varphi$ with $\xi \in O^*(\mathcal{U})$, then σ_W changes by the addition of the holomorphic form $(\mathrm{d}\xi)/\xi$. Thus, σ_W is a well defined global section of Q^1_M. If W_1, ..., W_l are distinct, reduced, and irreducible hypersurfaces, the sections $\sigma_{W_i} \in \Gamma(Q^1_M)$ are linearly independant.

2.3. Endomorphisms. Let g be a dominant endomorphism of a compact complex manifold M. Let $W \subset M$ be a totally invariant hypersurface, and W_1, ..., W_l be its irreducible components. There is a map $\iota : \{1,...,l\} \to \{1,...,l\}$ such that $g(W_i) = W_{\iota(i)}$ for all $i \in \{1,...,l\}$. The map ι is surjective, hence injective, and $g^{-1}(W_i) = W_{\iota^{-1}(i)}$ (as sets, not as divisors). Let $m_i - 1$ be the ramification index of g along $W_{\iota^{-1}(i)}$; then, by definition, we get the following equality of divisors

$$g^*W_i = m_i W_{\iota^{-1}(i)}.$$

In other words, if φ_i is a local equation of W_i, then $\varphi_i \circ g$ is locally equal to the m_i-th power of an equation of $W_{\iota^{-1}(i)}$. This implies that

$$g^*\sigma_{W_i} = m_i \sigma_{W_{\iota^{-1}(i)}},$$

as global sections of Q^1_M, hence modulo local holomorphic forms. In other words, the complex vector space $Z \subset \Gamma(Q^1_M)$ which is spanned by the σ_{W_i} has dimension l, is g^*-invariant, and the matrix of g^* in the basis $(\sigma_{W_i})_{1\le i\le l}$ is a permutation matrix with coefficients multiplied by positive integers m_i. Thus, there is a positive power of this matrix which is diagonal with positive integral entries. In particular, g^* is diagonalizable.

In what follows, to prove Theorem A, we use the following notations:

- g has k totally invariant hypersurfaces;
- the union of these hypersurfaces is a totally invariant hypersurface W;
- the number of irreducible components W_i of W is denoted by l; in particular, $l \geq k$;
- the vector space $Z \subset \Gamma(Q^1_M)$ is generated by the σ_{W_i}; its dimension is equal to l.

2.4. Invariant global meromorphic forms. The morphism

$$c : \Gamma(Q_M^1) \to H^1(M, \Omega_M^1),$$

that appears in the long exact sequence of section 2.1, is g^*-equivariant and its kernel K_M^1 is g^*-invariant. The intersection Z_0 of Z with K_M^1 is a g^*-invariant subspace, and the codimension of Z_0 in Z is at most $h^{1,1}(M)$. Since the subspace Z_0 is in the image of the surjective morphism $\Gamma(\mathcal{M}_M^1) \to \Gamma(Q_M^1)$, one can find a basis $(\eta_j)_{1\leq j\leq m}$ of Z_0 such that

(1) $m \geq l - h^{1,1}(M)$;
(2) η_j is a global meromorphic 1-form (i.e. a section of $\mathcal{M}_M^1$) ;
(3) η_j is an eigenvector of g^* (as an element of $\Gamma(Q_M^1)$).

The third assertion means that there exists a complex number λ_j such that $g^*\eta_j = \lambda_j\eta_j$ as sections of Q_M^1; this property follows from the fact that g^* is diagonalizable on Z and Z_0. This implies that the global meromorphic form η_j satisfies

$$g^*\eta_j = \lambda_j\eta_j + \xi_j$$

where ξ_j is a global holomorphic 1-form, i.e. an element of $\Gamma(\Omega_M^1)$.

Let $\hat{Z}_0$ be the preimage of Z_0 in $\Gamma(\mathcal{M}_M^1)$. The vector space $\Gamma(\Omega_M^1)$ embeds in $\hat{Z}_0$ as a g^*-invariant subspace, the dimension of which is equal to $h^{1,0}(M)$. Take a basis of $\Gamma(\Omega_M^1)$ and complete it with the vectors η_j to get a basis of $\hat{Z}_0$. With respect to this basis, the matrix of g^* is upper triangular by blocks: The two blocks on the diagonal are given by (i) the action of g^* on $\Gamma(\Omega_M^1)$ and (ii) the diagonal matrix with coefficients λ_j. The transposed matrix has m linearly independant eigenvectors corresponding to the last m vectors of the basis. This implies that g^* has m linearly independent eigenvectors $\nu_j \in \hat{Z}_0$ corresponding to the eigenvalues λ_j:

$$g^*\nu_j = \lambda_j\nu_j, \quad 1 \leq j \leq m.$$

Of course, some of these eigenvectors can be contained in $\Gamma(\Omega_M^1)$.

2.5. Conclusion. The dimension of the cotangent space T_x^*M, $x \in M$, is equal to $\dim(M)$. If m is strictly larger than the dimension of M, then the $\nu_j(x)$ are linearly dependent at all points x of M. Permuting the indices, we can assume that there exists an integer n, with

$$2 \leq n \leq \dim(M) < m,$$

such that ν_1, ... ν_n are linearly independent at the generic point x of M but

$$\nu_{n+1} = \sum_{j=1}^{j=n} b_j\nu_j$$

where the b_j are meromorphic functions, and the b_j are uniquely determined by this equation. Note that at least one of these functions is not constant because the ν_j are linearly independent in the complex vector space $\Gamma(\mathcal{M}_M^1)$.

Applying g^* and dividing by λ_{n+1} we get a second relation

$$\nu_{n+1} = \sum_{j=1}^{j=n} \frac{\lambda_j}{\lambda_{n+1}} b_j \circ g\, \nu_j.$$

By uniqueness of the b_j, this implies that there is a non constant meromorphic function Φ (equal to one of the b_j) and a complex number α (of type λ_j/λ_{n+1}) such that

$$\Phi \circ g = \alpha\Phi.$$

This completes the proof of Theorem A.

2.6. Complementary remarks.

2.6.1. In the proof of Theorem A, one can replace the sheaf of holomorphic forms by the sheaf of closed holomorphic forms, and get similar results. On compact Kähler manifolds, this does not change anything, because global holomorphic forms are always closed. To get an interesting example, consider the group $\mathsf{N}(\mathbf{C})$ of upper triangular complex matrices

$$B = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Define M as the quotient of $\mathsf{N}(\mathbf{C})$ by the cocompact discrete subgroup $\mathsf{N}(\mathbf{Z}[i])$ of matrices with coefficients in the set of Gaussian integers $\mathbf{Z}[i]$. Then

- $\omega = \mathrm{d}z - y\mathrm{d}x$ is a holomorphic 1-form on $\mathsf{N}(\mathbf{C})$, is invariant by right multiplication, and defines a 1-form on M that is not closed: $\mathrm{d}\omega = \mathrm{d}x \wedge \mathrm{d}y$.
- $g(x,y,z) = (qx, qy, q^2z)$, with $q \in \mathbf{Z} \setminus \{0\}$, is a dominant endomorphism with topological degree q^8; moreover, $g^*\omega = q^2\omega$.

Similar examples of non closed forms and automorphisms $g \neq Id$ can be constructed on $\mathsf{SL}_2(\mathbf{C})/\Lambda$ where Λ is a cocompact lattice in the complex Lie group $\mathsf{SL}_2(\mathbf{C})$ (for $\mathsf{SL}_2(\mathbf{C})$, any dominant endomorphism is an automorphism, see [5]).

2.6.2. When M is a compact Kähler manifold with $h^{1,0}(M) > 0$, the Albanese map provides a morphism $\alpha_M : M \to A_M$ where A_M is the compact complex torus

$$A_M = (H^0(M, \Omega^1_M))^*/H_1(M, \mathbf{Z}).$$

This morphism is equivariant under the action of g on M, the linear action of the transpose ${}^tg^*$ on the dual space $(H^0(M, \Omega^1_M))^*$ and the action of g_* on the lattice $H_1(M, \mathbf{Z})$. To study the image $\alpha_M(M)$ and the action of g on it, one can use the following fact: Let $B \subset A_M$ be the maximal connected subtorus such that $\alpha_M(M) + B = \alpha_M(M)$; let $\pi : A \to A/B$ be the quotient map; then $\pi(\alpha_M(M))$ is of general type and, consequently, all its dominant endomorphisms have finite order (see [8] chapter VIII). This strategy provides invariant fibrations as soon as $h^{1,0}(M) > 0$ and M is not a torus.

2.6.3. The proof of Theorem A produces $l - h^{1,1}(M)$ independant logarithmic 1-forms ν_j, with poles along the totally invariant hypersurfaces, that satisfy $g^*\nu_j = \lambda_j \nu_j$; as said above, we can assume this forms to be closed. If we have $\dim(M)$ of them, either there is an invariant, non constant, meromorphic function, or the forms ν_j define an invariant parallelism on the complement of the invariant hypersurfaces. With respect to this parallelism, g acts as an affine transformation (as Tchebychev and Lattès maps do).

2.7. Ueda's example (see [17]**).** Let us show that Theorem A can not be extended when the hypersurfaces W_i are just forward invariant, i.e. $g(W_i) = W_i$.

Let f be an endomorphism of the projective line $\mathbb{P}^1(\mathbf{C})$, with degree $d \geq 2$. Assume that f has two periodic points x_0 and y_0 which are not totally invariant and have distinct multipliers; here, by definition, the multiplier of a periodic point x with $f^k(x) = x$ is the non negative k-th root of the modulus of the derivative $|(f^k)'(x)|$.

Let $h \in \mathsf{End}(\mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C}))$ be the diagonal endomorphism $h = (f, f)$. Since h commutes to $\eta(x,y) = (y,x)$, it induces an endomorphism of the quotient space $(\mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C}))/\eta$. This complex surface parametrizes unordered pairs of points of $\mathbb{P}^1(\mathbf{C})$ and is isomorphic to the projective plane $\mathbb{P}^2(\mathbf{C})$; using affine coordinates, the natural projection $\pi\colon \mathbb{P}^1(\mathbf{C})\times\mathbb{P}^1(\mathbf{C})\to\mathbb{P}^2(\mathbf{C})$ can be identified with the map

$$\pi(x,y) = (xy, x+y).$$

The image of the diagonal Δ is the conic $C=\{(u,v);\, v^2=4u\}$; horizontal and vertical lines in $\mathbb{P}^1(\mathbf{C})\times\mathbb{P}^1(\mathbf{C})$ are mapped to the lines of $\mathbb{P}^2(\mathbf{C})$ which are tangent to C. The endomorphism g preserves C and the action of g on C is conjugate to the action of f on $\mathbb{P}^1(\mathbf{C})$ by $\pi\colon \Delta\to C$. Since h permutes horizontal (resp. vertical) lines,

- g permutes the set of lines tangent to the conic C.

In particular, each periodic point of f gives rise to a periodic tangent line. Thus,

- the number of g-periodic lines is infinite.

Let us now assume that there is a non constant meromorphic function $\Psi\colon \mathbb{P}^2(\mathbf{C}) \dashrightarrow \mathbb{P}^1(\mathbf{C})$ and an endomorphism $\overline{g}$ of $\mathbb{P}^1(\mathbf{C})$ such that

$$\Psi\circ g = \overline{g}\circ\Psi.$$

Let us lift Ψ to $\hat{\Psi}=\Psi\circ\pi : \mathbb{P}^1(\mathbf{C})\times\mathbb{P}^1(\mathbf{C}) \dashrightarrow \mathbb{P}^1(\mathbf{C})$. Then $\hat{\Psi}\circ h = \overline{g}\circ\hat{\Psi}$.

Let $m=(x,y)$ be a periodic point of h of period k. The differential $D(h^k)_m$ must preserve three directions: The horizontal direction, the vertical one, and the direction tangent to the level curve $F(m)$ of $\hat{\Psi}$ through m. If the multiplier of the periodic point x of f has a larger modulus than the multiplier of y, then the level curve $F(m)$ must have a horizontal tangent at m. Let us apply this remark to the point $m_0=(x_0,y_0)$. By assumption neither x_0 nor y_0 is totally invariant; taking preimages of the point m_0, we see that there exists a Zariski dense subset of points $m_i\in\mathbb{P}^1(\mathbf{C})\times\mathbb{P}^1(\mathbf{C})$ such that $F(m_i)$ has a horizontal tangent at m_i. This implies that $\hat{\Psi}$ is a non constant rational function of the second coordinate y. We get a contradiction because such a function is not invariant under the action of η. This contradiction shows that

- if $\Psi : \mathbb{P}^2(\mathbf{C}) \dashrightarrow \mathbb{P}^1(\mathbf{C})$ is a rational function, and $\overline{g}$ is an endomorphism of $\mathbb{P}^1(\mathbf{C})$ such that $\Psi\circ g=\overline{g}\circ\Psi$, then Ψ is constant.

In fact, the same proof shows that g does not preserve any algebraic foliation, but g preserves the algebraic 2-web of lines tangent to the conic C.

3. Meromorphic transformations

Let $g : M \dashrightarrow M$ be a meromorphic transformation of M. The indeterminacy set $\mathrm{Ind}(g)$ is a Zariski closed subset of M of codimension ≥ 2. The complement $\mathrm{Dom}(g)$ of $\mathrm{Ind}(g)$ is the domain of definition of g.

3.1. Strong total invariance. Let α be a holomorphic 1-form on an open subset U of M. Let V be the preimage of U under the holomorphic map $g:\mathrm{Dom}(g)\to M$, and V' be the interior of the closure of V in M (thus adding indeterminacy points of g to V). Then $g^*\alpha$ is a well defined holomorphic 1-form on V and Hartogs theorem shows that $g^*\alpha$ extends to a holomorphic section of the sheaf Ω^1_M on V'. This defines a linear map g^* from the set of sections of Ω^1_M on U to the set of sections on V'. In what follows, we assume that g is dominant: The image of $\mathrm{Dom}(g)$ is Zariski dense in M and, equivalently, g^* is an injection from $\Gamma(\Omega^1_M,U)$ to $\Gamma(\Omega^1_M,V')$ for all open subsets U of M.

The exceptional locus $\mathrm{Exc}(g)$ is the set of points m in M such that there is a curve through m which is mapped to a point by g. Since g is dominant, the codimension of $\mathrm{Exc}(g)$ is positive.

Let $W \subset M$ be a hypersurface. One says that W is *strongly totally invariant* if the total transform $g^*(W)$ of W by g is a multiple of W. This means that there is a positive integer m such that local equations φ of W are transformed into m-th power of local equations when composed with g. With this definition, Theorem A and its proof extend to strongly invariant hypersurfaces of rational maps.

Unfortunately, this definition is not appropriate to rational transformations, as the following example shows.

Example 3.1. Let C be a smooth cubic curve in the projective plane, and p be a point of C. Let $f_p : \mathbb{P}^2(\mathbf{C}) \dashrightarrow \mathbb{P}^2(\mathbf{C})$ be the birational involution that preserves the pencil of lines through p and fixes C pointwise. Its action on the plane can be described as follows. Let L be a generic line through p; it intersects C in three points p, a, and b. By definition, f_p preserves L, fixes a and b, and $(f_p)_{|L}$ is the unique non trivial involution with these properties. The line is isomorphic to $\mathbb{P}^1(\mathbf{C})$ and, choosing homogenous coordinates along L with $a = [0:1]$ and $b = [1:0]$, we have $f_p[x:y] = [-x:y]$. The indeterminacy points of f_p are contained in C; one of them is p, and the other are points $q \in C$ such that the line (pq) is tangent to C at q

In this example, all points of $C \setminus \mathrm{Ind}(f_p)$ are fixed by f_p, but C is not strongly invariant; its total transform contains all exceptional curves of f_p. Choosing different points p_i in C, and composing the f_{p_i}, we get birational transformations of the plane with infinite order and a cubic curve of fixed points. This behavior corresponds to a general fact: If a curve C is invariant under a birational transformation f of the plane, then C contains indeterminacy points of f (see [9] for example).

Remark 3.2 (see [2])**.** Let p_1, ... p_n be n generic points on the cubic curve C. Blowing up all the indeterminacy points of the involutions f_{p_i}, we get a surface X on which the f_{p_i} lift simultaneously to automorphisms. The composition of three of them $g = f_{p_1} \circ f_{p_2} \circ f_{p_3}$ is an automorphism with positive entropy and with a smooth genus one curve of fixed points. This answers the open question mentioned in [9], page 2987.

3.2. Total invariance. Let us define a more natural notion of invariance. One says that W is *totally invariant* if its strict transform by g is equal to W (as a set). For endomorphisms, this notion coincides with the definition given in the introduction.

Let $c(g)$ be the number of irreducible components of the codimension 1 part of the set of critical values of g. Let $e(g)$ be the number of irreducible components of $\mathrm{Exc}(g)$.

Theorem B. *Let M be a compact complex manifold, and g be a dominant meromorphic transformation of M. If the number of totally invariant hypersurfaces of M is strictly bigger than the sum*

$$c(g) + e(g) + \dim(M) + \dim(H^1(M, \Omega^1_M)),$$

then g preserves a non constant meromorphic function.

Unfortunately, the numbers $c(g)$ and $(e(g)$ depend on g, and are not bounded by the geometry of M; if M is projective, with a given polarization, $c(g) + e(g)$ can be bounded in term of the degree of g. A similar phenomenon appears in Ghys-Jouanolou theorem.

Corollary 3.3. *On a compact complex manifold, a dominant meromorphic transformation with an infinite number of totally invariant hypersurfaces preserves a non constant meromorphic function.*

Proof of Theorem B. We follow the same lines as in the proof of Theorem A. By assumption, g has k totally invariant hypersurfaces W^j, $1 \leq j \leq k$, with

$$k > c(g) + e(g) + \dim(M) + \dim(H^1(M, \Omega^1_M)).$$

For each W^j, we denote by W^j_i its irreducible components.

Let W be a totally invariant hypersurface and W_i, be its irreducible components. Then $g^*\sigma_{W_i} = m_i\sigma_{W_{\iota^{-1}(i)}} + \sigma_{F_i}$ where F_i is a hypersurface contained in $\mathrm{Exc}(g)$. The integer m_i is different from 1 if, and only if W_i is contained in the set of critical values of g. Thus, there are at least $k - c(g)$ totally invariant hypersurfaces W^j for which $m_i = 1$ for all their irreducible components W^j_i.

For each of these invariant hypersurfaces W^j, we can write

$$g^*\sigma_{W^j} = \sigma_{W^j} + \sigma_{F^j}$$

where, by definition,

$$\sigma_{W^j} = \sum_i \sigma_{W^j_i}$$

is a sum over the irreducible components W^j_i of W^j. The sections σ_{F^j} of Q^1_M are contained in the vector space which is spanned by the σ_{E_i} where E_i describes the set of irreducible components of $\mathrm{Exc}(g)$; the dimension of this complex vector space is equal to $e(g)$. Thus, taking linear combinations, we construct $k - c(g) - e(g)$ sections σ_j of Q^1_M such that

- $g^*\sigma_j = \sigma_j$ for all $1 \leq j \leq k - c(g) - e(g)$;
- the $k - c(g) - e(g)$ sections σ_j of Q^1_M are linearly independent.

The arguments of sections 2.4 and 2.5 can then be reproduced to complete the proof. Since the multiplicties m_i are all equal to 1, the meromorphic function Φ that is constructed is g-invariant: $\Phi \circ g = \Phi$. □

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IRMAR (UMR 6625 DU CNRS), UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX.
E-mail address: serge.cantat@univ-rennes1.fr