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REMARKS ON THE α–PERMANENT

PÉTER E. FRENKEL

ABSTRACT. We recall Vere-Jones's definition of the α–permanent and describe the connection between the (1/2)–permanent and the hafnian. We establish expansion formulae for the α–permanent in terms of partitions of the index set, and we use these to prove Lieb-type inequalities for the $\pm\alpha$–permanent of a positive semi-definite Hermitian $n \times n$ matrix and the $\alpha/2$–permanent of a positive semi-definite real symmetric $n \times n$ matrix if α is a nonnegative integer or $\alpha \geq n-1$. We are unable to settle Shirai's nonnegativity conjecture for α–permanents when $\alpha \geq 1$, but we verify it up to the 5×5 case, in addition to recovering and refining some of Shirai's partial results by purely combinatorial proofs.

1. Introduction

Following Vere-Jones [V1, V2], we define the α–permanent of the $n \times n$ matrix $A = (a_{i,j}) \in M_n(\mathbb{C})$ to be

$$\mathrm{per}_\alpha A = \sum_{\pi \in \mathfrak{S}_n} \alpha^{\nu(\pi)} \prod_{i=1}^{n} a_{i,\pi(i)},$$

where $\mathfrak{S}_n$ is the symmetric group on n elements and $\nu(\pi)$ is the number of disjoint cycles of the permutation π. In particular, $\alpha = 1$ yields the ordinary permanent and $\alpha = -1$ yields $(-1)^n$ times the determinant. For real symmetric matrices, the case $\alpha = 1/2$ recovers another known concept. Recall that the hafnian of a $2n \times 2n$ symmetric matrix $C = (c_{i,j})$ is defined by

$$\mathrm{haf}\ C = \frac{1}{n!2^n} \sum_{\pi \in \mathfrak{S}_{2n}} c_{\pi(1),\pi(2)} \cdots c_{\pi(2n-1),\pi(2n)} = \sum c_\Gamma,$$

where Γ runs over the 1-regular graphs (perfect matchings) on $[2n] = \{1, \ldots, 2n\}$ and

$$c_\Gamma = \prod_{e \in E(\Gamma)} c_e;$$

note that we write $c_e = c_{ij}$ if i and j are the endpoints of the edge e.

Proposition 1.1. *Let A be a real symmetric $n \times n$ matrix. Then*

$$\mathrm{per}_{1/2} A = \frac{1}{2^n} \mathrm{haf} \begin{pmatrix} A & A \\ A & A \end{pmatrix}. \tag{1}$$

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This is essentially known. Since both sides are polynomials in the entries of A, we may assume that A is positive semi-definite. Then we may consider centered, jointly Gaussian random variables $X_1, \ldots, X_n$ with covariance matrix A. The left hand side of (1) is known to be equal to $2^{-n}E(X_1^2 \cdot \cdots \cdot X_n^2)$, cf. Lu and Richards [LR] and Shirai [Sh]. The right hand side is the same quantity by the well-known [B, F, G, S, Z] Wick formula.

Nevertheless, a direct combinatorial proof may be of some interest.

Proof. Both sides of (1) are linear combinations of the a_Γ where Γ runs over the 2–regular graphs (with multiple edges and loops allowed) on the vertex set $[n]$. We shall compute the coefficients of each side.

Suppose that Γ has $\nu = \sum \nu_i$ connected components, of which ν_i are i–cycles. Then $n = \sum i\nu_i$.

The coefficient of a_Γ on the LHS of (1) is $2^{-\nu}$ times the number of (1,1)–regular directed graphs whose underlying undirected graph is Γ. This number is

$$2^{-\nu}2^{\nu-\nu_1-\nu_2} = 2^{-\nu_1-\nu_2},$$

since each cycle of length ≥ 3 has two non-isomorphic orientations, whereas each cycle of length 1 or 2 has only one.

Calculating the hafnian on the RHS of (1) involves studying 1–regular graphs on $[2n]$. Since we have a matrix with 2×2 blocks A, we consider the projection map $[2n] \simeq [n] \times [2] \to [n]$. Any graph on $[n] \times [2]$ projects to a graph on $[n]$; multiple edges and loops may be created when we project. By a lifting of a 2–regular graph given on a subset $S \subseteq [n]$, we mean a 1–regular graph on $S \times [2]$ that projects to the given graph.

The coefficient of a_Γ on the RHS of (1) is 2^{-n} times the number of liftings of Γ. A cycle of length 1 (i.e., a loop) in Γ has only one lifting. A cycle of length 2 (i.e., two parallel edges) has two liftings. A cycle of length $i \geq 3$ has 2^i liftings. Thus, the coefficient is

$$2^{-n}2^{\nu_2}\prod_{i\geq 3} 2^{i\nu_i} = 2^{-n+\nu_2+n-\nu_1-2\nu_2} = 2^{-\nu_1-\nu_2},$$

and the Proposition follows. □

2. Expansion formulae

In this section, we shall expand $\mathrm{per}_\alpha A$ in terms of certain β–permanents of diagonal submatrices of A. These expansions shall be described in terms of partitions of the set $[n]$. When A is the identity matrix, our arguments reduce more or less to some classical ideas of Gian-Carlo Rota related to enumerating set partitions [R].

Put $[n] = \{1, \ldots, n\}$. For an $n \times n$ matrix $A = (a_{i,j})$ and a subset I of $[n]$, we write $A[I] := (a_{i,j})_{i,j\in I}$. The symmetric group on I is written $\mathfrak{S}(I)$.

Lemma 2.1. *We have*

$$\mathrm{per}_{\beta_1+\cdots+\beta_m} A = \sum \prod_{j=1}^{m} \mathrm{per}_{\beta_j} A[I_j], \tag{2}$$

the summation being over all ordered partitions $(I_1, \ldots, I_m)$ of $[n]$ into m disjoint (possibly empty) subsets.

Proof. For any permutation π, let us write $\Pi(\pi) = \{C_1, \dots, C_{\nu(\pi)}\}$ for the unordered partition given by the cycles of π (these are non-empty subsets of $[n]$). Then

$$\begin{aligned}
\mathrm{per}_{\sum \beta_j} A &= \sum_\pi \Big(\sum \beta_j\Big)^{\nu(\pi)} \prod_i a_{i,\pi(i)} = \\
&= \sum_\pi \prod_{C\in\Pi(\pi)} \left(\sum \beta_j \prod_{i\in C} a_{i,\pi(i)}\right) = \\
&= \sum_\pi \sum_{f:\Pi(\pi)\to[m]} \prod_{C\in\Pi(\pi)} \left(\beta_{f(C)} \prod_{i\in C} a_{i,\pi(i)}\right) = \\
&= \sum_{f:[n]\to[m]} \sum_{\substack{\pi \\ f\circ\pi=f}} \prod_{C\in\Pi(\pi)} \left(\beta_{f(C)} \prod_{i\in C} a_{i,\pi(i)}\right) = \\
&= \sum_{(I_1,\dots,I_m)} \sum_{\pi_1\in\mathfrak{S}(I_1)} \cdots \sum_{\pi_m\in\mathfrak{S}(I_m)} \prod_{j=1}^m \left(\beta_j^{\nu(\pi_j)} \prod_{i\in I_j} a_{i,\pi(i)}\right) = \\
&= \sum \prod_{j=1}^m \mathrm{per}_{\beta_j} A[I_j].
\end{aligned}$$

□

We shall now apply the lemma to the case where $\beta_1 = \cdots = \beta_m$. It will be convenient to get rid of the empty subsets appearing in the partitions.

Let us define

(3) $$\mathrm{per}_\beta(A,k) = \sum_{(I_1,\dots,I_k)} \prod_{j=1}^k \mathrm{per}_\beta A[I_j],$$

the summation being over all ordered partitions $(I_1, \dots, I_k)$ of $[n]$ into k disjoint, *nonempty* subsets. We abbreviate per_1 to per and per_{-1} to $(-1)^n \det$.

As usual, we define $\binom{\alpha}{k} = \alpha(\alpha-1)\cdots(\alpha-k+1)/k!$.

Theorem 2.2. *For any numbers α and β, and any $n \times n$ matrix A, we have*

(4) $$\mathrm{per}_{\alpha\beta} A = \sum_{k=1}^n \binom{\alpha}{k} \mathrm{per}_\beta(A,k).$$

In particular,

(5) $$\mathrm{per}_\alpha A = \sum_{k=1}^n \binom{\alpha}{k} \mathrm{per}(A,k)$$

and

(6) $$\mathrm{per}_{-\alpha} A = (-1)^n \sum_{k=1}^n \binom{\alpha}{k} \det(A,k).$$

Also, if A is real and symmetric,

$$\mathrm{per}_{\alpha/2}A =$$

(7) $$= \frac{1}{2^n}\sum_{k=1}^{n}\binom{\alpha}{k}\sum\left\{\prod_{j=1}^{k}\mathrm{haf}\begin{pmatrix}A[I_j] & A[I_j]\\ A[I_j] & A[I_j]\end{pmatrix}\middle|\coprod_{j=1}^{k}I_j=[n],\forall I_j\neq\emptyset\right\}.$$

Proof. Both sides are polynomials in α, so we may assume that $\alpha = m$ is a nonnegative integer. By Lemma 2.1, we have

$$\mathrm{per}_{\alpha\beta}A = \mathrm{per}_{\beta+\cdots+\beta}A = \sum\prod_{j=1}^{\alpha}\mathrm{per}_{\beta}A[I_j],$$

where we are summing over ordered partitions with empty subsets allowed. However, the β–permanent of the empty matrix is 1. Thus we may restrict ourselves to ordered partitions with nonempty subsets, and such a partition, if it has k parts, will be obtained $\binom{\alpha}{k}$ times. Hence the result. □

3. Inequalities for positive semi-definite matrices

Throughout this section, A will be a positive semi-definite Hermitian $n\times n$ matrix.

Shirai and Takahashi [Sh, ShT] have conjectured that $\mathrm{per}_{\alpha}A\geq 0$ if $\alpha\geq 1$, and that $\mathrm{per}_{\alpha/2}A\geq 0$ if $\alpha\geq 1$ and A is real. Shirai [Sh] proves that $\mathrm{per}_{\alpha}A\geq 0$ if α is a nonnegative integer or $\alpha\geq \mathrm{rank}(A)-1$, proves that $\mathrm{per}_{\alpha/2}A\geq 0$ for real A if α is a nonnegative integer or $\alpha\geq n-1$, and proves also that $(-1)^n\mathrm{per}_{-\alpha}A\geq 0$ if α is a nonnegative integer.

The question of nonnegativity is motivated by problems from probability theory. See [EK, Sh, ShT, V1, V2] and references therein. Note that [EK, Sh, ShT] formulate everything in the terms of the α–determinant

$$\det_{\alpha}A = \alpha^n\mathrm{per}_{1/\alpha}A$$

rather than the α–permanent used by Vere-Jones and the present paper.

We shall now strengthen some of Shirai's nonnegativity results to obtain Lieb type inequalities when α is an integer or $\alpha\geq n-1$. Also, we verify that $\mathrm{per}_{\alpha}A\geq 0$ if $\alpha\geq 1$ and $n\leq 5$. Sadly, the conjectures of Shirai and Takahashi remain open in general. Nevertheless, we propose a stronger conjecture.

Suppose that the p.s.d.H. matrix A is partitioned as

(8) $$A = \begin{pmatrix}A' & B\\ B^* & A''\end{pmatrix}.$$

Put

(9) $$D = \begin{pmatrix}A' & 0\\ 0 & A''\end{pmatrix}.$$

Recall Lieb's inequality [L, D, Mi]

(10) $$\mathrm{per}A\geq\mathrm{per}D = \mathrm{per}A'\cdot\mathrm{per}A''$$

and the classical Fischer inequality

(11) $$\det A\leq\det D = \det A'\cdot\det A''.$$

We immediately deduce

$$\text{per}(A,k) \geq \text{per}(D,k) \tag{12}$$

and

$$\det(A,k) \leq \det(D,k). \tag{13}$$

When A is real, recall from [F] the the inequality

$$\text{haf}\begin{pmatrix} A & A \\ A & A \end{pmatrix} \geq \text{per}A. \tag{14}$$

We deduce

$$\sum \left\{ \prod_{j=1}^{k} \text{haf}\begin{pmatrix} A[I_j] & A[I_j] \\ A[I_j] & A[I_j] \end{pmatrix} \middle| \coprod_{j=1}^{k} I_j = [n], \forall I_j \neq \emptyset \right\} \geq \text{per}(A,k). \tag{15}$$

Theorem 3.1. *Suppose that α is either a nonnegative integer or a real number with $\alpha \geq n-1$. Then, for $A \in M_n(\mathbb{C})$ p.s.d.H. partitioned as in (8), we have*

$$\text{per}_\alpha A \geq \text{per}_\alpha D = \text{per}_\alpha A' \cdot \text{per}_\alpha A'' \tag{16}$$

and

$$0 \leq (-1)^n \text{per}_{-\alpha} A \leq (-1)^n \text{per}_{-\alpha} D = (-1)^n \text{per}_{-\alpha} A' \cdot \text{per}_{-\alpha} A''. \tag{17}$$

If, in addition, A is real, then

$$\text{per}_{\alpha/2} A \geq \frac{1}{2^n} \text{per}_\alpha A. \tag{18}$$

Proof. The assumption on α ensures that $\binom{\alpha}{k} \geq 0$ for $k = 1, \ldots, n$. Thus, by (5) and (12),

$$\begin{aligned} \text{per}_\alpha A = \sum_{k=1}^{n} \binom{\alpha}{k} \text{per}(A,k) \geq \\ \geq \sum_{k=1}^{n} \binom{\alpha}{k} \text{per}(D,k) = \text{per}_\alpha D = \text{per}_\alpha A' \cdot \text{per}_\alpha A''. \end{aligned}$$

Similarly, by (6) and (13),

$$0 \leq (-1)^n \text{per}_{-\alpha} A = \sum_{k=1}^{n} \binom{\alpha}{k} \det(A,k) \leq$$

$$\leq \sum_{k=1}^{n} \binom{\alpha}{k} \det(D,k) = (-1)^n \text{per}_{-\alpha} D = (-1)^n \text{per}_{-\alpha} A' \cdot \text{per}_{-\alpha} A''.$$

Finally, if A is real, then by (7) and (15),

$$\mathrm{per}_{\alpha/2}A =$$
$$= \frac{1}{2^n}\sum_{k=1}^{n}\binom{\alpha}{k}\sum\left\{\prod_{j=1}^{k}\mathrm{haf}\begin{pmatrix}A[I_j] & A[I_j]\\ A[I_j] & A[I_j]\end{pmatrix}\middle|\coprod_{j=1}^{k}I_j=[n],\forall I_j\neq\emptyset\right\}\geq$$
$$\geq \frac{1}{2^n}\sum_{k=1}^{n}\binom{\alpha}{k}\mathrm{per}(A,k)=\frac{1}{2^n}\mathrm{per}_\alpha A.$$

□

Corollary 3.2. *Suppose that α is a nonnegative integer or $\alpha \geq n-1$. Then, for $A \in M_n(\mathbb{C})$ p.s.d.H., we have*

$$\mathrm{per}_\alpha A \geq \alpha^n \prod_{i=1}^{n} a_{ii} \geq (-1)^n \mathrm{per}_{-\alpha} A. \tag{19}$$

If, in addition, A is real, then

$$\mathrm{per}_{\alpha/2} A \geq (\alpha/2)^n \prod_{i=1}^{n} a_{ii}. \tag{20}$$

Proof. Obvious induction to prove (19), then (18) and (19) to prove (20). □

Conjecture 3.3. *The condition on α can be relaxed to $\alpha \geq 1$ for all inequalities stated in Theorem 3.1 and its Corollary, except for the leftmost inequality in* (17)*.*

To support the conjecture, we prove the inequalities (19) of the above corollary for small matrices under the relaxed condition for α.

Theorem 3.4. *Suppose that $\alpha \geq 1$ and $n \leq 5$. Then, for $A \in M_n(\mathbb{C})$ p.s.d.H., the inequalities* (19) *hold.*

Proof. We may assume that $a_{ii} = 1$ for all i.

If the Theorem is true for a number α, then by Lemma 2.1, it is also true for $\alpha+1$. We may therefore assume that $1 \leq \alpha \leq 2$.

We may assume that $n = 5$ since the statement for A is equivalent to that for $A \oplus \mathbf{1}_{5-n}$.

From now on $\mathbf{1}$ is the 5×5 identity matrix.

The statement to be proven is $\mathrm{per}_{\pm\alpha} A \geq \mathrm{per}_{\pm\alpha}\mathbf{1}$.

In view of formula (4), it suffices to prove that

$$\sum_{k=1}^{3}\binom{\alpha}{k}\mathrm{per}_{\pm 1}(A,k) \geq \sum_{k=1}^{3}\binom{\alpha}{k}\mathrm{per}_{\pm 1}(\mathbf{1},k) \tag{21}$$

and

$$\sum_{k=4}^{5}\binom{\alpha}{k}\mathrm{per}_{\pm 1}(A,k) \geq \sum_{k=4}^{5}\binom{\alpha}{k}\mathrm{per}_{\pm 1}(\mathbf{1},k). \tag{22}$$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of 5 into k positive integer parts, we define $p_\pm(\lambda)$ to be the average value of

$$\prod_{j=1}^{k} \mathrm{per}_{\pm 1} A[I_j],$$

where we are averaging over the partitions $(I_1, \ldots, I_k)$ of [5] such that $|I_j| = \lambda_j$ for $j = 1, \ldots, k$. From the Lieb and Fischer inequalities, we get $p_\pm(\lambda) \geq p_\pm(\mu)$ if λ arises from μ by replacing two parts of μ by their sum.

Then (22) reduces to

$$10p_\pm(2,1,1,1) + (\alpha - 4)p_\pm(1,1,1,1,1) \geq \pm(10 + \alpha - 4),$$

which is true because

$$p_\pm(2,1,1,1) \geq p_\pm(1,1,1,1,1) = \pm 1.$$

Also, (21) reduces to

$$\begin{aligned} & p_\pm(5) + (\alpha-1)(5p_\pm(4,1) + 10p_\pm(3,2)) + \\ & +(\alpha-1)(\alpha-2)(10p_\pm(3,1,1) + 15p_\pm(2,2,1)) \geq \\ & \geq \pm(1 + (\alpha-1)(5+10) + (\alpha-1)(\alpha-2)(10+15)). \end{aligned} \tag{23}$$

By the Lieb and Fischer inequalities, we have

$$p_\pm(2,2,1) \leq p_\pm(3,2)$$

and

$$p_\pm(3,1,1) \leq \min(p_\pm(4,1), p_\pm(3,2)) \leq \frac{5}{6}p_\pm(4,1) + \frac{1}{6}p_\pm(3,2).$$

Thus, the LHS of (23) is at least

$$p_\pm(5) + 5(\alpha-1)\left(1 + \frac{5}{3}(\alpha-2)\right)(p_\pm(4,1) + 2p_\pm(3,2)). \tag{24}$$

Now

$$p_\pm(5) \geq \max(p_\pm(4,1), p_\pm(3,2)) \geq \frac{1}{3}(p_\pm(4,1) + 2p_\pm(3,2)),$$

so (24) is at least

$$\left(\frac{1}{3} + 5(\alpha-1)\left(1 + \frac{5}{3}(\alpha-2)\right)\right)(p_\pm(4,1) + 2p_\pm(3,2)).$$

Here the first factor is non-negative and the last factor is at least ± 3, whence the result. □

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References

[B] A. Barvinok, *Integration and optimization of multivariate polynomials by restriction onto a random subspace.* Found. Comput. Math. 7 (2007) no 2., 229–244.

[D] D. Ž. Đoković, *Simple proof of a theorem on permanents*, Glasgow Math. J. 10 (1969), 52–54.

[EK] N. Eisenbaum, H. Kaspi, *On permanental processes.* Stochastic Process. Appl. 119 (2009), no. 5, 1401–1415.

[F] P. E. Frenkel, *Pfaffians, Hafnians and products of real linear functionals.* Math. Res. Lett. 15 (2008), no. 2, 351–358.

[G] L. Gurvits, *Classical complexity and quantum entanglement.* J. Comput. System Sci. 69 (2004), no. 3, 448–484.

[L] E. H. Lieb, *Proofs of some conjectures on permanents,* J. Math. Mech. 16 (1966), 127–134.

[LR] I. L. Lu and D. P. Richards, *MacMahon's master theorem, representation theory, and moments of Wishart distributions,* Adv. Appl. Math. 27 (2001), 531–547.

[Mi] H. Minc, Permanents, *Encyclopedia of Mathematics and its Applications,* Addison-Wesley, 1978

[R] G.-C. Rota, *The number of partitions of a set,* Amer. Math. Monthly 71, no. 5 (May 1964), 498–504.

[Sh] T. Shirai, *Remarks on the positivity of alpha-determinants,* Kyushu J. Math. 61 (1) (2007), 169–189.

[ShT] T. Shirai and Y. Takahashi, *Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and boson point processes,* J. Funct. Anal. 205 (2003), 414–463.

[S] B. Simon, The $P(\phi)_2$ *Euclidean (Quantum) Field Theory,* Princeton Series in Physics, Princeton University Press, 1974

[V1] D. Vere-Jones, *A generalization of permanents and determinants,* Linear Alg. Appl. 111 (1998), 119–124.

[V2] D. Vere-Jones, *Alpha-permanents and their applications to multivariate Gamma, negative binomial and ordinary binomial distribution.* New Zealand J. of Math. 26 (1997), 125–149.

[Z] A. Zvonkin, *Matrix integrals and map enumeration: an accesible introduction, Combinatorics and physics* (Marseille, 1995), Math. Comput. Modelling 26 (1997), 281–304.

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, 1211 Genève 4
E-mail address: `frenkelp@renyi.hu`