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CM CYCLES AND NONVANISHING OF CLASS GROUP L–FUNCTIONS

Riad Masri

Abstract. Let K be a totally imaginary quadratic extension of a totally real number field F, and assume that F has narrow ideal class number 1. Let χ be a character of the ideal class group $\mathrm{CL}(K)$ of K, and let $L_K(\chi, s)$ be its associated L–function. In this paper we prove that for all $\epsilon > 0$,

$$\#\{\chi \in \mathrm{CL}(K)^\wedge : \ L_K(\chi, \frac{1}{2}) \neq 0\} \gg_{\epsilon, F} d_K^{\frac{1}{100}-\epsilon}$$

as the absolute discriminant $d_K \to \infty$.

1. Introduction and statement of results

It is an important problem in number theory to determine when the central value of an automorphic L–function is nonvanishing. Because it is difficult to determine whether an individual central value is nonvanishing, it is often useful to consider instead a family $\mathcal{F}$ of automorphic forms and study the average of the central values $L(f, \frac{1}{2})$ as f ranges over $\mathcal{F}$. One fruitful method for studying such averages combines period relations of Waldspurger type with the equidistribution of special points on varieties. For example, given an explicit enough period relation for the central value, one can often obtain an exact formula for the average as the sum of a fixed automorphic function evaluated as special points on the variety. If this automorphic function has nice properties (e.g. is smooth and compactly supported, or nonnegative), one can use the equidistribution of these special points to obtain an asymptotic formula or lower bound for the average as $\#\mathcal{F} \to \infty$. Such results can then be used to obtain information about the nonvanishing of central values in the family (see e.g. [C, Va1, Va2, MV, M, KMY, MY]).

In this paper, we will use a similar method to study the nonvanishing of L–functions associated to class group characters of CM number fields. Let K be a totally imaginary quadratic extension of a totally real number field F of degree $[F : \mathbb{Q}] = n$ over $\mathbb{Q}$. Such an extension K/F is called a *CM extension.* Let $\chi : \mathrm{CL}(K) \to \mathbb{C}^\times$ be a character of the ideal class group $\mathrm{CL}(K)$ of K. Then the class group L–function of χ is defined by

$$L_K(\chi, s) = {\sum_{\mathfrak{a}}}' \chi(\mathfrak{a}) N_{K/\mathbb{Q}}(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > 1$$

where the prime means the summation is taken over all nonzero integral ideals $\mathfrak{a}$ of K. There are h_K such L–functions, where h_K is the ideal class number of K.

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Let d_K be the absolute discriminant of K. The completed L–function

$$\Lambda_K(\chi, s) = \left(\frac{\sqrt{d_K}}{(2\pi)^n}\right)^s \Gamma(s)^n L_K(\chi, s)$$

has an analytic continuation to $\mathbb{C}$. It is entire if χ is nontrivial, and is meromorphic with a simple pole at $s = 1$ if χ is trivial. The L–function $\Lambda_K(\chi, s)$ satisfies the functional equation

$$\Lambda_K(\chi, s) = \Lambda_K(\chi, 1 - s),$$

and the number $L_K(\chi, \frac{1}{2})$ is the central value.

The main result of this paper is the following nonvanishing theorem for the central values $L_K(\chi, \frac{1}{2})$ as χ ranges over $\mathrm{CL}(K)^\wedge$ and $d_K \to \infty$.

Theorem 1.1. *Let K be a CM extension of a totally real number field F, and assume that F has narrow ideal class number 1. Then for all $\epsilon > 0$,*

$$\#\{\chi \in \mathrm{CL}(K)^\wedge : \ L_K(\chi, \frac{1}{2}) \neq 0\} \gg_{\epsilon, F} d_K^{\frac{1}{100} - \epsilon}$$

as $d_K \to \infty$. The implied constant is ineffective.

Remark 1.2. The exponent $\frac{1}{100} - \epsilon$ comes from a subconvexity bound of Venkatesh [V] for the L–function $L_K(\chi, s)$. The Lindelöf hypothesis would allow one to replace this exponent by $\frac{1}{2} - \epsilon$.

When K is an imaginary quadratic field, the central values of class group L–functions have been studied extensively. In [DFI], Duke, Friedlander, and Iwaniec used spectral methods to obtain an asymptotic formula for the second moment of the central values $L_K(\chi, \frac{1}{2})$ with a power savings in the error term, and used amplification to obtain a subconvexity bound for $L_K(\chi, \frac{1}{2})$ which improved the exponent in Burgess's subconvexity bound when χ is a real character.

In [B], Blomer used mollification along with a refinement of estimates in [DFI] to prove that for K imaginary quadratic,

$$\#\{\chi \in \mathrm{CL}(K)^\wedge : \ L_K(\chi, \frac{1}{2}) \neq 0\} \geq c \cdot h_K \prod_{p | d_K} \left(1 - \frac{1}{p}\right)$$

for some effective constant $c > 0$ and all sufficiently large d_K. Here one does not know how large d_K must be chosen for this lower bound to be valid because of an application of Siegel's theorem in the proof. If we use Siegel's lower bound for $L_K(\chi, 1)$, the bound of Blomer is (ineffectively) larger than $d_K^{\frac{1}{2} - \epsilon}$.

While many methods work quite well to obtain strong nonvanishing theorems when the base field is $\mathbb{Q}$, it is often very difficult to extend these methods to more general number fields. Here we will circumvent these difficulties by using an approach which combines period relations with the equidistribution of special points on varieties. To briefly describe this, fix a CM type Φ of K, and let $\mathcal{O}_F$ be the ring of integers of F. We will establish an exact formula for the mean square of the central values $L_K(\chi, \frac{1}{2})$ of the form

$$\frac{1}{h_K} \sum_{\chi \in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 = c_{K,F} \sum_{z \in \mathcal{CM}(K, \Phi, \mathcal{O}_F)} |\mathcal{E}'(z, \frac{1}{2})|^2 \tag{1.1}$$

where $c_{K,F} > 0$ is an explicit constant, $\mathcal{E}'(z, \frac{1}{2})$ is the central derivative of the weight zero, real-analytic Hilbert modular Eisenstein series for $\mathrm{SL}_2(\mathcal{O}_F)$, and $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ is the zero cycle of CM points on the open Hilbert modular variety $X_F = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ (see section 3). By work of Zhang [Z], which was made unconditional by subconvexity bounds of Venkatesh [V], it is known that the CM points $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ are equidistributed on X_F as $d_K \to \infty$. Because the function

$$f(z) := |\mathcal{E}'(z, \frac{1}{2})|^2 : \mathbb{H}^n \to \mathbb{R}_{\geq 0}$$

does not have rapid decay in the cusp of X_F, we cannot apply equidistribution directly to (1.1) to obtain an asymptotic formula for the mean square as $d_K \to \infty$. However, because $f(z)$ is nonnegative, we can multiply by a suitable smooth, compactly supported cutoff function and use equidistribution to obtain a lower bound of the form

$$\sum_{\chi \in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 \gg_{F,\epsilon} d_K^{\frac{1}{2}-\epsilon} \tag{1.2}$$

for any $\epsilon > 0$ and all sufficiently large d_K. The implied constant is ineffective due to an application of the Brauer-Siegel theorem. Finally, to obtain Theorem 1.1 we will combine (1.2) with the following subconvexity bound due to Venkatesh [V],

$$L_K(\chi, \frac{1}{2}) \ll_F d_K^{\frac{1}{4}-\frac{1}{200}}.$$

2. Hilbert modular Eisenstein series

Let F be a totally real number field of degree n with ring of integers $\mathcal{O}_F$ and real embeddings $\{\sigma_1, \ldots, \sigma_n\}$. We assume that F has ideal class number 1. Let $\mathbb{H}$ be the complex upper half-plane, and let $z = x + iy = (z_1, \ldots, z_n) \in \mathbb{H}^n$. Then $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathbb{H}^n$ by linear fractional transformation in each component,

$$\gamma \cdot z = (\sigma_1(\gamma)z_1, \ldots, \sigma_n(\gamma)z_n) \in \mathbb{H}^n.$$

For notational convenience, we let

$$N(y(z)) = \prod_{j=1}^{n} \mathrm{Im}(z_j) = \prod_{j=1}^{n} y_j$$

be the product of the imaginary products of the components of $z \in \mathbb{H}^n$. Define the real-analytic Hilbert modular Eisenstein series by

$$\mathcal{E}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathcal{O}_F)} N(y(\gamma z))^s, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \right\}.$$

Furthermore, we let

$$N(a+bz)=\prod_{j=1}^{n}(\sigma_j(a)+\sigma_j(b)z_j)$$

for $(a,b)\in\mathcal{O}_F\times\mathcal{O}_F$. Define the (related) real-analytic Hilbert modular Eisenstein series by

$$E(z,s)=\sum_{(a,b)\in\mathcal{O}_F\times\mathcal{O}_F/\mathcal{O}_F^\times}{}' \frac{N(y(z))^s}{|N(a+bz)|^{2s}},\quad z\in\mathbb{H}^n,\quad \mathrm{Re}(s)>1$$

where the sum is over a complete set of nonzero, nonassociated representatives of $\mathcal{O}_F\times\mathcal{O}_F$. Here we recall that (a,b) and (a',b') are *associated* if there exists a unit $\epsilon\in\mathcal{O}_F^\times$ such that $(a,b)=(\epsilon a',\epsilon b')$. The two Eisenstein series are related by

$$E(z,s)=\zeta_F(2s)\mathcal{E}(z,s) \tag{2.1}$$

where $\zeta_F(s)$ is the Dedekind zeta function of F.

The Eisenstein series $\mathcal{E}(z,s)$ has the Fourier expansion

$$\begin{aligned}\mathcal{E}(z,s)&=N(y(z))^s+\frac{1}{\sqrt{d_F}}\left[\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right]^n\frac{\zeta_F(2s-1)}{\zeta_F(2s)}N(y(z))^{1-s}\\ &+\frac{2^n}{\sqrt{d_F}}\left[\frac{\pi^s}{\Gamma(s)}\right]^n\frac{\sqrt{N(y(z))}}{\zeta_F(2s)}\quad\times\\ &\qquad\sum_{\tilde{a}\in\mathcal{O}_F^*}{}'\sum_{\substack{\tilde{a}=ab\\ a\in\mathcal{O}_F^*\\ b\in\mathcal{O}_F/\mathcal{O}_F^\times}}\left(\frac{N_{F/\mathbb{Q}}((a))}{N_{F/\mathbb{Q}}((b))}\right)^{s-\frac{1}{2}}\prod_{j=1}^{n}K_{s-\frac{1}{2}}(2\pi|\sigma_j(ab)|\,y_j)e^{2\pi iT(abx)}\end{aligned}$$

where $\mathcal{O}_F^*$ is the dual lattice,

$$T(ax)=\sum_{j=1}^{n}\sigma_j(a)x_j$$

is the trace, and $K_s(v)$ is the K-Bessel function.

From the Fourier expansion we see that $\mathcal{E}(z,s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with residue

$$\mathrm{Res}_{s=1}\mathcal{E}(z,s)=\frac{2^{n-1}\pi^nR_F}{\zeta_F(2)w_Fd_F}$$

where R_F is the regulator and w_F is the number of roots of unity.

Define the gamma factor

$$G(s)=d_F^{s/2}\left[\pi^{-s/2}\Gamma(\frac{s}{2})\right]^n$$

and the completed Eisenstein series

$$\mathcal{E}^*(z,s)=G(2s)\zeta_F(2s)\mathcal{E}(z,s).$$

Then $\mathcal{E}^*(z,s)$ satisfies the functional equation

$$\mathcal{E}^*(z,s)=\mathcal{E}^*(z,1-s).$$

In particular, we have that

$$\mathcal{E}(z,s) = \frac{G(2(1-s))\zeta_F(2(1-s))}{G(2s)\zeta_F(2s)}\mathcal{E}(z,1-s).$$

One can show by a calculation with Laurent expansions that

$$\lim_{s\to\frac{1}{2}} \frac{\zeta_F(2(1-s))}{\zeta_F(2s)} = -1.$$

Since $G(s)$ is holomorphic at $s=1$, this implies that

$$\mathcal{E}(z,\frac{1}{2}) = -\mathcal{E}(z,\frac{1}{2}),$$

and hence

$$\mathcal{E}(z,\frac{1}{2}) = 0$$

for all $z \in \mathbb{H}^n$. That is, $\mathcal{E}(z,\frac{1}{2})$ is identically zero.

It is important to observe that from the Fourier expansion for $\mathcal{E}(z,s)$, it is clear that the function

$$\mathcal{E}'(z,\frac{1}{2}) = \frac{\partial}{\partial s}\mathcal{E}(z,s)_{|_{s=\frac{1}{2}}}$$

does *not* have rapid decay in the cusp of X_F as $N(y(z)) \to \infty$.

3. CM zero cycles on Hilbert modular varieties

In this section we review some facts concerning CM points on Hilbert modular varieties. We follow closely the discussion in [BY, section 3]. Let F be a totally real number field of degree n. For $S \subset F$, let S^+ be the subset of S consisting of totally positive elements. For a fractional ideal $\mathfrak{f}_0$ of F, let

$$\Gamma(\mathfrak{f}_0) = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{f}_0) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) :\ a,d \in \mathcal{O}_F,\ b \in \mathfrak{f}_0,\ c \in \mathfrak{f}_0^{-1}\}.$$

Recall that $\Gamma(\mathfrak{f}_0)$ acts on $\mathbb{H}^n$ by

$$\gamma \cdot z = (\sigma_1(\gamma)z_1, \dots, \sigma_n(\gamma)z_n).$$

The quotient space

$$X(\mathfrak{f}_0) = \Gamma(\mathfrak{f}_0)\backslash\mathbb{H}^n$$

is the (open) Hilbert modular variety associated to $\mathfrak{f}_0$. The variety $X(\mathfrak{f}_0)$ parameterizes isomorphism classes of triples (A,i,m) where (A,i) is an abelian variety with real multiplication $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ and

$$m : \big(\mathfrak{M}_A, \mathfrak{M}_A^+\big) \to \big((\partial_F\mathfrak{f}_0)^{-1}, (\partial_F\mathfrak{f}_0)^{-1,+}\big)$$

is an $\mathcal{O}_F$-isomorphism from $\mathfrak{M}_A$ to $(\partial_F\mathfrak{f}_0)^{-1}$ which maps $\mathfrak{M}_A^+$ to $(\partial_F\mathfrak{f}_0)^{-1,+}$. Here $\mathfrak{M}_A$ is the polarization module of A and $\mathfrak{M}_A^+$ is its positive cone.

Let K be a CM extension of F and let $\Phi = (\sigma_1, \dots, \sigma_n)$ be a CM type of K. A point $z = (A,i,m) \in X(\mathfrak{f}_0)$ is a CM point of type (K,Φ) if one of the following equivalent definitions holds:

(1) As a point $z \in \mathbb{H}^n$, there is a point $\tau \in K$ such that

$$\Phi(\tau) = (\sigma_1(\tau), \ldots, \sigma_n(\tau)) = z$$

and

$$\Lambda_\tau = \mathfrak{f}_0 + \mathcal{O}_F \tau$$

is a fractional ideal of K.

(2) (A, i') is a CM abelian variety of type (K, Φ) with complex multiplication $i : \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$ such that $i = i'_{|\mathcal{O}_F}$.

Fix $\varepsilon_0 \in K^\times$ such that $\overline{\varepsilon_0} = -\varepsilon_0$ and $\Phi(\varepsilon_0) = (\sigma_1(\varepsilon_0), \ldots, \sigma_n(\varepsilon_0)) \in \mathbb{H}^n$. Let $\mathfrak{a}$ be a fractional ideal of K and $\mathfrak{f}_\mathfrak{a} = \varepsilon_0 \partial_{K/F} \mathfrak{a}\bar{\mathfrak{a}} \cap F$. By [BY, Lemma 3.1], the CM abelian variety $(A_\mathfrak{a} = \mathbb{C}^n/\Phi(\mathfrak{a}), i)$ defines a CM point on $X(\mathfrak{f}_0)$ if there exists an $r \in F^\times$ such that $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$. Thus any pair $(\mathfrak{a}, r)$ with $\mathfrak{a}$ a fractional ideal of K and $r \in F^\times$ with $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$ defines a CM point $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$ (we refer the reader to [BY] for a discussion of how the $\mathcal{O}_F$-isomorphism m depends on r). Two such pairs $(\mathfrak{a}_1, r_1)$ and $(\mathfrak{a}_2, r_2)$ are equivalent if there exists an $\alpha \in K^\times$ such that $\mathfrak{a}_2 = \alpha\mathfrak{a}_1$ and $r_2 = r_1\alpha\bar{\alpha}$. Write $[\mathfrak{a}, r]$ for the class of $(\mathfrak{a}, r)$ and identify it with its associated CM point $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$.

By [BY, Lemma 3.2], given a CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$, there is a decomposition

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathfrak{f}_0 \beta$$

with $z = \alpha/\beta \in K^\times \cap \mathbb{H}^n = \{z \in K^\times : \ \Phi(z) \in \mathbb{H}^n\}$. Moreover, z represents the CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$.

Let $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$ be the set of CM points $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$ which we view as a CM 0-cycle in $X(\mathfrak{f}_0)$. Let

$$\mathcal{CM}(K, \Phi) = \sum_{[\mathfrak{f}_0] \in \mathrm{CL}(F)^+} \mathcal{CM}(K, \Phi, \mathfrak{f}_0)$$

where $\mathrm{CL}(F)^+$ is the narrow ideal class group of F. The forgetful map

$$\mathcal{CM}(K, \Phi) \to \mathrm{CL}(K),$$
$$[\mathfrak{a}, r] \mapsto [\mathfrak{a}]$$

is surjective. Each fiber is indexed by $\epsilon \in \mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^\times$. Here $\#(\mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^\times)$ equals 1 or 2. In particular, it equals 1 if $\epsilon \in N_{K/F}\mathcal{O}_K^\times$.

Assume now that F has narrow ideal class number 1. Then

$$\mathcal{CM}(K, \Phi) = \mathcal{CM}(K, \Phi, \mathcal{O}_F),$$

and the forgetful map

$$\mathcal{CM}(K, \Phi) \to \mathrm{CL}(K)$$

is injective (hence bijective) since $N_{K/F}\mathcal{O}_K^\times = \mathcal{O}_F^\times$. We will repeatedly use this bijection to identify the 0-cycle of CM points $\mathcal{CM}(K, \Phi, \mathcal{O}_F) \subset X_F$ with

$$\{z_\mathfrak{a} \in K^\times \cap \mathbb{H}^n : [\mathfrak{a}] \in \mathrm{CL}(K)\}$$

where $z_\mathfrak{a}$ represents $[\mathfrak{a}, r] \in X_F$ as above. The reader should keep in mind that this latter set depends on Φ.

4. Periods of Eisenstein series

In the following proposition we express the class group L–function $L_K(\chi, s)$ as a twisted period of the Eisenstein series $\mathcal{E}(z, s)$ with respect to the CM 0-cycle $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$.

Proposition 4.1. *Let K be a CM extension of a totally real number field F of degree n, and assume that F has narrow ideal class number 1. Let Φ be a CM type of K. Then*

$$L_K(\chi, s) = \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{\zeta_F(2s)}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{z \in \mathcal{CM}(K,\Phi,\mathcal{O}_F)} \chi(z)\mathcal{E}(z, s).$$

Here $\chi(z)$ means $\chi([\mathfrak{a}])$, where $[\mathfrak{a}]$ is the ideal class of $\mathrm{CL}(K)$ *corresponding to z via the bijection $\mathcal{CM}(K, \Phi, \mathcal{O}_F) \to \mathrm{CL}(K)$.*

Proof. Let $C \in \mathrm{CL}(K)$ and fix an integral ideal $\mathfrak{a} \in C^{-1}$. As $\mathfrak{b}$ runs over integral ideals in C, $\mathfrak{a}\mathfrak{b} = (\omega)$ runs over principal ideals (ω) with $(\omega) \equiv 0 \mod \mathfrak{a}$. Then the partial Dedekind zeta function equals

$$\begin{aligned}\zeta_K(s, C) &= {\sum_{\mathfrak{b}\in C}}' N_{K/\mathbb{Q}}(\mathfrak{b})^{-s}\\ &= {\sum_{(\omega)\subset\mathfrak{a}}}' N_{K/\mathbb{Q}}(\mathfrak{a}^{-1}(\omega))^{-s}\\ &= N_{K/\mathbb{Q}}(\mathfrak{a})^s {\sum_{\omega\in\mathfrak{a}/\mathcal{O}_K^\times}}' N_{K/\mathbb{Q}}((\omega))^{-s}.\end{aligned}$$

Notice that

$${\sum_{\omega\in\mathfrak{a}/\mathcal{O}_K^\times}}' N_{K/\mathbb{Q}}((\omega))^{-s} = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} {\sum_{\omega\in\mathfrak{a}/\mathcal{O}_F^\times}}' N_{K/\mathbb{Q}}((\omega))^{-s}.$$

Thus we have

$$\zeta_K(s, C) = \frac{N_{K/\mathbb{Q}}(\mathfrak{a})^s}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} {\sum_{\omega\in\mathfrak{a}/\mathcal{O}_F^\times}}' N_{K/\mathbb{Q}}((\omega))^{-s}.$$

In section 3 we showed there exists a decomposition

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta$$

where $z_\mathfrak{a} = \beta/\alpha \in K^\times \cap \mathbb{H}^n$ and $z_\mathfrak{a}$ represents the CM point $[\mathfrak{a}, r] \in X_F$ (here $\mathfrak{f}_0 = \mathcal{O}_F$ since $\#\mathrm{CL}(F)^+ = 1$). Then

$$\begin{aligned}{\sum_{\omega\in\mathfrak{a}/\mathcal{O}_F^\times}}' N_{K/\mathbb{Q}}((\omega))^{-s} &= {\sum_{(a,b)\in\mathcal{O}_F\times\mathcal{O}_F/\mathcal{O}_F^\times}}' N_{K/\mathbb{Q}}((a\alpha + b\beta))^{-s}\\ &= N_{K/\mathbb{Q}}((\alpha))^{-s} {\sum_{(a,b)\in\mathcal{O}_F\times\mathcal{O}_F/\mathcal{O}_F^\times}}' N_{K/\mathbb{Q}}((a + bz_\mathfrak{a})).\end{aligned}$$

By a calculation with the CM type Φ we obtain

$$N_{K/\mathbb{Q}}((a + bz_\mathfrak{a})) = |N\,(a + bz_\mathfrak{a})|^2$$

where we have identified $z_{\mathfrak{a}}$ with $\Phi(z_{\mathfrak{a}}) \in \mathbb{H}^n$.

Furthermore, a calculation with determinants yields

$$N_{K/\mathbb{Q}}(\mathfrak{a}/(\alpha)) = N(y(z_{\mathfrak{a}}))\frac{2^n d_F}{\sqrt{d_K}}.$$

By combining the preceding calculations we obtain

$$\begin{aligned}\zeta_K(s,C) &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{(a,b)\in\mathcal{O}_F\times\mathcal{O}_F/\mathcal{O}_F^\times}{}' \frac{N(y(z_{\mathfrak{a}}))^s}{|N(a+bz_{\mathfrak{a}})|^{2s}} \\ &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} E(z_{\mathfrak{a}},s) \\ &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{\zeta_F(2s)}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \mathcal{E}(z_{\mathfrak{a}},s)\end{aligned}$$

where we have used the definition of $E(z,s)$ and the identity (2.1).

Finally, using that

$$L_K(\chi,s) = \sum_{C\in\mathrm{CL}(K)} \chi(C)\zeta_K(s,C)$$

we obtain

$$L_K(\chi,s) = \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{\zeta_F(2s)}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{[\mathfrak{a}]\in\mathrm{CL}(K)} \chi([\mathfrak{a}])\mathcal{E}(z_{\mathfrak{a}},s).$$

The result follows from the bijection $\mathcal{CM}(K,\Phi,\mathcal{O}_F) \to \mathrm{CL}(K)$. □

5. The mean square of $L_K(\chi,\frac{1}{2})$

In this section we use Proposition 4.1 to establish an exact formula for the mean square of the central values $L_K(\chi,\frac{1}{2})$.

Let r_F denote the residue of the Dedekind zeta function $\zeta_F(s)$ at $s=1$.

Proposition 5.1. *Let K be a CM extension of a totally real number field F of degree n, and assume that F has narrow ideal class number 1. Then*

$$\frac{1}{h_K}\sum_{\chi\in\mathrm{CL}(K)^\wedge} |L_K(\chi,\frac{1}{2})|^2 = \frac{2^n d_F}{4\sqrt{d_K}} \frac{r_F^2}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^2} \sum_{[\mathfrak{a}]\in\mathrm{CL}(K)} |\mathcal{E}'(z_{\mathfrak{a}},\frac{1}{2})|^2.$$

Proof. By the Laurent expansion of $\zeta_F(s)$ at $s=1$,

$$\zeta_F(2s) = \frac{r_F}{2(s-\frac{1}{2})} + a_0 + O(s-\frac{1}{2}),$$

and because $\mathcal{E}(z,s)$ vanishes identically at $s=1/2$ (see section 2),

$$\mathcal{E}(z,s) = \mathcal{E}'(z,\frac{1}{2})(s-\frac{1}{2}) + O(s-\frac{1}{2})^2.$$

Thus we have

$$\zeta_F(2s)\mathcal{E}(z,s) = \frac{r_F}{2}\mathcal{E}'(z,\frac{1}{2}) + O(s-\frac{1}{2}).$$

From Proposition 4.1 with $s = 1/2$ it follows that

$$L_K(\chi, \frac{1}{2}) = \frac{2^{n/2}\sqrt{d_F}}{2d_K^{1/4}} \frac{r_F}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} \chi([\mathfrak{a}])\mathcal{E}'(z_\mathfrak{a}, \frac{1}{2}),$$

from which we obtain

$$\sum_{\chi\in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 = \frac{2^n d_F}{4\sqrt{d_K}} \frac{r_F^2}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^2} \sum_{\chi\in \mathrm{CL}(K)^\wedge} \left| \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} \chi([\mathfrak{a}])\mathcal{E}'(z_\mathfrak{a}, \frac{1}{2})\right|^2.$$

By orthogonality, we have the relations

$$\sum_{\chi\in \mathrm{CL}(K)^\wedge} \chi([\mathfrak{a}])\overline{\chi([\mathfrak{b}])} = \begin{cases} h_K, & [\mathfrak{a}] = [\mathfrak{b}] \\ 0, & [\mathfrak{a}] \neq [\mathfrak{b}]. \end{cases}$$

Thus we obtain the identity

$$\sum_{\chi\in \mathrm{CL}(K)^\wedge} \left| \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} \chi([\mathfrak{a}])\mathcal{E}'(z_\mathfrak{a}, \frac{1}{2})\right|^2 = h_K \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} |\mathcal{E}'(z_\mathfrak{a}, \frac{1}{2})|^2.$$

□

6. Proof of Theorem 1.1

We can express Proposition 5.1 in the equivalent form

$$\sum_{\chi\in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 = \frac{2^n d_F}{4} \frac{r_F^2}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^2} \frac{h_K^2}{\sqrt{d_K}} \frac{1}{h_K} \sum_{[\mathfrak{a}]\in \mathrm{CL}(K)} |\mathcal{E}'(z_\mathfrak{a}, \frac{1}{2})|^2.$$

Define the function

$$f(z) = |\mathcal{E}'(z, \frac{1}{2})|^2.$$

Then $f \in C^\infty(X_F, \mathbb{R}_{\geq 0})$. Let μ_X be the invariant probability measure on X_F.

Lemma 6.1. *There exists a smooth, compactly supported function* $g : X_F \to [0, 1]$ *such that* $\mu_X(\mathrm{supp}(g) \cap \mathrm{supp}(f)) > 0$.

Proof. Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth, compactly supported function. Define the Poincaré series

$$\mathscr{P}(z) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathcal{O}_F)} \psi(N(y(\gamma z))), \quad z \in \mathbb{H}^n.$$

Then $\mathscr{P} \in C_c^\infty(X_F, \mathbb{R})$. Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth function supported on $[1, B]$ for some $B > 1$. Then

$$\phi(\mathscr{P}(z)) : X_F \to [0, 1]$$

is smooth, and $\mathrm{supp}(\phi(\mathscr{P})) \subset \mathrm{supp}(\mathscr{P})$. Hence $g := \phi(\mathscr{P})$ is compactly supported. Finally, it is clear that g can be constructed so that $\mu_X(\mathrm{supp}(g) \cap \mathrm{supp}(f)) > 0$. □

Let $g \in C_c^\infty(X_F, [0,1])$ be as in Lemma 6.1, and define $h := fg$. Because $f \geq 0$, we know that $f(z) \geq h(z)$ for all $z \in X_F$. From this we obtain the lower bound

$$\sum_{\chi \in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 \geq \frac{2^n d_F}{4} \frac{r_F^2}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|^2} \frac{h_K^2}{\sqrt{d_K}} \frac{1}{h_K} \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} h(z_{\mathfrak{a}}).$$

From the bijection

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) \to \mathrm{CL}(K)$$

we see that the set

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} : \ [\mathfrak{a}] \in \mathrm{CL}(K)\}$$

is an adelic toric orbit of CM points on X_F under the action of the adelic torus

$$\mathrm{CL}(K) = K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_K^\times.$$

It follows from work of Zhang [Z] and Venkatesh [V] that the CM points $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ are equidistributed on X_F as $d_K \to \infty$. In fact, this is proved in a quantitative form, which means that given a test function $\phi \in C_c^\infty(X_F)$, there exists an absolute constant $\eta > 0$ such that

$$\frac{1}{h_K} \sum_{[\mathfrak{a}] \in \mathrm{CL}(K)} \phi(z_{\mathfrak{a}}) = \int_{X_F} \phi(z) d\mu_X + O(d_K^{-\eta})$$

as $d_K \to \infty$.

Since $h \in C_c^\infty(X_F)$, we find that

$$\sum_{\chi \in \mathrm{CL}(K)^\wedge} |L_K(\chi, \frac{1}{2})|^2 \geq \frac{2^n d_F}{4} \frac{r_F^2}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|^2} \frac{h_K^2}{\sqrt{d_K}} \left(\int_{X_F} h(z) d\mu_X + o(1) \right)$$

as $d_K \to \infty$. Note that because $h = fg \geq 0$ is continuous and $\mu_X(\mathrm{supp}(g) \cap \mathrm{supp}(f)) > 0$, we have

$$\int_{X_F} h(z) d\mu_X \geq \int_{\mathrm{supp}(g) \cap \mathrm{supp}(f)} h(z) d\mu_X > 0.$$

The residue of $\zeta_K(s)$ at $s = 1$ equals

$$r_K = \frac{(2\pi)^n h_K R_K}{w_K \sqrt{d_K}}.$$

Write $K = F(\sqrt{\Delta})$ for $\Delta \in F^\times$. Since F has narrow ideal class number 1, we can choose Δ such that the relative discriminant $d_{K/F} = \Delta \mathcal{O}_F$. Thus

$$d_K = d_F^2 N_{F/\mathbb{Q}} d_{K/F} = d_F^2 \left| N_{F/\mathbb{Q}}(\Delta) \right|.$$

By [W, Proposition 4.16], one has

$$\frac{R_K}{R_F} = \frac{2^{n-1}}{\left|\mathcal{O}_K^\times : U_K \mathcal{O}_F^\times\right|}$$

where U_K is the group of roots of unity in K. Furthermore, by [W, Theorem 4.12], one has $\left|\mathcal{O}_K^\times : U_K\mathcal{O}_F^\times\right| = 1$ or 2. Thus

$$\begin{aligned} h_K &= \frac{w_K\sqrt{d_K}r_K}{(2\pi)^n(R_K/R_F)R_F} \\ &= \frac{w_K}{(2\pi)^n R_F}\frac{d_F\left|N_{F/\mathbb{Q}}(\Delta)\right|^{1/2}}{2^{n-1}}\left|\mathcal{O}_K^\times : U_K\mathcal{O}_F^\times\right| r_K \\ &\geq \frac{d_F}{(2\pi)^n R_F 2^{n-1}} r_K \left|N_{F/\mathbb{Q}}(\Delta)\right|^{1/2}. \end{aligned}$$

By the Brauer-Siegel theorem (see e.g. [S]), for all $\epsilon > 0$ one has

$$r_K \geq c(\epsilon) d_K^{-\epsilon}$$

where $c(\epsilon) > 0$ is ineffective. Thus

$$r_K \geq c(\epsilon)d_K^{-\epsilon} = c(\epsilon)d_F^{-2\epsilon}\left|N_{F/\mathbb{Q}}(\Delta)\right|^{-\epsilon},$$

which implies that

$$h_K \geq c(\epsilon)\frac{d_F^{1-2\epsilon}}{(2\pi)^n R_F 2^{n-1}}\left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{2}-\epsilon}.$$

Because $\sqrt{d_K} = d_F\left|N_{F/\mathbb{Q}}(\Delta)\right|^{1/2}$ it follows that

$$\frac{h_K^2}{\sqrt{d_K}} \geq \frac{c(\epsilon)^2 d_F^{1-4\epsilon}}{(2\pi)^{2n}2^{2n-2}R_F^2}\left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{2}-2\epsilon}.$$

Since w_K is bounded, one has

$$\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right| \ll_F 1.$$

We now obtain the lower bound

$$\begin{aligned} &\sum_{\chi\in \mathrm{CL}(K)^\wedge} |L_K(\chi,\tfrac{1}{2})|^2 \\ &\geq \frac{c(\epsilon)^2 d_F^{2-4\epsilon} r_F^2}{4(2\pi)^{2n}2^{n-2}R_F^2}\frac{1}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|^2}\left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{2}-2\epsilon}\left(\int_{X_F} h(z)d\mu_X + o(1)\right) \\ &\gg_{F,\epsilon} \left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{2}-2\epsilon} \end{aligned}$$

as $\left|N_{F/\mathbb{Q}}(\Delta)\right| \to \infty$. In particular, this shows that there exists at least one $\chi \in \mathrm{CL}(K)^\wedge$ such that

$$L_K(\chi,\frac{1}{2}) \neq 0$$

as $\left|N_{F/\mathbb{Q}}(\Delta)\right| \to \infty$.

By work of Venkatesh [V, Theorem 6.1], one has the subconvexity bound

$$L_K(\chi,\frac{1}{2}) \ll_F (N_{K/\mathbb{Q}}(\mathrm{cond}(\chi))d_K)^{\frac{1}{4}-\frac{1}{200}}.$$

Since $\mathrm{cond}(\chi) = \mathcal{O}_K$ and $d_K = d_F^2\left|N_{F/\mathbb{Q}}(\Delta)\right|$, one obtains

$$L_K(\chi,\frac{1}{2}) \ll_F \left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{4}-\frac{1}{200}}.$$

It follows that

$$\sum_{\chi\in \mathrm{CL}(K)^\wedge} |L_K(\chi,\frac{1}{2})|^2 \ll_F \left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{2}-\frac{1}{100}} \#\{\chi\in \mathrm{CL}(K)^\wedge:\ L_K(\chi,\frac{1}{2})\neq 0\}.$$

Finally, by combining inequalities, we find that for all $\epsilon > 0$,

$$\#\{\chi\in \mathrm{CL}(K)^\wedge:\ L_K(\chi,\frac{1}{2})\neq 0\} \gg_{F,\epsilon} \left|N_{F/\mathbb{Q}}(\Delta)\right|^{\frac{1}{100}-\epsilon}$$

as $\left|N_{F/\mathbb{Q}}(\Delta)\right| \to \infty$.

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: masri@math.wisc.edu