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FINITE BOUNDS FOR HÖLDER-BRASCAMP-LIEB MULTILINEAR INEQUALITIES

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Abstract. A criterion is established for the validity of multilinear inequalities of a class considered by Brascamp and Lieb, generalizing well-known inequalties of Rogers and Hölder, Young, and Loomis-Whitney.

1. Formulation

Consider multilinear functionals

$$\Lambda(f_1, f_2, \cdots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(\ell_j(y))\, dy \tag{1.1}$$

where each $\ell_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ is a surjective linear transformation, and $f_j : \mathbb{R}^{n_j} \to [0, +\infty]$. Let $p_1, \cdots, p_m \in [1, \infty]$. For which m-tuples of exponents and linear transformations is

$$\sup_{f_1, \cdots, f_m} \frac{\Lambda(f_1, f_2, \cdots, f_m)}{\prod_j \|f_j\|_{L^{p_j}}} < \infty? \tag{1.2}$$

The supremum is taken over all m-tuples of nonnegative Lebesgue measurable functions f_j having positive, finite norms. If $n_j = n$ for every index j then (1.2) is essentially a restatement of Hölder's inequality.[1] Other well-known particular cases include Young's inequality for convolutions and the Loomis-Whitney inequality[2] [15].

In this paper we characterize finiteness of the supremum (1.2) in linear algebraic terms, and discuss certain variants and a generalization. The problem has a long history, including the early work of Rogers [17] and Hölder [12]. In this level of generality, the question was to our knowledge first posed by Brascamp and Lieb [4]. A primitive version of the problem involving Cartesian product rather than linear algebraic structure was posed and solved by Finner [10]; see §7 below. In the case when the dimension n_j of each target space equals one, Barthe [1] characterized (1.2). Carlen, Lieb and Loss [7] gave an alternative characterization, closely related to ours, and an alternative proof for that case. [7] developed an inductive analysis

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[1]For a discussion of the history of Hölder's inequality, including its discovery by Rogers [17], see [16].

[2]Loomis and Whitney considered only the special case where each f_j is the characteristic function of a set.

closely related to that of Finner, whose argument in turn relied on a slicing and induction argument employed earlier by Loomis and Whitney [15] and Calderón [6] to treat special cases. [7] also introduced a version of the key concepts of critical and subcritical subspaces, a higher-dimensional reformulation of which is essential in our work.

An alternative line of analysis exists. Although rearrangement inequalities such as that of Brascamp, Lieb, and Luttinger [5] do not apply when the target spaces have dimensions greater than one, Lieb [14] nonetheless showed that the supremum in (1.2) equals the supremum over all m-tuples of Gaussian functions,[3] meaning those of the form $f_j = \exp(-Q_j(y,y))$ for some positive definite quadratic form Q_j. See [7] and references cited there for more on this approach. In a companion paper [3] we have given other proofs of our characterization of (1.2), by using heat flow to continuously deform arbitrary functions f_j to Gaussians while increasing the ratio in (1.2). That approach extends work of Carlen, Lieb, and Loss [7] via a method which they introduced.

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2. Results

Denote by $\dim(V)$ the dimension of a vector space V, and by $\mathrm{codim}_W(V)$ the codimension of a subspace $V \subset W$ in W. It is convenient to reformulate the problem in a more invariant fashion. Let $H, H_1, \ldots, H_m$ be Hilbert spaces of finite, positive dimensions. Each is equipped with a canonical Lebesgue measure, by choosing orthonormal bases, thus obtaining identifications with $\mathbb{R}^{\dim(H)}$, $\mathbb{R}^{\dim(H_j)}$. Let $\ell_j : H \to H_j$ be surjective linear mappings. Let $f_j : H_j \to \mathbb{R}$ be nonnegative. Then $\Lambda(f_1, \cdots, f_m)$ equals $\int_H \prod_{j=1}^m f_j \circ \ell_j(y)\, dy$.

Theorem 2.1. *For $1 \le j \le m$ let H, H_j be Hilbert spaces of finite, positive dimensions. For each index j let $\ell_j : H \to H_j$ be surjective linear transformations, and let $p_j \in [1, \infty]$. Then (1.2) holds if and only if*

$$\dim(H) = \sum_j p_j^{-1} \dim(H_j) \tag{2.1}$$

and

$$\dim(V) \le \sum_j p_j^{-1} \dim(\ell_j(V)) \quad \textit{for every subspace } V \subset H. \tag{2.2}$$

This equivalence is established by other methods in [3], Theorem 1.15.

Given that (2.1) holds, the hypothesis (2.2) can be equivalently restated as

$$\mathrm{codim}_H(V) \ge \sum_j p_j^{-1}\, \mathrm{codim}_{H_j}(\ell_j(V)) \quad \text{for every subspace } V \subset H; \tag{2.3}$$

[3] This situation should be contrasted with that of multilinear *operators* of the same general form, mapping $\otimes_j L^{p_j}$ to L^q. When $q \ge 1$, such multilinear operators are equivalent by duality to multilinear forms Λ. This is not so for $q < 1$, and Gaussians are then quite far from being extremal [8].

any two of these three conditions (2.1), (2.2), (2.3) imply the third. As will be seen through the discussion of variants below, (2.2) expresses a necessary condition governing large-scale geometry (compare Theorem 2.5), while (2.3) expresses a necessary condition governing small-scale geometry (compare Theorem 2.2). See also the discussion of necessary conditions for Theorem 2.3.

In the rank one case, when each target space H_j is one-dimensional, a necessary and sufficient condition for inequality (1.2) was first obtained by Barthe [1]. Carlen, Lieb, and Loss [7] gave a different proof of the inequality for the rank one case, and a different characterization which is closely related to ours. Write $\ell_j(x) = \langle x, v_j\rangle$. It was shown in [7] that (1.2) is equivalent, in the rank one case, to having $\sum_j p_j^{-1} = \dim(H)$ and $\sum_{j\in S} p_j^{-1} \le \dim(\mathrm{span}(\{v_j : j \in S\}))$ for every subset S of $\{1, 2, \cdots, m\}$; a set of indices S was said to be subcritical if this last inequality holds, and to be critical if it holds with equality. In the higher-rank case, we have formulated these concepts as properties of subspaces of H, rather than of subsets of $\{1, 2, \cdots, m\}$.

To elucidate the connection between the two formulations in the rank one case, define $W_S = \mathrm{span}\{v_j : j \in S\}$, and say that a set of indices S is maximal if there is no larger set $\tilde S$ of indices satisfying $W_{\tilde S} = W_S$. All sets of indices are subcritical, if and only if all maximal sets of indices are subcritical. If $j \in S$ then $\mathrm{codim}_{H_j}(\ell_j(W_S^\perp)) = 1$; if $j \notin S$ and S is maximal then $\mathrm{codim}_{H_j}(\ell_j(W_S^\perp)) = 0$; and $\mathrm{codim}(W_S^\perp) = \dim(\mathrm{span}(\{v_j : j \in S\}))$. Thus if S is maximal, then the subcriticality of S is equivalent to $\sum_{j=1}^n p_j^{-1}\,\mathrm{codim}_{H_j}(\ell_j(W_S^\perp)) \le \mathrm{codim}(W_S^\perp)$. As noted above, under the condition $\sum_{j=1}^n p_j^{-1} = \dim(H)$, this is equivalent to our subcriticality condition $\dim(V) \le \sum_j p_j^{-1} \dim(\ell_j(V))$ for the subspace $V = W_S^\perp$.

The necessity of (2.1) follows from scaling: if $f_j^\lambda(x_j) = g_j(\lambda x_j)$ for each $\lambda \in \mathbb{R}^+$ then $\Lambda(\{f_j^\lambda\})$ is proportional to $\lambda^{-\dim(H)}$, while $\prod_j \|f_j^\lambda\|_{p_j}$ is proportional to $\prod_j \lambda^{-\dim(H_j)/p_j}$. That (2.2) is also necessary will be shown in §5 in the course of the proof of the more general Theorem 2.3.

Remark 2.1. Λ can be alternatively expressed as a constant multiple of the integral $\int_\Sigma \prod_j f_j\,d\sigma$, where Σ is a linear subspace of $\oplus_j H_j$ and σ is Lebesgue measure on Σ. More exactly, Σ is the range of the map $H \ni x \mapsto \oplus_j \ell_j(x)$. Denote by π_j the restriction to Σ of the natural projection $\pi_j : \oplus_i H_i \to H_j$. Then condition (2.2) can be restated as

$$\dim(\tilde\Sigma) \le \sum_j p_j^{-1} \dim(\pi_j(\tilde\Sigma)) \text{ for every linear subspace } \tilde\Sigma \subset \Sigma. \tag{2.4}$$

A local variant is also natural. Consider

$$\Lambda_{\mathrm{loc}}(f_1, \cdots, f_m) = \int_{\{y\in H : |y|\le 1\}} \prod_j f_j \circ \ell_j(y)\,dy. \tag{2.5}$$

Theorem 2.2. *Let H, H_j, ℓ_j, and $f_j : H_j \to [0,\infty)$ be as in Theorem 2.1. Let $p_j \in [1,\infty]$ for $1 \le j \le m$. A necessary and sufficient condition for there to exist $C < \infty$ such that*

$$\Lambda_{\mathrm{loc}}(f_1, \cdots, f_m) \le C \prod_j \|f\|_{L^{p_j}} \tag{2.6}$$

for all nonnegative measurable functions f_j *is that every subspace* V *of* H *satisfies* (2.3): $\operatorname{codim}_H(V) \geq \sum_j p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V))$.

This is equivalent to Theorem 8.17 of [3], proved there by a different method.

Certain cases of Theorem 2.2 follow from Theorem 2.1; if there exist exponents r_j satisfying the hypotheses (2.1) and (2.2) of Theorem 2.1, such that $r_j \leq p_j$ for all j, then the conclusion of Theorem 2.2 follows directly from that of Theorem 2.1 by Hölder's inequality, since $\|f_j\|_{L^{r_j}} \leq C' \|f_j\|_{L^{p_j}}$. But not all cases of Theorem 2.2 are subsumed in Theorem 2.1 in this way. See Remark 7.1 for examples.

The next theorem, in which some but not necessarily all coordinates of y are constrained to a bounded set, unifies Theorems 2.1 and 2.2.

Theorem 2.3. *Let* $H, H_0, \cdots, H_m$ *be finite-dimensional Hilbert spaces and assume that* $\dim(H_j) > 0$ *for all* $j \geq 1$. *Let* $\ell_j : H \to H_j$ *be linear transformations for* $0 \leq j \leq m$, *which are surjective for all* $j \geq 1$. *Let* $p_j \in [1, \infty]$ *for* $1 \leq j \leq m$. *Then there exists* $C < \infty$ *such that*

$$\int_{\{y \in H : |\ell_0(y)| \leq 1\}} \prod_{j=1}^m f_j \circ \ell_j(y)\, dy \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}} \tag{2.7}$$

for all nonnegative Lebesgue measurable functions f_j *if and only if*

$$\dim(V) \leq \sum_{j=1}^m p_j^{-1} \dim(\ell_j(V)) \qquad \text{for all subspaces } V \subset \operatorname{kernel}(\ell_0) \tag{2.8}$$

and

$$\operatorname{codim}_H(V) \geq \sum_{j=1}^m p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V)) \quad \text{for all subspaces } V \subset H. \tag{2.9}$$

This subsumes Theorem 2.2, by taking $H_0 = H$ and $\ell_0 : H \to H$ to be the identity; (2.8) then only applies to $\{0\}$, for which it holds automatically, so that the only hypothesis is then (2.9). On the other hand, Theorem 2.1 is the special case $\ell_0 \equiv 0$ of Theorem 2.3. In that case $\operatorname{kernel}(\ell_0) = H$, so (2.8) becomes (2.2). In addition, the case $V = \{0\}$ of (2.9) yields the reverse inequality $\dim(H) \geq \sum_j p_j^{-1} \dim(H_j)$. Thus the hypotheses of Theorem 2.3 imply those of Theorem 2.1 when $\ell_0 \equiv 0$. The converse implication also holds, as was pointed out in the discussion of Theorem 2.2.

Our next result is one of several possible discrete analogues. Recall [13] that any finitely generated Abelian group G is isomorphic to $\mathbb{Z}^r \times H$ for some integer r and some finite Abelian group H; r is uniquely determined and is called the rank of G.

Theorem 2.4. *Let* G *and* $\{G_j : 1 \leq j \leq m\}$ *be finitely generated Abelian groups. Let* $\varphi_j : G \to G_j$ *be homomorphisms. Let* $p_j \in [1, \infty]$. *Then*

$$\operatorname{rank}(H) \leq \sum_j p_j^{-1} \operatorname{rank}(\varphi_j(H)) \text{ for every subgroup } H \text{ of } G \tag{2.10}$$

if and only if there exists $C < \infty$ *such that*

$$\sum_{y \in G} \prod_{j=1}^m (f_j \circ \varphi_j)(y) \leq C \prod_j \|f_j\|_{\ell^{p_j}(G_j)} \text{ for all } f_j : G_j \to [0, \infty). \tag{2.11}$$

Here the ℓ^{p_j} norms are defined with respect to counting measure.

A special case arises when G is isomorphic to $\mathbb{Z}^d$, G_j is isomorphic to $\mathbb{Z}^{d_j}$ for all j, and each φ_j is represented by a matrix with integer entries. The general case of Theorem 2.4 can be deduced directly from this special case, using the isomorphisms between e.g. G and $\mathbb{Z}^d \times H$ for some finite group H, and the fact that all ℓ^p norms are mutually equivalent on finite sets.

A related variant is as follows. In $\mathbb{R}^d$, for each $n \in \mathbb{Z}^d$ define $Q_n = \{x \in \mathbb{R}^d : |x-n| \le \sqrt{d}\}$. The space $\ell^p(L^\infty)(\mathbb{R}^d)$ is the space of all $f \in L^\infty(\mathbb{R}^d)$ for which $(\sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^\infty(Q_n)})^{1/p}$ is finite.

Theorem 2.5. *Let $m \ge 1$ be a positive integer, and for each $j \in \{1,2,\cdots,m\}$ let $\ell_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be a surjective linear transformation. Let $p_j \in [1,\infty]$. Then*

$$\dim(V) \le \sum_j p_j^{-1} \dim(\ell_j(V)) \text{ for every subspace } V \subset \mathbb{R}^d \tag{2.12}$$

if and only if there exists $C < \infty$ such that

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ \ell_j)(y)\, dy \le C \prod_j \|f_j\|_{\ell^{p_j}(L^\infty)(\mathbb{R}^{d_j})} \tag{2.13}$$

for all measurable $f_j : \mathbb{R}^{d_j} \to [0,\infty)$.

A related result is Corollary 8.11 of [3].

Yet another variant of our results, based on Cartesian product rather than linear algebraic or group theoretic structure, has been obtained earlier by Finner [10]; see also [11] for a discussion of some special cases from another point of view. Let $\{(X_i,\mu_i)_{i\in I}\}$ be a finite collection of measure spaces, and let $(X,\mu) = \prod_{i\in I}(X_i,\mu_i)$ be their product. Let J be another finite index set. For each $j \in J$, let S_j be some nonempty subset of I. Let $Y_j = \prod_{i\in S_j} X_i$, equipped with the associated product measure, and let $\pi_j : X \to Y_j$ be the natural projection map. Let $f_j : Y_j \to [0,\infty]$ be measurable. To avoid trivialities, we assume throughout the discussion that I, J are nonempty and that $\mu(X)$ is strictly positive. Define

$$\Lambda(f_j)_{j\in J} = \int_X \prod_{j\in J} f_j \circ \pi_j \, d\mu. \tag{2.14}$$

Denote by $|\cdot|$ the cardinality of a finite set.

Let $p_j \in [1,\infty]$ for each $j \in J$. Finner's theorem then asserts that if

$$1 = \sum_{j:i\in S_j} p_j^{-1} \text{ for all } i \in I \tag{2.15}$$

then

$$\Lambda(f_j)_{j\in J} \le \prod_{j\in J} \|f_j\|_{L^{p_j}(Y_j)}\,. \tag{2.16}$$

A modest generalization of Finner's theorem is discussed in §7.

The hypothesis (2.15) can be equivalently restated as

$$|K| = \sum_{j\in J} p_j^{-1} |S_j \cap K| \text{ for every subset } K \subset I, \tag{2.17}$$

or again as the conjunction of $|I| = \sum_{j\in J} p_j^{-1}|S_j|$ and $|K| \le \sum_{j\in J} p_j^{-1}|S_j \cap K|$ for every $K \subset I$. When each space X_i is some Euclidean space equipped with Lebesgue measure, the hypotheses in this last form are precisely those of Theorem 2.1, specialized to this limited class of linear mappings. The analogue of a subspace is now a subset $K \subset I$, and the analogue of criticality is (2.17); thus (2.16) holds if and only if every subset K is critical. This contrasts with the situations treated by Barthe [1], by Carlen, Lieb, and Loss [7], and in Theorem 2.1, where generic subspaces will be subcritical even if critical subspaces exist.

A special case treated by Calderón [6] is as follows: Let $1 \le k < n$. Let $x = (x_1, \cdots, x_n)$ be coordinates for $\mathbb{R}^n$. For each subset $S \subset \{1, 2, 3, \cdots, n\}$ of cardinality k let $\mathbb{R}^k_S$ be a copy of $\mathbb{R}^k$, with coordinates $(x_i)_{i\in S}$. Let $\pi_S : \mathbb{R}^n \to \mathbb{R}^k$ be the natural projections. Then for arbitrary nonnegative measurable functions,

$$\int_{\mathbb{R}^n} \prod_S f_S(\pi_S(x))\,dx \le \prod_S \|f_S\|_{L^p(\mathbb{R}^k_S)} \tag{2.18}$$

where $p = \binom{n-1}{k-1}$. A particular instance of Calderón's theorem is the Loomis-Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j \circ \pi_j(x)\,dx \le \prod_{j=1}^n \|f_j\|_{L^{n-1}}\,, \tag{2.19}$$

where $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the mapping that forgets the j-th coordinate.

Two quite distinct investigations motivated our interest in these problems. One derives from work [2] of three of us on multilinear versions of the Kakeya-Nikodym maximal functions. A second motivator was work [9] on multilinear operators with additional oscillatory factors; see Proposition 3.1 and Corollary 3.2 below.

3. An application to oscillatory integrals

Proposition 3.1. *Let $m > 1$. For $1 \le j \le m$ let $\ell_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be surjective linear mappings. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Let $\varphi \in C^1_0(\mathbb{R}^n)$ be a compactly supported, continuously differentiable cutoff function. For $\lambda \in \mathbb{R}$ and $f_j \in L^{p_j}(\mathbb{R}^{n_j})$ define $\Lambda_\lambda(f_1, \cdots, f_m) = \int_{\mathbb{R}^n} e^{i\lambda P(x)} \prod_{j=1}^m f_j(\ell_j(x))\,\varphi(x)\,dx$. Suppose that there exist $\delta > 0$ and $C < \infty$ such that for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$*

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \le C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^\infty}\,. \tag{3.1}$$

Let $(p_1, \cdots, p_m) \in [1, \infty]^m$, and suppose that for every proper subspace $V \subset \mathbb{R}^n$,

$$\operatorname{codim}_{\mathbb{R}^n}(V) > \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V)). \tag{3.2}$$

Then there exist $\delta > 0$ and $C < \infty$, depending on $(p_1, \cdots, p_m)$, such that

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \le C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^{p_j}} \tag{3.3}$$

for all parameters $\lambda \in \mathbb{R}$ and functions $f_j \in L^{p_j}(\mathbb{R}^{n_j})$.

By Theorem 2.2, the condition that $\operatorname{codim}_{\mathbb{R}^n}(V) \geq \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V))$ for every subspace $V \subset \mathbb{R}^n$ guarantees that the integral defining $\Lambda_\lambda(f_1, \cdots, f_m)$ converges absolutely for all functions $f_j \in L^{p_j}$, and is bounded by $C \prod_j \|f_j\|_{L^{p_j}}$. The conclusion of Proposition 3.1 then follows directly from this inequality and the hypothesis by complex interpolation.

A polynomial P is said [9] to be nondegenerate, relative to the collection $\{\ell_j\}$ of mappings, if P cannot be expressed as $P = \sum_j P_j \circ \ell_j$ for any collection of polynomials $P_j : \mathbb{R}^{n_j} \to \mathbb{R}$.

Corollary 3.2. *Let $\{\ell_j\}, P, \varphi$ be as in Proposition 3.1. Suppose that P is nondegenerate relative to $\{\ell_j\}$. Suppose that either (i) $n_j = 1$ for all j, $m < 2n$, and the family $\{\ell_j\}$ of mappings is in general position, or (ii) $n_j = n - 1$ for all j. Let $(p_1, \cdots, p_m) \in [1, \infty]^m$ and suppose that for every proper subspace $V \subset \mathbb{R}^n$, $\operatorname{codim}_{\mathbb{R}^n}(V) > \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V))$. Then there exists $\delta > 0$ such that for any $\varphi \in C_0^1$ there exists $C < \infty$ such that for all functions $f_j \in L^{p_j}(\mathbb{R}^{n_j})$,*

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \leq C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^{p_j}} .$$

Here general position means that for any subset $S \subset \{1, 2, \cdots, m\}$ of cardinality $|S| \leq n$, $\cap_{j \in S} \operatorname{kernel}(\ell_j)$ has dimension $n - |S|$.

By Theorems 2.1 and 2.2 of [9], the hypotheses imply (3.1). Proposition 3.1 then implies the Corollary.

4. Proof of sufficiency in Theorem 2.1

We begin with the proof of sufficiency of the hypotheses (2.1), (2.2) for the finiteness of the supremum in (1.2). Necessity will be established in the next section.

The next definition is made for the purposes of the discussion of Theorem 2.1; alternative notions of criticality are appropriate for the other theorems.

Definition 4.1. Relative to a set of exponents $\{p_j\}$, a subspace $V \subset H$ is said to be critical if

$$\dim(V) = \sum_j p_j^{-1} \dim(\ell_j(V)), \tag{4.1}$$

to be supercritical if the right-hand side is less than $\dim(V)$, and to be subcritical if the right-hand side is greater than $\dim(V)$.

In this language, the hypothesis (2.1) states that $V = H$ is critical relative to $\{p_j\}$, while (2.2) states that no subspace of H is supercritical.

Proof of sufficiency in Theorem 2.1. The proof proceeds by induction on the dimension of H. When $\dim(H) = 1$, necessarily $\dim(H_j) = 1$ for all j. The hypothesis of the theorem in this case is that $\sum_j p_j^{-1} = 1$, and the conclusion is simply a restatement of Hölder's inequality for functions in $L^{p_j}(\mathbb{R}^1)$.

Suppose now that $\dim(H) > 1$. There are two cases. Case 1 arises when there exists some proper nonzero critical subspace $W \subset H$. The analysis then relies on a factorization procedure visible in the work of Calderón [6], Finner [10], and Carlen, Lieb, and Loss [7]. Express $H = W^\perp \oplus W$ where $W^\perp$ is the orthocomplement of W, with coordinates $y = (y', y'') \in W^\perp \oplus W$; we will identify $(y', 0)$ with y' and $(0, y'')$ with y''. Define $U_j \subset H_j$ to be

$$U_j = \ell_j(W). \tag{4.2}$$

Define $\tilde{\ell}_j = \ell_j|_W : W \to U_j$, which is surjective. For $y' \in W^\perp$ and $x_j \in U_j$ define

$$g_{j,y'}(x_j) = f_j(x_j + \ell_j(y')). \tag{4.3}$$

Then

$$f_j(\ell_j(y', y'')) = f_j(\ell_j(y') + \tilde{\ell}_j(y'')) = g_{j,y'}(\tilde{\ell}_j(y'')). \tag{4.4}$$

Now

$$\begin{aligned}\Lambda(f_1, \cdots, f_m) &= \int_{W^\perp} \int_W \prod_j f_j(\ell_j(y', y''))\, dy''\, dy' \\ &= \int_{W^\perp} \int_W \prod_j g_{j,y'}(\tilde{\ell}_j(y''))\, dy''\, dy',\end{aligned}$$

so

$$\Lambda(f_1, \cdots, f_m) = \int_{W^\perp} \tilde{\Lambda}(g_{1,y'}, \cdots, g_{m,y'})\, dy' \tag{4.5}$$

where

$$\tilde{\Lambda}(g_1, \cdots, g_m) = \int_W \prod_j g_j(\tilde{\ell}_j(y''))\, dy''. \tag{4.6}$$

We claim that

$$\tilde{\Lambda}(g_1, \cdots, g_m) \le C \prod_j \|g_j\|_{p_j}. \tag{4.7}$$

Since W has dimension strictly less than $\dim(H)$, this follows from the induction hypothesis provided that W is critical and no subspace $V \subset W$ is supercritical, relative to the mappings $\tilde{\ell}_j$ and exponents p_j. But since $\tilde{\ell}_j$ is the restriction of ℓ_j to W, this condition is simply the specialization of the original hypothesis from arbitrary subspaces of H to those subspaces contained in W, together with the criticality of W hypothesized in Case 1. Thus

$$\Lambda(f_1, \cdots, f_m) = \int_{W^\perp} \tilde{\Lambda}(g_{1,y'}, \cdots, g_{m,y'})\, dy' \le C \int_{W^\perp} \prod_j \|g_{j,y'}\|_{L^{p_j}(U_j)}\, dy'. \tag{4.8}$$

We will next show how this last integral is another instance of the original problem, with H replaced by the lower-dimensional vector space $W^\perp$. For $z_j \in U_j^\perp$ define

$$F_j(z_j) = \Big(\int_{U_j} f_j(x_j + z_j)^{p_j}\, dx_j \Big)^{1/p_j}, \tag{4.9}$$

recalling that $f_j \ge 0$, with $F_j(z_j) = \text{ess sup}_{\,x_j \in U_j} f_j(x_j + z_j)$ if $p_j = \infty$. Thus[4]

$$\|F_j\|_{L^{p_j}(U_j^\perp)} = \|f_j\|_{L^{p_j}(H_j)}. \tag{4.10}$$

[4]If $U_j = \{0\}$ then the domain of F_j is H_j, and $F_j \equiv f_j$. If $U_j = H_j$ then the domain of F_j is $\{0\}$, and $\|F_j\|_{p_j}$ is by definition $F_j(0) = \|f_j\|_{p_j}$.

Denote by $\pi_{U_j^\perp}: H_j \to U_j^\perp$ and $\pi_{U_j}: H_j \to U_j$ the orthogonal projections. Define $L_j: W^\perp \to U_j^\perp$ by

$$L_j = \pi_{U_j^\perp} \circ \ell_j. \tag{4.11}$$

Decomposing $\ell_j(y') = L_j(y') + u_j$ where $u_j = \pi_{U_j}(\ell_j(y'))$, and making the change of variables $\tilde{x}_j = x_j + u_j$ in U_j, gives (if $p_j < \infty$)

$$\|g_{j,y'}\|_{L^{p_j}(U_j)}^{p_j} = \int_{U_j} |g_{j,y'}(x_j)|^{p_j}\, dx_j = \int_{U_j} |f_j(x_j + \ell_j(y'))|^{p_j}\, dx_j \tag{4.12}$$
$$= \int_{U_j} |f_j(x_j + u_j + L_j(y'))|^{p_j}\, dx_j = \int_{U_j} |f_j(\tilde{x}_j + L_j(y'))|^{p_j}\, d\tilde{x}_j = F_j(L_j(y'))^{p_j}.$$

Consequently we have shown thus far that

$$\Lambda(f_1, \cdots, f_m) \leq C \int_{W^\perp} \prod_j F_j \circ L_j \tag{4.13}$$

where $\|F_j\|_{L^{p_j}(U_j^\perp)} = \|f_j\|_{L^{p_j}(H_j)}$. Since $\ell_j: H \to H_j$ is surjective, H_j is spanned by $\ell_j(W) = U_j$ together with $\ell_j(W^\perp)$; thus the orthogonal projection of $\ell_j(W^\perp)$ onto $U_j^\perp$ is all of $U_j^\perp$; thus each $L_j: W^\perp \to U_j^\perp$ is surjective.

To complete the argument for Case 1 we need only show that

$$\int_{W^\perp} \prod_j F_j \circ L_j \leq C \prod_j \|F_j\|_{L^{p_j}(U_j^\perp)}. \tag{4.14}$$

By induction on the ambient dimension, this follows from the next lemma, which appears in [7] in the special case when $\dim(H_j) = 1$ for all j. Although there are no additional complications in the general case, we include a proof for the sake of completeness.

Lemma 4.1. *Fix an m-tuple $(p_1, \cdots, p_m)$ of exponents in $[1, \infty]$. Suppose that with respect to these exponents, H is critical with respect to these exponents, H has no supercritical subspaces, and $W \subset H$ is a nonzero proper critical subspace. Define surjective linear transformations $L_j = \pi_{\ell_j(W)^\perp} \circ \ell_j : W^\perp \to \ell_j(W)^\perp$. Then for any subspace $V \subset W^\perp$, $\dim(V) \leq \sum_j p_j^{-1} \dim(L_j(V))$.*

Proof. Let V be any subspace of H contained in $W^\perp$. Associate to V the subspace $V + W \subset H$. Since $V \subset W^\perp$, $\dim(V + W) = \dim(V) + \dim(W)$. Moreover, for any j,

$$\dim(\ell_j(V + W)) = \dim(L_j(V)) + \dim(\ell_j(W)), \tag{4.15}$$

since $L_j = \pi_{\ell_j(W)^\perp} \circ \ell_j$.

Therefore

$$\begin{aligned}
\sum_j p_j^{-1} \dim(L_j(V)) &= \sum_j p_j^{-1} \dim(\ell_j(V + W)) - \sum_j p_j^{-1} \dim(\ell_j(W)) \\
&= \sum_j p_j^{-1} \dim(\ell_j(V + W)) - \dim(W) \\
&\geq \dim(V + W) - \dim(W) = \dim(V),
\end{aligned}$$

by the criticality of W and subcriticality of $V+W$. Thus V is not supercritical.

When $V=W^\perp$, one has $V+W=H$, whence $\sum_j p_j^{-1}\dim(\ell_j(V+W))=\dim(V+W)$ since H is assumed to be critical. With this information the final inequality of the preceding display becomes an equality, demonstrating that $W^\perp$ is critical. □

The proof of Case 1 of Theorem 2.1 is complete. Turn next to Case 2, in which every nonzero proper subspace of H is subcritical. ∞^{-1} is to be interpreted as zero throughout the discussion.

Consider the set K of all m-tuples $t=(t_1,\cdots,t_m)\in[0,1]^m$ such that relative to the exponents $p_j=t_j^{-1}$, H is critical and has no supercritical subspace. Thus K equals the intersection of $[0,1]^m$ with the hyperplane defined by the equation $\dim(H)=\sum_j t_j\dim(H_j)$, and with all of the closed half-spaces defined by the inequalities $\dim(V)\le\sum_j t_j\dim(\ell_j(V))$, as V ranges over all subspaces of H. Therefore K is comvex and compact.

While the number of such subspaces V is infinite, the number of $m+1$-tuples $(\dim(V),\dim(\ell_1(V)),\cdots,\dim(\ell_m(V)))$ is finite. The set of all distinct inequalities induced by subspaces of H is in one-to-one corresondence with the set of all such $m+1$-tuples. Thus K is the intersection of $[0,1]^m$ with a hyperplane and with finitely many closed half-spaces. Therefore K has finitely many extreme points. Since K is compact and convex, K consequently equals the convex hull of its extreme points.

We will show that for any extreme point t of K, there exists a finite constant C such that $\Lambda(f_1,\cdots,f_m)\le C\prod_j\|f_j\|_{L^{1/t_j}}$ for all nonnegative measurable functions f_j. Granting such inequalities, let $\{t^{(i)}\}_i$ be the set of all extreme points of K, and let C_i be constants for which the corresponding inequalities hold. Any $t=(t_1,\cdots,t_m)\in K$ can be expressed as $t=\sum_i\lambda_i t^{(i)}$ for some scalars $\lambda_i\in[0,1]$ satisfying $\sum_i\lambda_i=1$. Write $t^{(i)}=(t_1^{(i)},\cdots,t_m^{(i)})$. A direct application of complex interpolation shows that

$$\Lambda(f_1,\cdots,f_m)\le\prod_i C_i^{\lambda_i}\prod_j\|f_j\|_{L^{1/t_j}}^{\lambda_i}=C\prod_j\|f_j\|_{L^{1/t_j}}$$

for all nonnegative measurable functions f_j, where $C=\prod_i C_i^{\lambda_i}$.

At an arbitrary extreme point t of K, at least one of the inequalities defining K must become an equality. Therefore some nonzero proper subspace of H must be critical relative to t, or $t_i\in\{0,1\}$ for at least one index i.

Consider the set $\tilde K$ of all $t\in[0,\infty)^m$ for which $\dim(H)=\sum_j t_j\dim(H_j)$ and $\dim(V)\le\sum_j t_j\dim(\ell_j(V))$ for all subspaces V of H. Thus $K=\tilde K\cap[0,1]^m\subset\tilde K$. We claim that $\tilde K=K$. Indeed, for any $t\in\tilde K$, the homogeneity and subcriticality conditions imply that $\operatorname{codim}_H(V)\ge\sum_j t_j\operatorname{codim}_{H_j}(\ell_j(V))$ for all subspaces $V\subset H$. Consider any index i and let V be the nullspace of ℓ_i. Then

$$\begin{aligned}(4.16)\qquad \dim(H_i)=\operatorname{codim}_H(V)&\ge\sum_j t_j\operatorname{codim}_{H_j}(\ell_j(V))\\ &\ge t_i\operatorname{codim}_{H_i}\{0\}=t_i\dim(H_i).\end{aligned}$$

Therefore $t_i\le1$ for all i, so $t\in K$. Thus $\tilde K\subset K$, as claimed.

Since $\tilde K=K$, if t is any extreme point of K, then either equality must hold in at least one of the inequalities defining $\tilde K$, or the number m of indices j must be 1 (so that the hypothesis $\dim(H) = \sum_j p_j^{-1} \dim(H_j)$ specifies the single exponent p_1^{-1}, reducing K to a single point). Thus some nonzero proper subspace of H must be critical relative to t, or at least one coordinate t_i must equal 0, or $m = 1$. In the first subcase we are in Case 1, which has already been treated above. For the case $m = 1$, see the second paragraph below.

In the second subcase, we may proceed by induction on m, for an inequality $\Lambda(f_1, \cdots, f_m) \le C \|f_i\|_{L^\infty} \prod_{j \ne i} \|f_j\|_{L^{p_j}}$ is equivalent to

$$\Lambda(f_1, \cdots, f_{i-1}, 1, f_{i+1}, \cdots, f_m) \le C \prod_{j \ne i} \|f_j\|_{L^{p_j}}. \tag{4.17}$$

The hypotheses of Theorem 2.1 are inherited by this multilinear operator of one lower degree, acting on $\{f_j : j \ne i\}$, whence the desired inequality follows by induction.

This induction is founded by the subcase in which $m = 1$. In this case the hypothesis $\dim(H) = \sum_j p_j^{-1} \dim(H_j)$ becomes $\dim(H) = p_1^{-1} \dim(H_1)$. Since $\ell_1 : H \to H_1$ is assumed to be surjective, $\dim(H_1) \le \dim(H)$, so this forces both $p_1 = 1$ and $\dim(H_1) = \dim(H)$. Since $\ell_1 : H \to H_1$ is surjective, it must be invertible. Therefore $\Lambda(f_1) = \int_H f_1 \circ \ell_1 = c \int_{H_1} f_1 = c\|f_1\|_{L^1}$ for some finite constant c, which certainly implies the desired inequality $\Lambda(f_1) \le C\|f_1\|_{L^1}$. □

Remark 4.1. When $\dim(H_j) = 1$ for all j, every extreme point $(p_1^{-1}, \cdots, p_m^{-1})$ of K has each $p_j^{-1} \in \{0, 1\}$ [1],[7]. This is not the case in general; in the Loomis-Whitney inequality (2.19) for $\mathbb{R}^n$, K consists of a single point, with $p_j = n - 1$ for all j.

5. Proof of Theorem 2.3

Consider $\int_{\{y \in H : |\ell_0(y)| \le 1\}} \prod_{j=1}^m f_j \circ \ell_j \, dy$ where the linear transformation ℓ_0 has domain H and range H_0, with $\dim(H_0)$ possibly equal to zero. Thus some components of y are constrained to a bounded set, while the rest are free. Set

$$\mathcal{V} = \text{kernel}(\ell_0); \tag{5.1}$$

the component of y lying in $\mathcal{V}$ is completely unconstrained, while the component in $\mathcal{V}^\perp$ is constrained to a bounded set.

Proof of necessity of (2.8) *and* (2.9). For any subspace $V \subset H$ define $V_{\text{big}} = V \cap \mathcal{V}$ and $V_{\text{small}} = V \ominus V_{\text{big}} = V \cap (V_{\text{big}})^\perp$, so that $V = V_{\text{small}} \oplus V_{\text{big}}$. Denote by π_V the orthogonal projection of either H or some H_j onto a subspace V.

Let $r \le 1 \le R$ be arbitrary. Define $f_j = f_j(x_j)$ to be the characteristic function of the set S_j of all $x_j \in H_j$ such that

$$|\pi_{\ell_j(V_{\text{big}})}(x_j)| \le R, |\pi_{\ell_j(V) \cap (\ell_j(V_{\text{big}}))^\perp}(x_j)| \le 1, \text{ and } |\pi_{(\ell_j(V))^\perp}(x_j)| \le r.$$

Let $c_0 > 0$ be a small constant independent of r, R, and define $S \subset H$ to be the set of all y such that $|\pi_{V^\perp}(y)| \le c_0 r$, $|\pi_{V_{\text{small}}}(y)| \le c_0$, and $|\pi_{V_{\text{big}}}(y)| \le c_0 R$.

Fix a constant $C < \infty$ such that $|\ell_j(y)| \le C|y|$ for all y, j. Provided that $c_0 < 1/3C$, $y \in S \Rightarrow f_j(\ell_j(y)) = 1$ for all indices j. Indeed, for any $y \in S \cap V^\perp$, $|\ell_j(y)| \le C|y| \le Cc_0 r < r/3$, so $\ell_j(y) \in \frac{1}{3} S_j$. If on the other hand $y \in S \cap V_{\text{small}}$, then $|\ell_j(y)| \le C|y| \le Cc_0 < \frac{1}{3}$, so since $\ell_j(y) \in \ell_j(V)$, $\ell_j(y) \in \frac{1}{3} S_j$. Finally if $y \in S \cap V_{\text{big}}$ then $|\ell_j(y)| \le Cc_0 R$, which implies that $\ell_j(y) \in \frac{1}{3} S_j$ since $\ell_j(y) \in \ell_j(V_{\text{big}})$. Any $y \in S$ admits an orthogonal decomposition $y = u + v + w$ where $u \in S \cap V^\perp$, $v \in S \cap V_{\text{small}}$, and $w \in S \cap V_{\rm big}$. These components satisfy $|u| \le c_0 r$, $|v| \le c_0$, and $|w| \le c_0 R$, by definition of S. $\ell_j(y)$ is thus a sum of three terms in $\frac{1}{3}S_j$, so $\ell_j(y) \in S_j$.

Moreover $y \in S \Rightarrow |\ell_0(y)| \le 1$. Therefore

$$\tilde{\Lambda}_{\rm loc}(\{f_j\}) \ge |S| \sim R^{\dim(V_{\rm big})} \cdot r^{\operatorname{codim}_H(V)} \tag{5.2}$$

while

$$\|f_j\|_{p_j} \sim R^{p_j^{-1}\dim(\ell_j(V_{\rm big}))} r^{p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V))}. \tag{5.3}$$

Suppose that the ratio $\tilde{\Lambda}_{\rm loc}(\{f_j\})/\prod_j \|f_j\|_{p_j}$ is bounded uniformly as a function of r, R. By letting $R \to \infty$ while r remains fixed, we conclude that

$$\dim(V_{\rm big}) \le \sum_j p_j^{-1} \dim(\ell_j(V_{\rm big})).$$

Letting $r \to 0$ with R fixed gives

$$\operatorname{codim}_H(V) \ge \sum_j p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V)).$$

$\square$

The following lemma will be used in the proof of Theorem 2.3.

Lemma 5.1. *Suppose that* $\operatorname{codim}_H(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V))$ *for every subspace* $V \subset H$*, and that* $W \subset H$ *is a subspace satisfying* $\operatorname{codim}_H(W) = \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(W))$*. Then for any subspace* $V \subset W$*,*

$$\operatorname{codim}_W(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)}(\ell_j(V)).$$

Likewise for any subspace $V \subset W^\perp$*,*

$$\operatorname{codim}_{W^\perp}(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)^\perp}(L_j(V)).$$

Proof. For the first conclusion,

$$\begin{aligned} \operatorname{codim}_W(V) &= \dim(W) - \dim(V) = \operatorname{codim}_H(V) - \operatorname{codim}_H(W) \\ &\ge \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V)) - \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(W)) \\ &= \sum_j p_j^{-1}(\dim(\ell_j(W)) - \dim(\ell_j(V))) = \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)}(\ell_j(V)). \end{aligned} \tag{5.4}$$

For the second conclusion,

$$
\begin{aligned}
\operatorname{codim}_{W^\perp}(V) &= \dim(H) - \dim(W) - \dim(V) \\
&= \operatorname{codim}_H(V+W) \\
&\ge \sum_j p_j^{-1} \operatorname{codim}_{H_j}(V+W) \\
&= \sum_j p_j^{-1} \big(\dim(H_j) - \dim(\ell_j(W)) - \dim(L_j(V)) \big) \\
&= \sum_j p_j^{-1} \big(\dim(L_j(W^\perp)) - \dim(L_j(V)) \big). \\
&= \sum_j p_j^{-1} \operatorname{codim}_{L_j(W^\perp)}(L_j(V)).
\end{aligned}
\tag{5.5}
$$

The identity $\dim(H_j) = \dim(\ell_j(W)) + \dim(L_j(W^\perp))$ used to obtain the final line is (4.15) specialized to $V = W^\perp$. □

Proof of sufficiency in Theorem 2.3. The proof follows the inductive scheme of the proof of Theorem 2.1. To simplify notation set $t_j = p_j^{-1} \in [0,1]$. Case 1 now breaks down into two subcases. Case 1A arises when there exists a nonzero proper subspace W of H that is contained in $\mathcal{V}$ and is critical in the sense of (2.8), that is,[5] $\sum_j t_j \dim(\ell_j(W)) = \dim(W)$.

With coordinates (y', y'') for $W^\perp \oplus W$, ℓ_0 is independent of y'', and for every subspace $V \subset W$, $\sum_j t_j \dim(\ell_j(V)) \ge \dim(V)$ by (2.8). Thus the collection of mappings $\{\ell_j|_W\}$ satisfies the hypothesis of Theorem 2.1, whence $\int_W \prod_j f_j \circ \ell_j(y', y'')\, dy'' \le C \prod_j F_j(y')$ where $\|F_j\|_{L^{p_j}(W^\perp)} \le C \|f_j\|_{L^{p_j}(H_j)}$.

It remains to bound $\int_{W^\perp} \chi_B \circ \ell_0(y', 0) \prod_j F_j \circ L_j(y')\, dy'$, where B denotes the characteristic function of a ball of finite radius. Theorem 2.3 can be invoked by induction on the ambient dimension, provided that (2.8) and (2.9) hold for the data $W^\perp, \mathcal{V} \cap W^\perp, \{U_j^\perp, L_j, p_j\}$. We will write $(2.8)_H$, $(2.8)_W$, and $(2.8)_{W^\perp}$ to distinguish between this hypothesis for the three different data that arise in the discussion; likewise for (2.9).

$(2.9)_W$ is the condition that $\operatorname{codim}_{W^\perp}(V) \ge \sum_j t_j \operatorname{codim}_{L_j(W^\perp)}(L_j(V))$ for every subspace $V \subset W^\perp$, which is the second conclusion of Lemma 5.1. $(2.8)_W$ is the condition

$$
\dim(V) \le \sum_j t_j \dim(L_j(V)) \quad \text{for all subspaces } V \subset \mathcal{V} \cap W^\perp. \tag{5.6}
$$

Since V, W are both contained in $\mathcal{V}$ so is $V+W$, so $\sum_j t_j \dim(\ell_j(V+W)) \ge \dim(V+W) = \dim(V) + \dim(W)$ by $(2.8)_H$. This together with the previously established identity $\dim(\ell_j(V+W)) = \dim(\ell_j(W)) + \dim(L_j(V))$ and the criticality condition $\sum_j t_j \dim(\ell_j(W)) = \dim(W)$ yields (5.6). Thus Case 1A is treated by applying Theorem 2.1 for W and the induction hypothesis for $W^\perp$.

Case 1B arises when there exists a nonzero proper subspace $W \subset H$ that is critical in the sense of (2.9), that is, $\operatorname{codim}_H(W) = \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(W))$. The analysis

[5] All summations with respect to j are taken over $1 \le j \le m$.

follows the same inductive scheme. Lemma 5.1 guarantees that $(2.9)_W$ holds, while $(2.8)_W$ is simply the specialization of $(2.8)_H$ to subspaces $V \subset W \cap \mathcal{V}$. Thus Theorem 2.3 may be applied by induction to $W, \{\ell_j(W)\}, \{\ell_j|_W\}, \{p_j\}$.

This reduces matters to $\int_{W^\perp \cap \{|L_0(y')| \le 1\}} \prod_j F_j \circ L_j \, dy'$, where the nullspace $\tilde{V}$ of L_0 is the set of all $y' \in W^\perp$ for which there exists $y'' \in W$ such that $\ell_0(y', y'') = 0$; thus the subspace $\mathcal{V} \subset H$ is now replaced by $\pi_{W^\perp}\mathcal{V} \subset W^\perp$.

Now it is natural to expect to use $(2.8)_H$ to establish $(2.8)_{W^\perp}$, but the latter pertains to certain subspaces not contained in $\mathcal{V}$, about which the former says nothing. Luckily the inequality in (5.6) holds for arbitrary subspaces $V \subset W^\perp$, not merely those contained in $\pi_{W^\perp}\mathcal{V}$. Indeed,

$$\begin{aligned}
\sum_j t_j \dim(L_j(V)) &= \sum_j t_j \dim(\ell_j(V+W)) - \sum_j t_j \dim(\ell_j(W)) \\
&= \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(W)) - \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V+W)) \\
&= \operatorname{codim}_H(W) - \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V+W)) \\
&\ge \operatorname{codim}_H(W) - \operatorname{codim}_H(V+W) \\
&= \dim(V).
\end{aligned}$$

The assumption that W is critical in the sense that equality holds in $(2.9)_H$ implies $(2.9)_W^\perp$, by the second conclusion of Lemma 5.1. Thus by induction on the dimension, Theorem 2.3 may be applied to the integral over $W^\perp$, concluding the proof for Case 1B.

Case 2 arises when no subspace W is critical in either sense. Consider the set $K \subset [0,1]^m$ of all $(t_1, \cdots, t_m)$ such that $\sum_j t_j \dim(\ell_j(V)) \ge \dim(V)$ for all subspaces $V \subset \mathcal{V} = \operatorname{kernel}(\ell_0)$, and $\operatorname{codim}_H(V) \ge \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V))$ for all subspaces $V \subset H$. As in the proof of Theorem 2.1, K is a compact convex set with finitely many extreme points, and consequently equals the convex hull of the set of all of those extreme points. It suffices to prove that $\int_H \chi_B \circ \ell_0 \prod_{j \ge 1} f_j \circ \ell_j \le C \prod_j \|f_j\|_{q_j}$ for every extreme point $(t_1, \cdots, t_m)$ of K, where $q_j = t_j^{-1}$. Consider such an extreme point. If there exists a nonzero proper subspace $V \subset \mathcal{V}$ that is critical in the sense that $\sum_j t_j \dim(\ell_j(V)) = \dim(V)$, or a nonzero proper subspace $V \subset H$ that is critical in the sense that $\operatorname{codim}_H(V) = \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V))$, then Case 1A or Case 1B apply.

There are other cases in which equality might hold in (2.8) or (2.9), besides those subsumed under Case 1. If equality holds for $V = \{0\}$ in (2.9) with $p_j^{-1} = t_j$, then $\dim(H) = \sum_j t_j \dim(H_j)$, which is the first hypothesis of Theorem 2.1. In conjunction with (2.9) this implies that (2.8) holds for every subspace $V \subset H$, which is the second hypothesis of Theorem 2.1. Therefore the conclusion (2.7) of Theorem 2.3 holds without the restriction $|\ell_0(y)| \le 1$ in the integral, by Theorem 2.1.

If on the other hand $H = \mathcal{V} = \operatorname{kernel}(\ell_0)$ and equality holds for $V = H$ in (2.8) with $p_j^{-1} = t_j$, then $\dim(H) = \sum_j t_j \dim(H_j)$, so Theorem 2.1 applies once more.

Therefore matters reduce to the case where equality holds in (2.8) for no subspace of $\mathcal{V}$ except $V = \{0\}$, and where furthermore equality holds in (2.9) for no subspace of H except for $V = H$ itself. Equality always holds in both of those cases, so they play no part in defining K.

t satisfies $\operatorname{codim}_H(V) \ge \sum_j t_j \operatorname{codim}_{H_j}(V))$ for every subspace $V \subset H$. Therefore as in Case 2 of the proof of Theorem 2.1, every remaining extreme point $(t_1, \cdots, t_m)$ of K must have $t_i = 0$ for at least one index i.

By induction on m, it therefore suffices to treat the case $m = 1$, with $p_1 = \infty$. By (2.8) applied to $V = \operatorname{kernel}(\ell_0)$, $\dim(\operatorname{kernel}(\ell_0)) \le 0 \dim(H_1) = 0$, so ℓ_1 has no kernel. Therefore the restriction $|\ell_0(y)| \le 1$ constrains y to a bounded region, whence $\int_{|\ell_0(y)|\le 1} f_1 \circ \ell_1(y)\, dy \le C\|f_1\|_{L^\infty}$ for some finite constant C. □

6. Proof of Theorem 2.4

This proof contains no significant new elements. We denote the identity element of a group by 0. $\times$ will denote the Abelian direct product, that is, the Cartesian product of two Abelian groups, equipped with the Abelian group structure associated naturally to its factors. A subgroup H of G is said to be of finite index if the quotient group G/H is finite. If H, H' are subgroups of G, then $H + H'$ denotes the subgroup of G generated by $H \cup H'$.

A very few properties of finitely generated Abelian groups will be used in the proof. See for instance [13] pp. 76-80, especially Theorem 2.6. Let G be any finitely generated Abelian group. G is isomorphic to $\mathbb{Z}^r \times H$ for some integer r and some finite Abelian group H for a unique nonnegative integer r, called the rank of G. H is uniquely determined up to isomorphism, and is isomorphic to the subgroup of G consisting of all elements of finite order, which is called the torsion subgroup of G. Thus G is finite if and only if $\operatorname{rank}(G) = 0$. Any subgroup H of G is finitely generated, and satisfies $\operatorname{rank}(H) \le \operatorname{rank}(G)$. If H_1, H_2 are subgroups of G, and if $H_1 \cap H_2 = \{0\}$, then $\operatorname{rank}(H_1 + H_2) = \operatorname{rank}(H_1) + \operatorname{rank}(H_2)$. For any homomorphism φ, $\operatorname{rank}(\varphi(G)) \le \operatorname{rank}(G)$. Any subgroup H is of course normal, so G/H is also a finitely generated Abelian group; however, in contrast to the theory of vector spaces, G is not in general isomorphic to the direct product of $H \times (G/H)$. However, the following weaker property does hold, and will serve as a substitute in our analysis: If $\pi : G \to G/H$ denotes the natural projection then for any subgroup H' of G/H, $\operatorname{rank}(\pi^{-1}(H')) = \operatorname{rank}(H) + \operatorname{rank}(H')$. In particular, $\operatorname{rank}(G) = \operatorname{rank}(H) + \operatorname{rank}(G/H)$.

Let groups G, G_j, homomorphisms φ_j, and exponents p_j satisfy the hypotheses of Theorem 2.4. Consider first the case where there exists a subgroup $G' \subset G$, satisfying $0 < \operatorname{rank}(G') < \operatorname{rank}(G)$, which is critical in the sense that $\sum_j p_j^{-1} \operatorname{rank}(\varphi_j(G')) = \operatorname{rank}(G')$. Define $G'_j = \varphi_j(G') \subset G_j$. Since every subgroup of G inherits the hypothesis of the theorem, we may conclude by induction on the rank that

$$\sum_{y\in G'} \prod_j (f_j \circ \varphi_j)(y) \le C \prod_j \|f_j\|_{\ell^{p_j}(G'_j)}. \tag{6.1}$$

Define $F_j \in \ell^{p_j}(G_j/G'_j)$ by

$$F_j(x + G'_j) = (\sum_{z\in G'_j} |f_j(x+z)|^{p_j})^{1/p_j}.$$

Then $\|F_j\|_{\ell^{p_j}(G_j/G'_j)} = \|f_j\|_{\ell^{p_j}(G_j)}$. Define homomorphisms $\psi_j : G/G' \to G_j/G'_j$ by composing each φ_j with the quotient map from G_j to G'_j. Then

$$\begin{aligned}\sum_{y\in G}\prod_j (f_j\circ\varphi_j)(y) &= \sum_{x\in G/G'}\sum_{z\in G'}\prod_j (f_j\circ\varphi_j)(x+z)\\ &\le C\sum_{x\in G/G'}\prod_j (F_j\circ\psi_j)(x);\end{aligned} \tag{6.2}$$

the inequality follows from an invocation of (6.1). It suffices to show that the homomorphisms ψ_j inherit the hypothesis of Theorem 2.4, which may then be applied by induction on the rank to yield the desired bound $O(\prod_j \|F_j\|_{\ell^{p_j}})$. This hypothesis is verified using the criticality of G' and the additivity of ranks, just as additivity of dimensions was used in the proof of Theorem 2.1. Thus Theorem 2.4 is proved in the special case where there exists a critical subgroup $G' \subset G$ satisfying $0 < \mathrm{rank}\,(G') < \mathrm{rank}\,(G)$.

In the general case of Theorem 2.4, consider the compact convex set K of all $(t_1, \cdots, t_m) \in [0,1]^m$ for which $\mathrm{rank}\,(H) \le \sum_j t_j\,\mathrm{rank}\,(\varphi_j(H))$ for all subgroups $H \subset G$. As in the proof of Theorem 2.1, the set of all extreme points of K is finite, and K is equal to its convex hull.[6] It suffices to prove that $\sum_{y\in G}\prod_j (f_j\circ\varphi_j)(y) \le C\prod_j \|f_j\|_{1/t_j}$ for all extreme points $(t_1, \cdots, t_m)$ of K.

If $(t_1, \cdots, t_m)$ is an extreme point then either $\sum_j t_j\,\mathrm{rank}\,(\varphi_j(G')) = \mathrm{rank}\,(G')$ for some subgroup G' satisfying $0 < \mathrm{rank}\,(G') < \mathrm{rank}\,(G)$, or $t_j \in \{0,1\}$ for all indices j, or the total number m of indices j equals one. The first case has already been treated above.

Suppose that $(t_1, \cdots, t_m) \in K$ and $t_j \in \{0,1\}$ for all j. Let $S = \{j : t_j = 1\}$, and consider the subgroup $G' = \cap_{j:t_j=1}\,\mathrm{kernel}\,(\varphi_j)$. The hypothesis (2.10) states that $0 = \sum_{j\in S}\mathrm{rank}\,(\varphi_j(G')) \ge \mathrm{rank}\,(G')$, so G' has rank 0, hence is finite. For any point $z = (z_j)_{j\in S} \in \prod_{j\in S} G_j$, the cardinality of $\{y \in G : \phi_j(y) = z_j \ \forall j \in S\}$ is $\le |G'|$. Therefore

$$\begin{aligned}\sum_{y\in G}\prod_j (f_j\circ\varphi_j)(y) &\le \prod_{j\notin S}\|f_j\|_{\ell^\infty}\sum_{y\in G}\prod_{j\in S}(f_j\circ\varphi_j)(y)\\ &= \prod_{j\notin S}\|f_j\|_{\ell^\infty}\sum_{z\in\prod_{j\in S}G_j}|\{y : \varphi_j(y) = z_j \ \forall j\in S\}|\prod_{j\in S} f_j(z_j)\\ &\le |G'|\prod_{j\notin S}\|f_j\|_{\ell^\infty}\prod_{j\in S}\|f_j\|_{\ell^1},\end{aligned}$$

which is the desired inequality.

If $m = 1$ then the hypothesis $\mathrm{rank}\,(G) = p_1^{-1}\,\mathrm{rank}\,(\varphi_1(G))$ forces $p_1 = 1$ and $\mathrm{rank}\,(\varphi_1(G)) = \mathrm{rank}\,(G)$. Therefore the kernel of φ_1 has rank 0, that is, it is a finite group. The required inequality $\sum_{y\in G}(f_1\circ\varphi_1)(y) \le C\,\|f_1\|_{\ell^1}$ is then immediate. □

[6] There is a subtlety here. The set of all subspaces of fixed dimension of a finite-dimensional vector space naturally carries the structure of a compact topological space. The set of all subgroups of a finitely generated Abelian group lacks such structure. However, the inequalities which define K here are in one-to-one corresondence with the set of all m-tuples $(\mathrm{rank}\,(H), \mathrm{rank}\,(\varphi_1(H)), \cdots, \mathrm{rank}\,(\varphi_m(H)))$, as H varies over all subgroups of G. Since all ranks belong to the finite set $[0, \mathrm{rank}\,(G)] \cap \mathbb{Z}$, only finitely many such inequalities arise.

Conversely, necessity of the hypothesis that $\operatorname{rank}(H) \le \sum_j p_j^{-1} \operatorname{rank}(\varphi_j(H))$ for all subgroups H of G is routine. Observe first that if the conclusion of Theorem 2.4 holds, then it holds with G replaced by any subgroup H; this follows simply by restricting the sum over all $y \in G$ on the left-hand side of (2.11) to $y \in H$. Therefore in order to prove necessity of the hypothesis, it suffices to prove that $\operatorname{rank}(G) \le \sum_j p_j^{-1} \operatorname{rank}(\varphi_j(G))$.

It is no loss of generality to replace G_j by $\varphi_j(G)$, so we may assume that each homomorphism φ_j is surjective. Each group G_j is isomorphic to $\mathbb{Z}^{r_j} \times T_j$ where $r_j = \operatorname{rank}(G_j)$, and T_j is some finite group. Write (x_j, t_j) for coordinates on $\mathbb{Z}^{r_j} \times T_j$, and define $\|(x_j, t_j)\| = |x_j|$. Define a similar function $G \ni y \mapsto \|y\|$. Let R be an arbitrary large positive real number, and set $f_j(x_j, t_j) = 1$ if $|x_j| \le R$, and $= 0$ otherwise. Then $\|f_j\|_{p_j} \sim R^{r_j/p_j}$ for large R. On the other hand, there exists $c > 0$ independent of R such that $\prod_j (f_j \circ \varphi_j)(y) = 1$ for all $y \in G$ satisfying $\|y\| \le cR$. The number of such points $y \in G$ is $\ge c' R^{\operatorname{rank}(G)}$. By letting $R \to \infty$ we conclude that if (2.11) holds, then $\operatorname{rank}(G) \le \sum_j p_j^{-1} \operatorname{rank}(G_j)$. $\square$

Outline of proof of Theorem 2.5. This argument requires no essentially new ideas. Necessity of the hypothesis for a subspace V is proved by defining each f_j to be the intersection of $\{x_j : |x_j| \le R\}$ with $\{x_j : \operatorname{distance}(x_j, \ell_j(V)) \le C_0\}$, and letting $R \to \infty$ while C_0 remains fixed.

The proof of sufficiency is based on an inductive argument for the critical case, in which there exists a proper subspace of positive dimension satisfying $\dim(V) = \sum_j p_j^{-1} \dim(\ell_j(V))$, and a direct verification for the case in which all $p_j \in \{0, 1\}$. The latter is straightforward.

The former is more awkward than the corresponding step in our other proofs, because the intersection of $\mathbb{Z}^d$ with a subspace of $\mathbb{R}^d$ is rarely a cocompact lattice. Let V be a nonzero critical subspace of $\mathbb{R}^d$. For each index j, define W_j to be the orthocomplement in $\mathbb{R}^{d_j}$ of $\ell_j(V)$. Choose a sublattice $\mathcal{L}_j$ of W_j of rank $\dim(W_j)$, and a sublattice $\mathcal{L}'_j$ of $\ell_j(V)$ of rank equal to $\dim(\ell_j(V))$. Let $A, A' < \infty$ be large constants. For $y \in W_j$ define

$$F_j(y) = \Big(\sum_{n \in \mathcal{L}'} \sup_{|y-n| \le A} |f_j(y+n)|^{p_j} \Big)^{1/p_j}.$$

Define $\|F_j\|_{\ell^{p_j}(L^\infty)}$ by first taking the L^∞ norm over $\{y : |y-n| < A'\}$ for each $n \in \mathcal{L}$, then taking an ℓ^{p_j} norm with respect to n. Such a norm, associated to any sublattice of full rank, is comparable to such a norm associated to any other sublattice of full rank, provided that the constants A' are chosen to be sufficiently large, depending on the sublattices in question. For any $A, A' < \infty$ there exists $C < \infty$ such that

$$\|F_j\|_{\ell^{p_j}(L^\infty)(\ell_j(V))} \le C \, \|f_j\|_{\ell^{p_j}(L^\infty)(\mathbb{R}^{d_j})}.$$

If A is sufficiently large then uniformly for all $y \in V^\perp$,

$$\int_V \prod_j (f_j \circ \ell_j)(y+z)\, dz \le C_{A'} \prod_j (F_j \circ L_j)(y)$$

where $L_j : V^\perp \to W_j$ is $\ell_j|_{V^\perp}$ followed by orthogonal projection onto W_j. Thus

$$\Lambda(f_1, \cdots, f_m) \le C_A \tilde{\Lambda}(F_1, \cdots, F_m) = \int_{V^\perp} \prod_j (F_j \circ L_j).$$

If the constant A' appearing in the definition of the $\ell^{p_j}(L^\infty)$ norms of the functions F_j is chosen to be sufficiently large, then there exists $C < \infty$ such that

$$\tilde{\Lambda}(F_1, \cdots, F_m) \le C \prod_j \|F_j\|_{\ell^{p_j}(L^\infty)}.$$

The rest of the argument is as in our other proofs. □

7. Variants based on product structure

Our next result is analogous to a unification of Theorems 2.3 and 2.5. We say that a measure space (X, μ) is atomic if there exists $\delta > 0$ such that $\mu(E) \ge \delta$ for every measurable set E having strictly positive measure.

Proposition 7.1. *Suppose that the index set I is a disjoint union $I = I_0 \cup I_\infty \cup I_\star$, where X_i is a finite measure space for each $i \in I_0$, is atomic for each $i \in I_\infty$, and is an arbitrary measure space for each $i \in I_\star$. Then a sufficient condition for the inequality* (2.16) *is that*

$$1 \ge \sum_{j : i \in S_j} p_j^{-1} \text{ for all } i \in I_0 \tag{7.1}$$

$$1 \le \sum_{j : i \in S_j} p_j^{-1} \text{ for all } i \in I_\infty \tag{7.2}$$

$$1 = \sum_{j : i \in S_j} p_j^{-1} \text{ for all } i \in I_\star. \tag{7.3}$$

That these sufficient conditions are also necessary, in general, is a consequence of the necessity of the hypotheses of Theorem 2.3.

Remark 7.1. Consider the case where each X_i is a finite measure space. If $(p_j)_{j \in J}$ satisfies the hypothesis (2.15), and if $q_j \ge p_j$ for all $j \in J$, then $\Lambda(f_j)_{j \in J} \le C \prod_j \|f_j\|_{p_j} \le C' \prod_j \|f_j\|_{q_j}$ by Finner's theorem and Hölder's inequality. However, there are situations[7] in which $(q_j)_{j \in J}$ satisfies (7.1) yet there exists no $(p_j)_{j \in J}$ satisfying (2.15) with $q_j \ge p_j$ for all $j \in J$.

To construct an example, begin with any situation where there is an extreme point $(q_j^{-1})_{j \in J}$ of $K = \{(t_j)_{j \in J} \in [0,1]^J : 1 = \sum_{j : i \in S_j} t_j \text{ for all } i \in I\}$, such that $q_j^{-1} < 1$ for all j; for instance, the Loomis-Whitney example. Augment I by adding a single new index i', choose one index j' already in J, and replace $S_{j'}$ by $S_j \cup \{i'\}$, while keeping S_j unchanged for all $j \ne j'$. Thus $\sum_{j : i' \in S_j} q_j^{-1} = q_{j'}^{-1} < 1$; $(q_j)_{j \in S}$ satisfies (7.1). However no $(p_j)_{j \in J}$. For if $p_j \ge q_j^{-1}$ for all j with strict inquality for some index k, choose some $i \in S_k$. Then $\sum_{j : i \in S_j} p_j^{-1} > \sum j : i \in S_j q_j^{-1} = 1$, so that (7.1) fails for $(p_j)_{j \in S}$.

[7]The special case of Proposition 7.1 in which all X_i are finite measure spaces is stated in [10], p. 1898, but no proof is given.

Proposition 7.1 can be proved by repeating Case 1 of the proofs of Theorems 2.1 and 2.3, arguing by induction on $|I|$, and integrating with respect to the m-th coordinate in $\prod_{i\in I} X_i$ while all other coordinates are held constant. The basis case $m = 1$ is Hölder's inequality. Indeed, this is the argument given in [10] for the special case when $I = I_\star$.

Alternatively, when I_0 is empty,[8] Proposition 7.1 can be reduced to the case where each X_i is $\mathbb{R}^1$ equipped with Lebesgue measure, by approximating general functions by finite linear combinations of characteristic functions of product sets, and then embedding any particular situation measure-theoretically into a (product of copies of) $\mathbb{R}^1$. The inequality (2.16) then follows from an application of Theorem 2.1.

8. A final remark

We have assumed throughout the discussion that all exponents satisfy $p_j \geq 1$. In Theorems 2.1, 2.2, and 2.3, the inequalities in question are false if some $p_j < 1$. To see this, fix one index j. Take f_i to be the characteristic function of a fixed ball centered at the origin for each $i \neq j$, take f_j to be the characteristic function of a ball of measure δ centered at the origin, and let $\delta \to 0$. Then $\tilde{\Lambda}(f_1, \cdots, f_m)$ has order of magnitude δ, while $\prod_i \|f_i\|_{L^{p_i}}$ has order of magnitude $\delta^{1/p_j} \ll \delta$.

Valid inequalities can hold in Theorems 2.4 and 2.5 with some exponents strictly less than one, but they are always implied by stronger inequalities already contained in those theorems. More precisely, if the inequality holds for some m-tuple $(p_1, \cdots, p_m)$, then it also holds with each p_i replaced by $\max(p_i, 1)$. In the case of Theorem 2.4, that p_j can be replaced by 1 if $p_j < 1$ can be shown by considering the case when the support of f_i is a single point, then exploiting linearity and symmetry.

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