Math. Res. Lett. **17** (2010), *no.* 04, 601–612

STABILITY UNDER DEFORMATIONS OF EXTREMAL ALMOST-KÄHLER METRICS IN DIMENSION 4.

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Abstract. Given a path of almost-Kähler metrics compatible with a fixed symplectic form on a compact 4-manifold such that at time zero the almost-Kähler metric is an extremal Kähler one, we prove, for a short time and under a certain hypothesis, the existence of a smooth family of extremal almost-Kähler metrics compatible with the same symplectic form, such that at each time the induced almost-complex structure is diffeomorphic to the one induced by the path.

1. Introduction

An almost-Kähler metric on a $2n$-dimensional symplectic manifold (M, ω) is induced by an almost-complex structure J compatible with ω in the sense that the tensor field $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is symmetric and positive definite and thus it defines a Riemannian metric on M. The almost-Kähler metric is Kähler if the almost-complex structure J is integrable. Given an almost-Kähler metric, one can define a canonical hermitian connection ∇ (see e.g. [16, 24]). The hermitian scalar curvature s^∇ is then obtained by taking a trace and contracting the curvature of ∇ with ω. In the Kähler case, the hermitian scalar curvature coincides with the Riemannian scalar curvature.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of the infinite dimensional group $Ham(M, \omega)$ of hamiltonian symplectomorphisms on the space AK_ω of ω-compatible almost-Kähler metrics is hamiltonian with moment map $\mu : AK_\omega \to (Lie(Ham(M, \omega)))^*$ given by $\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}$. The critical points of the norm $\int_M \left(s^\nabla\right)^2 \frac{\omega^n}{n!}$ are called *extremal almost-Kähler metrics*. It turns out that the symplectic gradient of s^∇ of such metrics is a holomorphic vector field in the sense that its flow preserves the corresponding almost-complex structure. In particular, extremal Kähler metrics in the sense of Calabi [7] and almost-Kähler metrics with constant hermitian scalar curvature are extremal.

The GIT formal picture in [9] suggests the existence and the uniqueness of an extremal almost-Kähler metric, modulo the action of $Ham(M, \omega)$, in each 'stable complexified' orbit of the action of $Ham(M, \omega)$. However, in this formal infinite dimensional setting, a natural complexification of $Ham(M, \omega)$ does not exist. When $H^1(M, \mathbb{R}) = 0$, an identification of the 'complexified' orbit of a Kähler metric $(J, g) \in AK_\omega$ is given by considering all Kähler metrics $(J, \tilde{g})$ in the Kähler class $[\omega]$ and applying Moser's Lemma [9]. In this setting, Fujiki–Schumacher [14] and LeBrun–Simanca [21] showed, in the abscence of holomorphic vector fileds, that the existence of

Received by the editors April 26, 2010.

an extremal Kähler metric is an open condition on the space of such orbits. Moreover, Apostolov–Calderbank–Gauduchon–Friedman [3] generalized this result by fixing a maximal torus T in the reduced automorphism group of (M, J) and considering T-invariant ω-compatible Kähler metrics. In general, for an almost-Kähler metric, a description of these 'complexified' orbits is not avaible, see however [10] for the toric case. Nevertheless, the formal picture suggests that the existence of an extremal Kähler metric should persist for smooth almost-Kähler metrics close to an extremal one.

Thus motivated, we consider in this paper the 4-dimensional case where one can introduce a notion of almost-Kähler potential related to the one defined by Weinkove [27, 28]. In the spirit of [14, 21], we shall apply the Banach Implicit Function Theorem for the hermitian scalar curvature of T-invariant ω-compatible almost-Kähler metrics where T is a maximal torus in $Ham(M, \omega)$. The main technical problem is the regularity of a family of Green operators involved in the definition of the almost-Kähler potential. Using a Kodaira–Spencer result [19, 20], one can resolve this problem if we suppose that the dimension of g_t-harmonic J_t-anti-invariant 2-forms, denoted by $h^-_{J_t}$ (see [12]), satisfies the condition $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ along the path $(J_t, g_t) \in AK^T_\omega$ in the space of T-invariant ω-compatible almost-Kähler metrics. So, our main theorem claims the following

Theorem 1.1. *Let (M, ω) be a 4-dimensional compact symplectic manifold and T a maximal torus in $Ham(M, \omega)$. Let (J_t, g_t) be any smooth family of almost-Kähler metrics in AK^T_ω such that (J_0, g_0) is an extremal Kähler metric. Suppose that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, there exists a smooth family $(\tilde{J}_t, \tilde{g}_t)$ of extremal almost-Kähler metrics in AK^T_ω, defined for sufficiently small t, with $(\tilde{J}_0, \tilde{g}_0) = (J_0, g_0)$ and such that $\tilde{J}_t$ is equivariantly diffeomorphic to J_t.*

Remark 1.2. (i) The condition that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ is satisified in the following cases:

(1) When J_t are integrable almost-complex structures for each t. Then, $h^-_{J_t} = 2h^{2,0}(M, J_t) = b^+(M) - 1$ by a well-known result of Kodaira [5]. On the other hand, it is unknown whether or not, for an ω-compatible *non-integrable* almost-complex J on a compact 4-dimensional symplectic manifold M with $b^+(M) \geq 3$, the equality $h^-_J = b^+(M) - 1$ is possible (see [12]).

(2) When $b^+(M) = 1$, $h^-_{J_t} = 0$ for each t. This condition is satisfied when (M, ω) admits a non trivial torus in $Ham(M, \omega)$ [17].

(ii) Theorem 1.1 holds under the weaker assumption that the torus $T \subset Ham(M, \omega)$ is maximal in $Ham(M, \omega) \cap Isom_0(M, g_0)$, where $Isom_0(M, g_0)$ denotes the connected component of the isometry group of the initial metric g_0. By a known result of Calabi [8], any extremal Kähler metric is invariant under a maximal connected compact subgroup of $Ham(M, \omega) \cap \widetilde{Aut}(M, J_0)$, where $\widetilde{Aut}(M, J_0)$ is the reduced automorphism group of (M, J_0). Hence, Theorem 1.1 generalizes [14, 21] in the 4-dimensional case.
(iii) It was kindly pointed out to us by T. Drăghici that using a recent result of Donaldson and Remarks (i) and (ii) above, one can further extend Theorem 1.1 in the case when $b^+(M) = 1$ as follows: Let (M, ω_0, J_0, g_0) be a compact 4-dimensional extremal Kähler manifold with $b^+(M) = 1$ and T be a maximal torus in $Ham(M, \omega) \cap$

$Isom_0(M, g_0)$. Then, for any smooth family of T-invariant almost-complex structures $J(t)$ with $J(t) = J_0$, $J(t)$ is compatible with an extremal almost-Kähler metric g_t for $t \in (-\epsilon, \epsilon)$. Indeed, as $J(t)$ are tamed by ω_0 for $t \in (-\epsilon, \epsilon)$ and $b^+(M) = 1$, one can use the openess result of Donaldson [11, Proposition 1] (see also [12, Sec. 5]) to show that there exists a smooth family of $J(t)$-invariant symplectic forms ω_t with $[\omega_t] = [\omega_0]$. Averaging ω_t over the compact group T and using the equivariant Moser Lemma, we obtain a family J_t of T-invariant ω_0-compatible almost-complex structures such that J_t is T-equivariantly diffeomorphic to $J(t)$. We can then apply Theorem 1.1 to produce compatible extremal metrics

Kim and Sung [18] showed that, in any dimension, if one starts with a Kähler metric of constant scalar curvature with no holomorphic vector fields, one can construct infinite dimensional families of almost-Kähler metrics of constant hermitian scalar curvature which concide with the initial metric away from an open set. Similar existence result was presented in [22] when the initial Kähler metric is locally toric.

2. Preliminaries

Let (M, ω) be a compact symplectic manifold of dimension $2n$. An almost-complex structure J is compatible with ω if the tensor field $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M; then, (J, g) is called an (ω-compatible) *almost-Kähler* metric on (M, ω). If, additionally, the almost-complex structure J is *integrable*, then (J, g) is a *Kähler* metric on (M, ω).

The almost-complex structure J acts on the cotangent bundle $T^*(M)$ by $J\alpha(X) = -\alpha(JX)$, where α is a 1-form and X a vector field on M. Any section ψ of the bundle $\otimes^2 T^*(M)$ admits an orthogonal splitting $\psi = \psi^{J,+} + \psi^{J,-}$, where $\psi^{J,+}$ is the J-invariant part and $\psi^{J,-}$ is the J-anti-invariant part, given by

$$\psi^{J,+}(\cdot,\cdot) = \frac{1}{2}\left(\psi(\cdot,\cdot) + \psi(J\cdot, J\cdot)\right) \text{ and } \psi^{J,-}(\cdot,\cdot) = \frac{1}{2}\left(\psi(\cdot,\cdot) - \psi(J\cdot, J\cdot)\right).$$

In particular, the bundle of 2-forms decomposes under the action of J

$$\Lambda^2(M) = \mathbb{R}\,.\,\omega \oplus \Lambda_0^{J,+}(M) \oplus \Lambda^{J,-}(M), \tag{2.1}$$

where $\Lambda_0^{J,+}(M)$ is the subbundle of the *primitive* J-invariant 2-forms (i.e. 2-forms pointwise orthogonal to ω) and $\Lambda^{J,-}(M)$ is the subbundle of J-anti-invariant 2-forms. Hence, the subbundle of primitive 2-forms $\Lambda_0^2(M)$ admits the splitting

$$\Lambda_0^2(M) = \Lambda_0^{J,+}(M) \oplus \Lambda^{J,-}(M).$$

For an ω-compatible almost-Kähler metric (J, g), the canonical hermitian connection on the complex tangent bundle $(T(M), J, g)$ is defined by

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J\,(D^g_X J)\,Y,$$

where D^g is the Levi-Civita connection with respect to g and X, Y are vector fields on M. Denote by R^∇ the curvature of ∇. Then, the *hermitian Ricci form* ρ^∇ is the trace of $R^\nabla_{X,Y}$ viewed as an anti-hermitian linear operator of $(T(M), J, g)$, i.e.

$$\rho^\nabla(X, Y) = -tr(J \circ R^\nabla_{X,Y}).$$

Hence, the 2-form ρ^∇ is a closed (real) 2-form and it is a deRham representative of $2\pi c_1(T(M), J)$ in $H^2(M, \mathbb{R})$, where $c_1(T(M), J)$ is the first (real) Chern class. If the almost-complex structure J is compatible with a symplectic form $\tilde{\omega}$ such that $\tilde{\omega}^n = e^F \omega^n$ for some smooth real-valued function F on M, then [26, 27]

$$\tilde{\rho}^\nabla = -\frac{1}{2} dJdF + \rho^\nabla, \tag{2.2}$$

where $\tilde{\rho}^\nabla$ is the hermitian Ricci form of the almost-Kähler metric $(J, \tilde{g})$ (here $\tilde{g}(\cdot,\cdot) = \tilde{\omega}(\cdot, J\cdot)$ is the induced Riemannian metric).

We define the *hermitian scalar curvature* s^∇ of an almost-Kähler metric (J, g) as the trace of ρ^∇ with respect to ω, i.e.

$$s^\nabla \omega^n = 2n \left(\rho^\nabla \wedge \omega^{n-1}\right). \tag{2.3}$$

The (Riemannian) *Hodge operator* $*_g : \Lambda^p(M) \to \Lambda^{2n-p}(M)$ is defined to be the unique isomorphism such that $\psi_1 \wedge (*_g \psi_2) = g(\psi_1, \psi_2) \frac{\omega^n}{n!}$, for any p-forms ψ_1, ψ_2. Then, the codifferential δ^g, defined as the formal adjoint of the exterior derivative d with respect to g, is related to d by the relation [6, 15]

$$\delta^g = - *_g d *_g .$$

It follows that

$$d = *_g \delta^g *_g . \tag{2.4}$$

In dimension $2n = 4$, the bundle of 2-forms decomposes as

$$\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M),$$

where $\Lambda^\pm(M)$ correspond to the eigenvalue (± 1) under the action of the Hodge operator $*_g$. This decomposition is related to the splitting (2.1) as follows

$$\Lambda^+(M) = \mathbb{R}\,.\,\omega \oplus \Lambda^{J,-}(M) \text{ and } \Lambda^-(M) = \Lambda_0^{J,+}(M). \tag{2.5}$$

3. Extremal almost-Kähler metrics

Let (M, ω) be a compact and connected symplectic manifold of dimension $2n$. Any ω-compatible almost-complex structure is identified with the induced Riemannian metric.

Denote by AK_ω the Fréchet space of ω-compatible almost-complex structures. The space AK_ω comes naturally equipped with a formal Kähler structure. Let $Ham(M, \omega)$ be the group of hamiltonian symplectomorphisms of (M^{2n}, ω). The Lie algebra of $Ham(M, \omega)$ is identified with the space of smooth functions on M with zero mean value.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of $Ham(M, \omega)$ on AK_ω is hamiltonian with momentum given by the hermitian scalar curvature. More precisely, the moment map $\mu : AK_\omega \to (Lie(Ham(M, \omega)))^*$ is

$$\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}$$

where s^∇ is the hermitian scalar curvature of (J, g) and f is a smooth function with zero mean value viewed as an element of $Lie(Ham(M, \omega))$. The square-norm of the hermitian scalar curvature defines a functional on AK_ω

$$J \mapsto \int_M \left(s^\nabla\right)^2 \frac{\omega^n}{n!}. \tag{3.1}$$

Definition 3.1. The critical points (J, g) of the functional (3.1) are called *extremal almost-Kähler metrics.*

Proposition 3.2. *An almost-Kähler metric (J, g) is a critical point of (3.1) if and only if $grad_\omega s^\nabla$ is a Killing vector field with respect to g.*

A proof of Proposition 3.2 is given in [4, 15, 22].

3.1. The extremal vector field. We fix a maximal torus T in $Ham(M, \omega)$ and denote by $\mathfrak{t}_\omega$ the finite dimensional space of real-valued smooth functions on M which are hamiltonians with zero mean value of elements of $\mathfrak{t} = Lie(T)$. Denote by Π^T_ω the L^2-orthogonal projection of T-invariant smooth functions on $\mathfrak{t}_\omega$ with respect to the volume form $\frac{\omega^n}{n!}$. Let AK^T_ω be the space of ω-compatible T-invariant almost-complex structures. Given any $J \in AK^T_\omega$, we define $z^T_\omega := \Pi^T_\omega s^\nabla$, where s^∇ is the hermitian scalar curvature of (J, g). Then, we have the following (for more details see [3, 15, 22])

Proposition 3.3. *The potential z^T_ω is independant of (J, g). Furthermore, a ω-compatible T-invariant almost-Kähler metric (J, g) is extremal if and only if*

$$\mathring{s}^\nabla = z^T_\omega,$$

where $\mathring{s}^\nabla$ is the integral zero part of the hermitian scalar curvature s^∇ of (J, g).

Definition 3.4. The vector field $Z^T_\omega := grad_\omega z^T_\omega$ is called the *extremal vector field* relative to T.

Proposition 3.5. *The vector field Z^T_ω is invariant under T-invariant isotopy of ω.*

Remark 3.6. The assumption that $T \subset Ham(M, \omega)$ is a maximal torus is used only in the second part of Proposition 3.3. Indeed, the arguments in [22] show that $z^T_\omega = \Pi^T_\omega s^\nabla$ is independent of (J, g) for any torus $T \subset Ham(M, \omega)$ and Proposition 3.5 still holds true for the corresponding vector field $Z^T_\omega = grad_\omega z^T_\omega$.

4. Almost-Kähler potentials in dimension 4

Let (M, ω) be a compact symplectic manifold of dimension $2n = 4$ and (J, g) a ω-compatible almost-Kähler metric. In order to define the almost-Kähler potentials, we consider the following second order linear differential operator [23] on the smooth sections $\Omega^{J,-}(M)$ of the bundle of J-anti-invariant 2-forms.

$$\begin{array}{rccc} P: & \Omega^{J,-}(M) & \longrightarrow & \Omega^{J,-}(M) \\ & \psi & \longmapsto & (d\delta^g\psi)^{J,-}, \end{array}$$

where δ^g is the codifferential with respect to the metric g.

Lemma 4.1. *P is a self-adjoint strongly elliptic linear operator with kernel the g-harmonic J-anti-invariant 2-forms.*

Proof. The principal symbol of P is given by the linear map $\sigma(P)_\xi(\psi) = -\frac{1}{2}|\xi|^2\psi$, $\forall \xi \in T^*_x(M), \psi \in \Omega^{J,-}(M)$. So, P is a self-adjoint elliptic linear operator with respect to the global inner product $\langle \cdot,\cdot\rangle = \int_M g(\cdot,\cdot)\,\frac{\omega^2}{2}$. Now, let $\psi \in \Omega^{J,-}(M)$ and suppose that $P(\psi) = 0$. Then, $0 = \big\langle (d\delta^g\psi)^{J,-}, \psi\big\rangle = \langle d\delta^g\psi, \psi\rangle = \langle \delta^g\psi, \delta^g\psi\rangle$ which means that $\delta^g\psi = 0$. It follows from (2.5) and since ψ is J-anti-invariant that $*_g\psi = \psi$. Using the relation (2.4), we obtain $d\psi = *_g\delta^g *_g \psi = *_g\delta^g\psi = 0$. Hence, $d\psi = \delta^g\psi = 0$ and thus ψ is a g-harmonic J-anti-invariant 2-form. □

Corollary 4.2. *For $f \in C^\infty(M,\mathbb{R})$, there exist a unique $\psi_f \in \Omega^{J,-}(M)$ orthogonal to the kernel of P such that $(d\delta^g\psi_f)^{J,-} = (dJdf)^{J,-}$.*

Proof. For a smooth real-valued function $f \in C^\infty(M,\mathbb{R})$ and any α in the kernel of P, we have $\big\langle (dJdf)^{J,-}, \alpha\big\rangle = \langle dJdf, \alpha\rangle = \langle Jdf, \delta^g\alpha\rangle = 0$. By a standard result of elliptic theory [6, 29] and since P is self-adjoint, there exist a smooth section $\psi_f \in \Omega^{J,-}(M)$ such that $P(\psi_f) = (dJdf)^{J,-}$. Moreover, ψ_f is unique if one requires ψ_f be orthogonal to the kernel of P.

□

From Corollary 4.2, it follows that, for $f \in C^\infty(M,\mathbb{R})$, the symplectic form $\omega_f = \omega + d(Jdf - \delta^g\psi_f)$ is a J-invariant closed 2-form. Then, the function f is called an *almost-Kähler potential* if the induced symmetric tensor $g_f(\cdot,\cdot) := \omega_f(\cdot, J\cdot)$ is a Riemannian metric. This notion of almost-Kähler potential is closely related but different (in general) from the one defined by Weinkove in [28]. More precisely, if the almost-complex structure J is compatible with a symplectic form $\tilde\omega$ which is cohomologous to ω i.e. $\tilde\omega - \omega = d\alpha$ (for some 1-form α), then the almost-Kähler potential defined by Weinkove is given by the function $\tilde f$ which is uniquely determined (up to the addition of constant) by the Hodge decomposition of α with respect to the (self-adjoint elliptic) *twisted Laplace operator* $\tilde\Delta^c = J\Delta^{\tilde g}J^{-1}$, where $\Delta^{\tilde g}$ is the (Riemannian) Laplace operator with respect to the induced metric $\tilde g(\cdot,\cdot) = \tilde\omega(\cdot, J\cdot)$. In other words, we have the decomposition $\alpha = \alpha_{H^c} + \tilde\Delta^c\tilde{\mathbb{G}}\,\alpha$, where $\tilde{\mathbb{G}}$ is the *Green operator* associated to $\tilde\Delta^c$ and α_{H^c} is the harmonic part of α with respect to $\tilde\Delta^c$. Thus, $\tilde f = -\delta^{\tilde g}J\,\tilde{\mathbb{G}}\alpha$, where $\delta^{\tilde g}$ is the codifferential with respect to the metric $\tilde g$.

Note that $(dJdf)^{J,-} = D^g_{(df)^{\sharp_g}}\omega$ (see e.g. [15]), where $\sharp_g$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by g^{-1}. Hence, in the Kähler case, $(dJdf)^{J,-} = 0$ which implies that $\psi_f = 0$ and thus this almost-Kähler potential coincides with the usual Kähler one.

5. Main Theorem

Let (M,ω) be a compact and connected symplectic manifold of dimension $2n = 4$ and $J_t \in AK_\omega$ be a smooth path of ω-compatible almost-complex structures. We define the following family of differential operators associated to J_t

$$\begin{array}{rccl} P_t: & \Omega^2_0(M) & \longrightarrow & \Omega^2_0(M) \\ & \psi & \longmapsto & \frac{1}{2}\Delta^{g_t}\psi - \frac{1}{4}g_t(\Delta^{g_t}\psi,\omega)\omega, \end{array}$$

where $\Omega^2_0(M)$ is the space of smooth sections of the bundle $\Lambda^2_0(M)$ of primitive 2-forms (pointwise orthogonal to ω) and Δ^{g_t} is the (Riemannian) Laplacian with respect to the metric $g_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$ (here we use the convention $g_t(\omega,\omega) = 2$).

One can easily check that P_t preserves the decomposition

$$\Omega_0^2(M) = \Omega_0^{J,+}(M) \oplus \Omega^{J,-}(M).$$

Furthermore,

$$P_t|_{\Omega^{J_t,-}(M)}(\psi) = (d\delta^{g_t}\psi)^{J_t,-} \text{ and } P_t|_{\Omega_0^{J_t,+}(M)}(\psi) = \frac{1}{2}\Delta^{g_t}\psi.$$

It follows that the kernel of P_t consists of primitive harmonic 2-forms which splits as *anti-selfdual* and J_t-anti-invariant ones so we have

$$\dim \ker(P_t) = b^-(M) + h^-_{J_t},$$

where $h^-_{J_t}$ is introduced by Drăghici–Li–Zhang in [12].

Moreover, $P_t - \frac{1}{2}\Delta^{g_t}$ is a linear differential operator of order 1. Indeed, a direct computation shows that

$$\begin{aligned}\left(P_t - \frac{1}{2}\Delta^{g_t}\right)(\psi) &= \frac{1}{2}\left[\frac{1}{2}\delta^{g_t}\left(D^{g_t}\omega(\psi)\right) - \frac{1}{2}g_t(D^{g_t}\psi, D^{g_t}\omega)\right.\\ &+ \left.\frac{s_{g_t}}{6}g_t(\omega,\psi) - W^{g_t}(\omega,\psi)\right]\omega,\end{aligned}$$

where W^{g_t} stands for the *Weyl tensor* (see e.g. [6]), D^{g_t} (resp. δ^{g_t}) for the Levi-Civita connection (resp. the codifferential) with respect to the metric g_t and s_{g_t} for the Riemannian scalar curvature defined as the trace of the (Riemannian) tensor.

The operator P_t is a self-adjoint strongly elliptic linear operator of order 2. We obtain then a family of Green operators $\mathbb{G}_t$ associated to P_t. If $h^-_{J_t} = h^-_{J_0} = b^+(M)-1$ for $t \in (-\epsilon,\epsilon)$, then $\mathbb{G}_t$ is C^∞ *differentiable* in $t \in (-\epsilon,\epsilon)$ [19, 20], meaning that $\mathbb{G}_t(\psi_t)$ is a smooth family of sections of $\Lambda_0^2(M)$ for any smooth sections ψ_t.

To show Theorem 1.1, we consider the extension of $\mathbb{G}_t$ to the Sobolev spaces $W^{k,p}(M,\Lambda_0^2(M))$ involving derivatives up to k.

Lemma 5.1. *Let $\mathbb{G}_t : \Omega_0^2(M) \to \Omega_0^2(M)$ the family of the above Green operators associated to P_t and suppose that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon,\epsilon)$. Then, the extension of $\mathbb{G}_t$ to Sobolev spaces, still denoted by $\mathbb{G}_t$, defines a C^1 map $\mathbb{G} : (-\epsilon,\epsilon) \times W^{p,k}(M,\Lambda_0^2(M)) \to W^{p,k+2}(M,\Lambda_0^2(M))$*

Proof. Denote by Π_t the L^2-orthogonal projection to the kernel of P_t with respect to $\langle\cdot,\cdot\rangle_{L^2_{g_t}} = \int_M g_t(\cdot,\cdot)\frac{\omega^2}{2}$. We claim that $\mathbb{G}_t\circ\Pi_0$ and $\Pi_0\circ\mathbb{G} : (-\epsilon,\epsilon)\times W^{k,p}(M,\Lambda_0^2(M)) \to W^{k+2,p}(M,\Lambda_0^2(M))$ are C^1 maps. Indeed, let $\{\psi_0^i\}$ be an orthonormal basis of the kernel of P_0 with respect to $\langle\cdot,\cdot\rangle_{L^2_{g_0}}$. Note that ψ_0^i are smooth since P_0 is elliptic.

Then, we have

$$
\begin{aligned}
(\mathbb{G}_t \circ \Pi_0)(\psi) &= \sum_i \left\langle \psi, \psi_0^i \right\rangle_{L^2_{g_0}} \mathbb{G}_t(\psi_0^i), \\
(\Pi_0 \circ \mathbb{G}_t)(\psi) &= \sum_i \left\langle \mathbb{G}_t(\psi), (\psi_0^i)^{J_0,+} + (\psi_0^i)^{J_0,-} \right\rangle_{L^2_{g_0}} \psi_0^i \\
&= \sum_i \left(\int_M -\mathbb{G}_t(\psi) \wedge (\psi_0^i)^{J_0,+} + \mathbb{G}_t(\psi) \wedge (\psi_0^i)^{J_0,-} \right) \psi_0^i \\
&= \sum_i \left(\int_M -\mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,+}\right)^{J_t,+} - \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,+}\right)^{J_t,-} \right. \\
&+ \left. \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,-}\right)^{J_t,+} + \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,-}\right)^{J_t,-} \right) \psi_0^i \\
&= \sum_i \left[\left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,+}\right)^{J_t,+} \right) \right\rangle_{L^2_{g_t}} - \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,+}\right)^{J_t,-} \right) \right\rangle_{L^2_{g_t}} \right. \\
&- \left. \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,-}\right)^{J_t,+} \right) \right\rangle_{L^2_{g_t}} + \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,-}\right)^{J_t,-} \right) \right\rangle_{L^2_{g_t}} \right] \psi_0^i
\end{aligned}
$$

(in the latter equality, we used the fact that $\mathbb{G}_t$ is self-adjoint with respect to $L^2_{g_t}$). The claim follows from the result of Kodaira–Spencer [19, 20].

Denote by $W^{k,p}(M, \Lambda_0^2(M))^{\perp}$ the space of 2-forms in $W^{k,p}(M, \Lambda_0^2(M))$ which are orthogonal to the kernel of P_0 with respect to $L^2_{g_0}$ and consider the map

$$
\begin{array}{rrcl}
\Phi: & (-\epsilon, \epsilon) \times W^{k+2,p}(M, \Lambda_0^2(M))^{\perp} & \longrightarrow & (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_0^2(M))^{\perp} \\
& (t, \psi) & \longmapsto & (t, (Id - \Pi_0) P_t(\psi)),
\end{array}
$$

Clearly, the map Φ is of class C^1 and its differential at $(0, \psi)$ is an isomorphism so by the *inverse function theorem* for Banach spaces there exist a neighboorhood V of $(0, \psi)$ such that $\Phi|_V$ admits an inverse of class C^1. By the The Kodaira–Spencer result [19, 20], the map $\Pi : (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_0^2(M)) \to W^{k,p}(M, \Lambda_0^2(M))$ is C^1 and thus the map $P_t(Id - \Pi_0)\mathbb{G}_t(Id - \Pi_0) = (Id - \Pi_t)(Id - \Pi_0) - P_t(\Pi_0 \mathbb{G}_t)(Id - \Pi_0)$ is clearly C^1 since it is a composition of such operators. Then, the map

$$
\begin{aligned}
\Phi|_V^{-1}\left(t, (Id - \Pi_0) P_t (Id - \Pi_0) \mathbb{G}_t (Id - \Pi_0)\right) &= (t, (Id - \Pi_0)\mathbb{G}_t(Id - \Pi_0)) \\
&= (t, \mathbb{G}_t - \Pi_0 \mathbb{G}_t - \mathbb{G}_t \Pi_0 + \Pi_0 \mathbb{G}_t \Pi_0)
\end{aligned}
$$

is C^1 and hence $\mathbb{G}_t$ is C^1.

□

Proof of Theorem 1.1 Let (M, ω) be a 4-dimensional compact and connected symplectic manifold and T a maximal torus in $Ham(M, \omega)$. Let (J_t, g_t) a smooth family of ω-compatible almost-Kähler metrics in AK_ω^T such that (J_0, g_0) is an extremal Kähler metric.

Following [21], we consider the almost-Kähler deformations

$$\omega_{t,f} = \omega + d(J_t df - \delta^{g_t} \psi_f^t),$$

where f belongs to the Fréchet space $\widetilde{C}_T^{\infty}(M, \mathbb{R})$ of T-invariant smooth functions (with zero integral), which are L^2-orthogonal, with respect to $\frac{\omega^2}{2}$, to $\mathfrak{t}_\omega$ and where the 2-form ψ_f^t is given by Corollary 4.2.

Let $\mathcal{U}$ be an open set in $\mathbb{R} \times \widetilde{C}^{\infty}_T(M, \mathbb{R})$ containing $(0, 0)$ such that the symmetric tensor $g_{t,f}(\cdot, \cdot) := \omega_{t,f}(\cdot, J_t \cdot)$ is a Riemannian metric.

By possibly replacing $\mathcal{U}$ with a smaller open set, we may assume as in [21] that the kernel of the operator $(Id - \Pi^T_\omega) \circ (Id - \Pi^T_{\omega_{t,f}})$ is equal to the kernel of $(Id - \Pi^T_{\omega_{t,f}})$. Indeed, let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{t} = Lie(T)$. Then, the corresponding hamiltonians with zero mean value $\{\xi^1_\omega, \cdots, \xi^n_\omega\}$ resp. $\{\xi^1_{\omega_{t,f}}, \cdots, \xi^n_{\omega_{t,f}}\}$, with respect to ω resp. $\omega_{t,f}$, form a basis of $\mathfrak{t}_\omega$ resp. $\mathfrak{t}_{\omega_{t,f}}$. Let $\{\tilde{\xi}^1_\omega, \cdots, \tilde{\xi}^n_\omega\}$ resp. $\{\tilde{\xi}^1_{\omega_{t,f}}, \cdots, \tilde{\xi}^n_{\omega_{t,f}}\}$ the corresponding orthonormal basis obtained by the Gram–Schmidt procedure. Since $\det\left[\left\langle \tilde{\xi}^i_\omega, \tilde{\xi}^j_{\omega_{t,f}} \right\rangle\right]$ defines a continuous function on $\mathcal{U}$, then we may suppose that $\det\left[\left\langle \tilde{\xi}^i_\omega, \tilde{\xi}^j_{\omega_{t,f}} \right\rangle\right] \neq 0$ on an eventually smaller open set than $\mathcal{U}$ (here $\langle \cdot, \cdot \rangle$ denotes the L^2 product with respect to the volume form $\frac{\omega^2_{t,f}}{2}$). So, if

$$u \in ker\left((Id - \Pi^T_\omega) \circ (Id - \Pi^T_{\omega_{t,f}})\right)$$

then $v \in \mathfrak{t}_\omega \cap \left(\mathfrak{t}_{\omega_{t,f}}\right)^{\perp_{g_{t,f}}}$, where $v = (Id - \Pi^T_{\omega_{t,f}})u$. But the hypothesis

$$\det\left[\left\langle \tilde{\xi}^i_\omega, \tilde{\xi}^j_{\omega_{t,f}} \right\rangle\right] \neq 0$$

implies that $v \equiv 0$ and then $ker\left((Id - \Pi^T_\omega) \circ (Id - \Pi^T_{\omega_{t,f}})\right) = ker(Id - \Pi^T_{\omega_{t,f}})$.

We then consider the map:

$$\begin{array}{llll} \Psi: & \mathcal{U} & \longrightarrow & \mathbb{R} \times \widetilde{C}^{\infty}_T(M, \mathbb{R}) \\ & (t, f) & \longmapsto & \left(t, (Id - \Pi^T_\omega) \circ (Id - \Pi^T_{\omega_{t,f}})(\mathring{s}^{\nabla_{t,f}})\right), \end{array}$$

where $\mathring{s}^{\nabla_{t,f}}$ is the zero integral part of the hermitian scalar curvature $s^{\nabla_{t,f}}$ of $(J_t, g_{t,f})$.

It follows from Proposition 3.3 that $\Psi(t, f) = (t, 0)$ if and only if $(J_t, g_{t,f})$ is an extremal almost-Kähler metric. In particular, $\Psi(0, 0) = (0, 0)$.

Let $\alpha_{t,f} = J_t df - \delta^{g_t}\psi^t_f = J_t df - \delta^{g_t}\mathbb{G}_t\left((dJ_t df)^{J_t,-}\right) = J_t df - \delta^{g_t}\mathbb{G}_t(D^{g_t}_{df^{\sharp_{g_t}}}\omega)$, where $\mathbb{G}_t$ is the Green operator associated to the elliptic operator $P_t : \Omega^{J_t,-}(M) \to \Omega^{J_t,-}(M)$. In order to extend the map Ψ to Sobolev spaces, we give an explicit expression of $(Id - \Pi^T_{\omega_{t,f}})(s^{\nabla_{t,f}})$. A direct computation using (2.2) shows that

$$s^{\nabla_{t,f}} = \Delta^{g_{t,f}} F_{t,f} + g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}), \tag{5.1}$$

where $F_{t,f} = \log\left(\frac{1}{2}\left((1 + g_t(d\alpha_{t,f}, \omega))^2 + 1 - g_t(d\alpha_{t,f}, d\alpha_{t,f})\right)\right)$ satisfying the relation $\omega^2_{t,f} = e^{F_{t,f}}\omega^2$. Then

$$(Id - \Pi^T_{\omega_{t,f}})(s^{\nabla_{t,f}}) = \Delta^{g_{t,f}} F_{t,f} + g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}) - \sum_j \left\langle s^{\nabla_{t,f}}, \tilde{\xi}^j_{\omega_{t,f}} \right\rangle \tilde{\xi}^j_{\omega_{t,f}}. \tag{5.2}$$

Let $\widetilde{W}^{p,k}_T$ be the completion of $\widetilde{C}^{\infty}_T(M, \mathbb{R})$ with respect to the Sobolev norm $\| \cdot \|_{p,k}$ involving derivatives up to order k. We choose p, k such that $pk > 2n$ and the corresponding Sobolev space $\widetilde{W}^{p,k}_T \subset C^3_T(M, \mathbb{R})$ so that all coefficients are $C^0_T(M, \mathbb{R})$. Since $\widetilde{W}^{p,k}_T$ form an algebra relative to the standard multiplication of functions [1], we dedudce from the expression (5.2) that the extension of Ψ to the Sobolev completion of $\widetilde{C}^{\infty}_T(M, \mathbb{R})$ is a map $\Psi^{(p,k)} : \widetilde{\mathcal{U}} \subset \mathbb{R} \times \widetilde{W}^{p,k+4}_T \longrightarrow \mathbb{R} \times \widetilde{W}^{p,k}_T$.

Clearly $\Psi^{(p,k)}$ is a C^1 map (in a small enough open around $(0,0)$). Indeed, it is obtained by a composition of C^1 maps by Lemma 5.1 and (5.2).

As in [21] and using Proposition 3.3, the differential of $\Psi^{(p,k)}$ at $(0,0)$ is given by

$$\left(\mathbf{T}_{(0,0)}\Psi^{(p,k)}\right)(t,f) = \left(t, t\delta^{g_0}\delta^{g_0}h - 2\delta^{g_0}\delta^{g_0}(D^{g_0}df)^{J_0,-}\right),$$

where $h = \frac{d}{dt}|_{t=0}\, g_t$.

The operator $L := \frac{\partial\Psi}{\partial f}|_{(0,0)}$ given by $L(f) = -2\delta^{g_0}\delta^{g_0}(D^{g_0}df)^{J_0,-}$ is called the *Lichnerowicz operator*. It is a 4-th order self-adjoint T-invariant elliptic linear operator leaving invariant $(\mathfrak{t}_\omega)^\perp$ since $L(f) = 0$ for any $f \in \mathfrak{t}_\omega$. By a known result of the elliptic theory [6, 29], we obtain the L^2-orthogonal splitting $\widetilde{C}^\infty_T(M,\mathbb{R}) = ker(L) \oplus Im(L)$. Following the argument in [3, Lemma 4], any $f \in ker(L)$ gives rise to a Killing vector field in the centralizer of $\mathfrak{t} = Lie(T)$. By the maximality of the torus T, $f \in \mathfrak{t}_\omega$. It follows that L is an isomorphism of $\widetilde{C}^\infty_T(M,\mathbb{R})$ and also from $\widetilde{W}^{p,k+4}_T$ to $\widetilde{W}^{p,k}_T$. Thus, $\mathbf{T}_{(0,0)}\Psi^{(p,k)}$ is an isomorphism from $\mathbb{R} \oplus \widetilde{W}^{p,k+4}_T$ to $\mathbb{R} \oplus \widetilde{W}^{p,k}_T$. It follows from the *inverse function theorem* for Banach manifolds that $\Psi^{(p,k)}$ determines an isomorphism from an open neighbourhood V of $(0,0)$ to an open neighbourhood of $(0,0)$. In particular, there exists $\mu > 0$ such that for $|t| < \mu$, $\Psi^{(p,k)}\left(\Psi^{(p,k)}|_V^{-1}(t,0)\right) = (t,0)$. By Sobolev embedding, we can choose a k large enough, such that $\widetilde{W}^{p,k+4}_T \subset \widetilde{C}^6_T(M,\mathbb{R})$. Thus, for $|t| < \mu$, $(J_t, g_{\Psi^{(p,k)}|_V^{-1}(t,0)})$ is an extremal almost-Kähler metric of regularity at least C^4 (so we ensure, in this case, that $grad_\omega s^{\nabla_{t,f}}$ is of regularity C^1).

By Proposition 3.5, the extremal vector field $Z^T_{\omega_{t,f}} = Z^T_\omega$ is smooth for any almost-Kähler metric $(J_t, g_{t,f})$. In particular, for an extremal almost-Kähler metric $(J_t, g_{t,f})$ of regularity C^4, the dual $ds^{\nabla_{t,f}}$ of Z^T_ω with respect to $\omega_{t,f}$ is of regularity C^4, then the hermitian scalar curvature $s^{\nabla_{t,f}}$ of $(J_t, g_{t,f})$ is of regularity C^5. From (5.1), it follows that the hermitian scalar curvature is given by the pair of equations

$$s^{\nabla_{t,f}} - g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}) = \Delta^{g_{t,f}}(u), \tag{5.3}$$

$$e^u = \frac{\omega^2_{t,f}}{\omega^2}. \tag{5.4}$$

From (5.3), using the ellipticity [6] of the (Riemannian) Laplacian $\Delta^{g_{t,f}}$ and since the l.h.s of (5.3) is of Hölder class $C^{3,\beta}$ for any $\beta \in (0,1)$, it follows that u is of class $C^{5,\beta}$. Following [11, 28], the linearisation of the equation (5.4) $(\omega + d\alpha) \wedge d\dot{\alpha} = 0$ together with the constraints $\delta^{g_t}\dot{\alpha} = 0$ and $(d\dot{\alpha})^{J_t,-} = 0$ form a linear elliptic system in $\dot{\alpha}$. Elliptic theory [2, 6] ensures that the almost-Kähler metric $g_{t,f}$ is of class $C^{5,\beta}$ as the volume form and we can prove that any extremal almost-Kähler metric of regularity C^4 is smooth by a bootstraping argument (in the Kähler case see [21]).

We obtain then a smooth family of T-invariant extremal almost-Kähler structures $(J_t, \omega_t = \omega + d\alpha_t)$ defined for $|t| < \mu$. The main theorem follows from the Moser Lemma [25].

Acknowledgements

The author thanks V. Apostolov for his invaluable help and judicious advices and T. Drăghici for valuable suggestions. He is very grateful to P. Gauduchon, P. Guan and T. J. Li for useful discussion.

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