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GEOGRAPHY OF SIMPLY CONNECTED SPIN SYMPLECTIC 4-MANIFOLDS

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Abstract. We present an algorithm that produces new families of closed simply connected spin symplectic 4-manifolds with nonnegative signature that are interesting with respect to the symplectic geography problem. In particular, for each odd integer q satisfying $q \geq 275$, we construct infinitely many pairwise nondiffeomorphic irreducible smooth structures on the topological 4-manifold $q(S^2 \times S^2)$, the connected sum of q copies of $S^2 \times S^2$.

1. Introduction

The geography problem for symplectic 4-manifolds was originally posed by McCarthy and Wolfson in [15], and the first systematic study was carried out by Gompf in [12]. In this paper we will be concerned with the geography problem for closed simply connected spin symplectic 4-manifolds. Our geography problem is the symplectic analogue of the geography problem for closed simply connected spin complex surfaces that was studied by Persson, Peters and Xiao in [19]. The negative signature case has been completely solved by the second author and Szabó in [17], so we will focus our attention only on the nonnegative signature case. For analogous results in the nonspin case, we refer the reader to our joint paper with Hughes [2].

Let M be a closed simply connected spin symplectic 4-manifold with nonnegative signature. Let $e(M)$ and $\sigma(M)$ denote the Euler characteristic and the signature of M, respectively. We define

$$\chi_h(M) = \frac{e(M)+\sigma(M)}{4} \quad \text{and} \quad c_1^2(M) = 2e(M) + 3\sigma(M).$$

In [20], Rohlin showed that

$$c_1^2(M) - 8\chi_h(M) = \sigma(M) \equiv 0 \pmod{16}. \tag{1}$$

Our geography problem asks which ordered pairs of positive integers satisfying (1) can be realized as the pair $(\chi_h(M), c_1^2(M))$ for some closed simply connected spin symplectic 4-manifold M with nonnegative signature.

According to the classification of symmetric bilinear integral forms that are indefinite and unimodular (cf. [16]), the intersection form of such M is of the form

$$pE_8 \oplus qH, \tag{2}$$

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where $p = \sigma(M)/8$ is a nonnegative even integer and $q = b_2^-(M)$ is a positive odd integer. Here,

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The famous 11/8 conjecture, originally due to Y. Matsumoto (cf. [14, 10]), speculates that

$$b_2(M) \geq \frac{11}{8}|\sigma(M)|,$$

or equivalently $q \geq \frac{3}{2}p$ in (2). To state our results, it will be convenient to introduce the following terminology.

Definition 1. We say that a 4-manifold M has ∞^2-*property* if there exist infinitely many pairwise nondiffeomorphic irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to M. We also say that a symmetric bilinear form has ∞^2-*property* if it is the intersection form of infinitely many pairwise nondiffeomorphic simply connected irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic simply connected irreducible nonsymplectic 4-manifolds.

A spin 4-manifold cannot contain any surface with odd self-intersection and hence is minimal. Moreover, the irreducibility condition always holds in our situation.

Lemma 2. *Every simply connected spin symplectic* 4-*manifold is irreducible.*

Proof. Let M be a simply connected spin symplectic 4-manifold. Suppose $M = M_1 \# M_2$ is a connected sum of two smooth 4-manifolds M_1 and M_2. Then both M_1 and M_2 are simply connected and the intersection forms of M_1 and M_2 are both even.

If $b_2^+(M_1)$ and $b_2^+(M_2)$ are both strictly positive, then the Seiberg-Witten invariant of $M_1 \# M_2$ is trivial (cf. [22]). But since $b_2^+(M) = b_2^+(M_1) + b_2^+(M_2) > 1$, the Seiberg-Witten invariant of the symplectic 4-manifold M cannot be trivial by Taubes's theorem in [21]. This contradiction shows that one of $b_2^+(M_1)$ and $b_2^+(M_2)$ is 0.

Without loss of generality, assume $b_2^+(M_1) = 0$. If $b_2(M_1) = b_2^-(M_1) > 0$, then the intersection form of M_1 is a nontrivial negative definite form, so by Donaldson's theorem in [6], it is equivalent to the diagonal form $b_2(M_1)\langle -1 \rangle$. But this contradicts the fact that the intersection form of M_1 is even. Thus we conclude that $b_2(M_1) = 0$. Since M_1 is simply connected, M_1 must be homeomorphic to S^4 by Freedman's theorem in [9]. □

Given a nonnegative even integer p, it is now well known (cf. [18]) that $pE_8 \oplus qH$ has ∞^2-property when the odd integer q is larger than some constant that depends on p.

Definition 3. For an even integer $p \geq 0$, let Λ_p denote the smallest positive odd integer such that $pE_8 \oplus qH$ has ∞^2-property for every odd integer $q \geq \Lambda_p$.

From the 11/8 conjecture, we immediately obtain a conjectural lower bound $\Lambda_p \geq \frac{3}{2}p$. It may not be too optimistic to make the following conjecture.

Conjecture 4. Λ_p is the smallest positive odd integer that is greater than or equal to $\frac{3}{2}p$.

Unfortunately, Conjecture 4, as well as the 11/8 conjecture, seems quite difficult to tackle with existing techniques. More modestly, the main goal of this paper is to provide new upper bounds on Λ_p that are easily computable. We refer to Corollary 10 below for the precise statement. In particular, we will prove the following in Section 4.

Theorem 5. *Let $p \geq 0$ be an even integer. If $n \geq 3$ is any odd integer satisfying $p \leq \frac{1}{3}n(n^2-1) - 2$, then $\Lambda_p \leq 10n^3 + 2n - 1 - 10p$.*

For example, Theorem 5 implies that $\Lambda_0 \leq 275$, i.e., $q(S^2 \times S^2)$ has ∞^2-property for every odd integer $q \geq 275$. This is a significant improvement over the upper bound $\Lambda_0 \leq 2N + 1$ in [18], where $N = 267145kx^2 + 70$ for some sufficiently large integers k and x which were not explicitly computed.

2. General construction algorithm

Given a surface bundle over a surface with nice properties, we will show how it can be used to construct closed simply connected spin symplectic 4-manifolds with nonnegative signature.

Let Σ_b be a genus b surface. Let X be a closed 4-manifold that is the total space of a genus f surface bundle over Σ_b. Assume that X is spin and that $\sigma(X) = 16s$. Further assume that X has a section $\Sigma_b \to X$ whose image is a genus b surface in X of self-intersection $-2t$ for some integer t. By symplectically resolving the double point of the union of a fiber Σ_f and the image of a section, we obtain a symplectic submanifold Σ_{f+b} in X of genus $f+b$ and self-intersection $2-2t$.

Suppose that r is a positive integer satisfying

$$1 - t \leq r \leq \min\{s,\ f + b + 1 - t\}.$$

Let K be a fibered knot of genus $g(K) = f + b + 1 - t - r$ in S^3. Let $E(2r)_K$ denote the homotopy elliptic surface with signature $-16r$ that was constructed by Fintushel and Stern in [7]. A sphere section of $E(2r)$ gives rise to a symplectic submanifold $S_{g(K)}$ of genus $g(K)$ and self-intersection $-2r$ in $E(2r)_K$. By symplectically resolving $r+t-1$ double points of the union of $r+t-1$ fibers and $S_{g(K)}$, we obtain a symplectic submanifold Σ'_{f+b} of genus $f+b$ and self-intersection $2t-2$.

Let $Z = X\#_{\Sigma_{f+b}=\Sigma'_{f+b}}E(2r)_K$ be a generalized fiber sum (cf. [12, 15]) of X and $E(2r)_K$ along symplectic submanifolds Σ_{f+b} and Σ'_{f+b}. Then Z is a spin symplectic

4-manifold satisfying

$$\begin{aligned}\sigma(Z) &= \sigma(X) + \sigma(E(2r)_K) = 16(s-r) \geq 0,\\ e(Z) &= e(X) + e(E(2r)_K) - 2e(\Sigma_{f+b})\\ &= 4(f-1)(b-1) + 24r + 4(f+b-1) = 4fb + 24r,\\ \chi_h(Z) &= fb + 4s + 2r,\\ c_1^2(Z) &= 8fb + 48s.\end{aligned}$$

Note that $c_1^2(Z)$ is independent of r.

Lemma 6. *Z is simply connected and has ∞^2-property* (*cf. Definition* 1).

Proof. Let $\nu\Sigma_{f+b}$ and $\nu\Sigma'_{f+b}$ denote the open tubular neighborhoods of Σ_{f+b} and Σ'_{f+b} inside X and $E(2r)_K$, respectively, such that

$$Z = (X \setminus \nu\Sigma_{f+b}) \cup (E(2r)_K \setminus \nu\Sigma'_{f+b}). \tag{3}$$

From the homotopy long exact sequence for fibration, we have

$$\pi_1(\Sigma_f) \xrightarrow{\iota_*} \pi_1(X) \longrightarrow \pi_1(\Sigma_b) \longrightarrow 1,$$

where $\iota : \Sigma_f \hookrightarrow X$ is the inclusion of a fiber. Thus $\pi_1(X)$ is generated by $\iota_*(\pi_1(\Sigma_f))$ and the image of $\pi_1(\Sigma_b)$ under a section. It follows that

$$\pi_1(X)/\langle\pi_1(\Sigma_{f+b})\rangle = 1, \tag{4}$$

where $\langle\pi_1(\Sigma_{f+b})\rangle$ is the normal subgroup of $\pi_1(X)$ that is generated by the image of $\pi_1(\Sigma_{f+b})$ under the inclusion.

Let $\partial(\nu\Sigma_{f+b})$ denote the boundary of $\nu\Sigma_{f+b}$, which is a circle bundle over Σ_{f+b} with Euler number $2-2t$. It is well known (cf. Proposition 10.4 in [8]) that

$$\pi_1(\partial(\nu\Sigma_{f+b})) = \langle \alpha_j, \beta_j, \mu \mid \textstyle\prod_{j=1}^{f+b}[\alpha_j, \beta_j] = \mu^{2-2t},\ \alpha_j\mu\alpha_j^{-1} = \mu,\ \beta_j\mu\beta_j^{-1} = \mu\rangle,$$

where the index j ranges over $1, \dots, f+b$. Here, μ is represented by a fiber circle which is a meridian of Σ_{f+b}, and α_j, β_j are the parallel push-offs of the standard generators of $\pi_1(\Sigma_{f+b})$. Let $\langle\pi_1(\partial(\nu\Sigma_{f+b}))\rangle$ denote the normal subgroup of $\pi_1(X \setminus \nu\Sigma_{f+b})$ that is generated by the image of $\pi_1(\partial(\nu\Sigma_{f+b}))$ under the inclusion of $\partial(\nu\Sigma_{f+b}) = \partial(X \setminus \nu\Sigma_{f+b})$ into $X \setminus \nu\Sigma_{f+b}$. Then $\langle\pi_1(\partial(\nu\Sigma_{f+b}))\rangle$ is normally generated by the meridians of Σ_{f+b} and the image of $\pi_1(\Sigma_{f+b})$ under the push-off homomorphism. It now follows from Seifert-Van Kampen theorem and (4) that $\pi_1(X \setminus \nu\Sigma_{f+b})/\langle\pi_1(\partial(\nu\Sigma_{f+b}))\rangle = 1$.

Next recall from [7] that $\pi_1(E(2r)_K) = 1$. Since Σ'_{f+b} transversely intersects once a topological sphere in $E(2r)_K$ coming from a cusp fiber of $E(2r)$, a meridian of Σ'_{f+b} bounds a disk and hence $\pi_1(E(2r)_K \setminus \nu\Sigma'_{f+b}) = 1$. By applying Seifert-Van Kampen theorem to (3), we conclude that

$$\pi_1(Z) = \frac{\pi_1(X \setminus \nu\Sigma_{f+b})}{\langle\pi_1(\partial(\nu\Sigma_{f+b}))\rangle} = 1$$

as well.

To obtain infinite families of pairwise nondiffeomorphic 4-manifolds that are homeomorphic to Z, we recall from [13] that $E(2)$ contains three disjoint copies of Gompf nucleus (cf. [11]). We view $E(2r)$ as the fiber sum of r copies of $E(2)$ so that $E(2r)_K$ contains at least two nuclei N_1 and N_2 that are disjoint from Σ'_{f+b}. By performing a second knot surgery (cf. [7]) in one of these two nuclei, say N_1, inside Z, we obtain an irreducible 4-manifold $Z_{K'}$ that is homeomorphic to Z. By varying our choice of the knot K', we can realize infinitely many pairwise nondiffeomorphic 4-manifolds, either symplectic or nonsymplectic. □

Note that by the classification of indefinite unimodular integral forms (cf. [16]), the intersection form of Z is given by

$$2(s-r)E_8 \oplus (2fb+20r-8s-1)H. \tag{5}$$

3. New families of simply connected spin 4-manifolds

In this section, we apply the construction algorithm in the previous section to some concrete surface bundles that are found in the literature.

Example 7. For any pair of integers $g, n \geq 2$, let $X_{g,n}$ be a genus gn surface bundle over a genus $g(g-1)n^{2g-2}+1$ surface $\widetilde{D}$ in Theorem 1.1 of [4]. Recall from [4] that $\varphi_{g,n} : X_{g,n} \to \widetilde{D} \times C$ is an n-fold cyclic branched cover whose branch locus is the union of two disjoint surfaces Γ_1 and Γ_2 in $\widetilde{D} \times C$. Here, C is a genus g surface, and each Γ_i is the graph of a map $\widetilde{D} \to C$ such that the homology class $[\Gamma_1] - [\Gamma_2]$ is divisible by n. Our surface bundle is given by the composition $\mathrm{pr}_1 \circ \varphi_{g,n} : X_{g,n} \to \widetilde{D} \times C \to \widetilde{D}$, where pr_1 denotes the projection onto the first factor. Note that $X_{g,n}$ has two disjoint sections whose images are $\varphi_{g,n}^{-1}(\Gamma_i)$, $i = 1, 2$, both with self-intersection equal to $\frac{1}{n}[\Gamma_i]^2 = -2g(g-1)n^{2g-3}$. We have

$$\sigma(X_{g,n}) = \frac{4}{3} g(g-1)(n^2-1)n^{2g-3},$$

and by the work of Brand [3], we have

$$\begin{aligned} w_2(X_{g,n}) &\equiv \varphi_{g,n}^{*}\Big(w_2(\widetilde{D} \times C) + \frac{n-1}{n} PD([\Gamma_1] - [\Gamma_2])\Big) \\ &\equiv \frac{n-1}{n}\, \varphi_{g,n}^{*}\big(PD([\Gamma_1] - [\Gamma_2])\big) \pmod 2, \end{aligned}$$

where PD denotes the Poincaré dual. If n is odd, then $w_2(X_{g,n}) \equiv 0 \pmod 2$, and consequently $X_{g,n}$ is spin.

Applying the algorithm in Section 2, for any triple of positive integers g, n and r satisfying

$$\begin{aligned} & g \geq 2, \quad n \geq 3, \quad n \equiv 1 \pmod 2, \\ & r \leq \frac{1}{12} g(g-1)(n^2-1)n^{2g-3}, \\ & r \leq 2 + gn + g(g-1)(n-1)n^{2g-3}, \end{aligned}$$

we get a closed simply connected spin symplectic 4-manifold $Z = Z^r_{g,n}$ having ∞^2-property and satisfying

$$\begin{aligned}
\sigma(Z^r_{g,n}) &= \frac{4}{3}g(g-1)(n^2-1)n^{2g-3} - 16r,\\
e(Z^r_{g,n}) &= 4gn\big(g(g-1)n^{2g-2}+1\big) + 24r,\\
\chi_h(Z^r_{g,n}) &= gn + \frac{1}{3}g(g-1)n^{2g-3}\big((3g+1)n^2-1\big) + 2r,\\
c_1^2(Z^r_{g,n}) &= 8gn + 4g(g-1)n^{2g-3}\big((2g+1)n^2-1\big).
\end{aligned}$$

Table 1 below lists the smallest $Z^r_{g,n}$'s with signature lying between 0 and 160. In particular, by Freedman's classification theorem in [9], $Z^4_{2,3}$ is homeomorphic to $275(S^2 \times S^2)$. Since $Z^4_{2,3}$ is irreducible, it is not diffeomorphic to $275(S^2 \times S^2)$.

TABLE 1.

	$Z^4_{2,3}$	$Z^3_{2,3}$	$Z^2_{2,3}$	$Z^1_{2,3}$	$Z^{16}_{2,5}$	$Z^{15}_{2,5}$	$Z^{14}_{2,5}$	$Z^{13}_{2,5}$	$Z^{12}_{2,5}$	$Z^{11}_{2,5}$	$Z^{10}_{2,5}$
σ	0	16	32	48	64	80	96	112	128	144	160
e	552	528	504	480	2424	2400	2376	2352	2328	2304	2280
χ_h	138	136	134	132	622	620	618	616	614	612	610
c_1^2	1104	1104	1104	1104	5040	5040	5040	5040	5040	5040	5040

We note that the ratio $c_1^2(Z^r_{g,n})/\chi_h(Z^r_{g,n})$ is a decreasing function of r and an increasing function of n. It follows that

$$\frac{c_1^2(Z^r_{g,n})}{\chi_h(Z^r_{g,n})} < \lim_{n\to\infty} \frac{c_1^2(Z^1_{g,n})}{\chi_h(Z^1_{g,n})} = \frac{4(2g+1)}{\frac{1}{3}(3g+1)} \le \frac{60}{7} \approx 8.5714286.$$

Hence $Z^r_{g,n}$'s lie below the Bogomolov-Miyaoka-Yau (BMY) line, $c_1^2 = 9\chi_h$.

Example 8. For each integer $n \ge 2$, let X_n be a genus $3n$ surface bundle over a genus $2n^2+1$ surface $\widetilde{B}$ in Theorem 3.1 of [5]. Recall from [5] that $\varphi_n : X_n \to \widetilde{B} \times B$ is an n-fold cyclic branched cover whose branch locus is the union of two disjoint surfaces Γ_π and Γ'_π in $\widetilde{B} \times B$. Here, B is a genus 3 surface, and Γ_π and Γ'_π are the graphs of certain maps $\widetilde{B} \to B$ such that the homology class $D = [\Gamma_\pi] - [\Gamma'_\pi]$ is divisible by n. Our surface bundle is then given by the composition $\mathrm{pr}_1 \circ \varphi_n : X_n \to \widetilde{B} \times B \to \widetilde{B}$, where pr_1 denotes the projection onto the first factor. Note that X_n has two disjoint sections whose images are $\varphi_n^{-1}(\Gamma_\pi)$ and $\varphi_n^{-1}(\Gamma'_\pi)$, both with self-intersection equal to $\frac{1}{2n}D^2 = -4n$. We have

$$\begin{aligned}
\sigma(X_n) &= \frac{8}{3}n(n-1)(n+1),\\
w_2(X_n) &\equiv \frac{n-1}{n}\,\varphi_n^*\big(PD([\Gamma_\pi]-[\Gamma'_\pi])\big) \pmod 2.
\end{aligned}$$

If n is odd, then $w_2(X_n) \equiv 0 \pmod 2$, and hence X_n is spin.

Applying the algorithm in Section 2, for any pair of positive integers n and r satisfying

$$n \geq 3, \quad n \equiv 1 \pmod 2,$$
$$r \leq \min\left\{\frac{1}{6}n(n-1)(n+1),\, 2n^2+n+2\right\},$$

we get a closed simply connected spin symplectic 4-manifold $Z = V_n^r$ having ∞^2-property and satisfying

$$\sigma(V_n^r) = \frac{8}{3}n(n-1)(n+1) - 16r,$$
$$e(V_n^r) = 12n(2n^2+1) + 24r,$$
$$\chi_h(V_n^r) = \frac{1}{3}n(20n^2+7) + 2r,$$
$$c_1^2(V_n^r) = 8n(7n^2+2).$$

Table 2 below lists the smallest V_n^r's with signature lying between 0 and 160. In particular, V_3^4 is homeomorphic but not diffeomorphic to $389(S^2 \times S^2)$.

Table 2.

	V_3^4	V_3^3	V_3^2	V_3^1	V_5^{16}	V_5^{15}	V_5^{14}	V_5^{13}	V_5^{12}	V_5^{11}	V_5^{10}
σ	0	16	32	48	64	80	96	112	128	144	160
e	780	756	732	708	3444	3420	3396	3372	3348	3324	3300
χ_h	195	193	191	189	877	875	873	871	869	867	865
c_1^2	1560	1560	1560	1560	7080	7080	7080	7080	7080	7080	7080

We note that the ratio $c_1^2(V_n^r)/\chi_h(V_n^r)$ is a decreasing function of r and an increasing function of n. For fixed r, we have

$$\lim_{n\to\infty} \frac{c_1^2(V_n^r)}{\chi_h(V_n^r)} = 8.4$$

and thus V_n^r's lie well below the BMY line $c_1^2 = 9\chi_h$.

4. Geography of spin 4-manifolds

The following theorem is a spin analogue of Theorem 5.3 in [1] and Theorem 6.2 in [12].

Theorem 9. *Let Z be a closed spin symplectic 4-manifold that contains a symplectic torus T of self-intersection 0. Let νT be a tubular neighborhood of T and $\partial(\nu T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(\nu T)) \to \pi_1(Z \setminus \nu T)$ induced by the inclusion is trivial. Then for any pair of integers (χ, c) satisfying*

$$\chi \geq 1, \quad 0 \leq c \leq 8\chi \quad \textit{and} \quad c - 8\chi \equiv 0 \pmod{16}, \tag{6}$$

there exists a closed spin symplectic 4-manifold Y with $\pi_1(Y) = \pi_1(Z)$,

$$\chi_h(Y) = \chi_h(Z) + \chi \quad \textit{and} \quad c_1^2(Y) = c_1^2(Z) + c.$$

Proof. Recall from Proposition 3.2 in [17] that for any pair of integers $n \geq 0$ and $\rho \geq 0$, there exists a closed simply connected spin symplectic 4-manifold $M_n(\rho)$ satisfying

$$\chi_h(M_n(\rho)) = 2(\rho+1) + n \quad \text{and} \quad c_1^2(M_n(\rho)) = 8n.$$

Moreover, each $M_n(\rho)$ contains a Gompf nucleus of $E(2)$ (cf. proof of Theorem 1.3 in [17]), and hence there is a symplectic torus $F \subset M_n(\rho)$ of self-intersection 0 such that $\pi_1(M_n(\rho) \setminus \nu F) = 1$.

For each integer $\chi \geq 1$, recall from the proof of Theorem 5.3 in [1] that there exists a closed spin symplectic 4-manifold W with the following properties.

(i) $\chi_h(W) = \chi$ and $c_1^2(W) = 8\chi$.

(ii) W contains a symplectic torus T' of self-intersection 0 such that the complement $W \setminus T'$ does not contain any symplectic sphere of self-intersection -1.

(iii) $\pi_1(W \setminus \nu T')/G = 1$, where G is the normal subgroup of $\pi_1(W \setminus \nu T')$ that is generated by the image of the inclusion induced homomorphism $\pi_1(\partial(\nu T')) \to \pi_1(W \setminus \nu T')$.

Given a pair of integers (χ, c) satisfying (6), we write

$$c - 8\chi = -16(\rho+1)$$

for some integer $\rho \geq -1$. If $\rho = -1$, then let Y be the generalized fiber sum $Z \#_{T=T'} W$. If $\rho \geq 0$, then let Y be the generalized fiber sum $Z \#_{T=F} M_n(\rho)$, where $n = \chi - 2(\rho+1)$.

An easy application of Seifert-Van Kampen theorem gives $\pi_1(Y) = \pi_1(Z \setminus \nu T) = \pi_1(Z)$. Other properties of Y can also be immediately verified. □

From the above theorem, we can deduce the following upper bound on Λ_p (cf. Definition 3).

Corollary 10. *Let X be a spin 4-manifold that is the total space of a genus f surface bundle over a genus b surface. Assume that $\sigma(X) = 16s$, and X has a section whose image is a genus b surface of self-intersection $-2t$ for some integer t. Let r be a positive integer satisfying*

$$1 - t \leq r \leq \min\{s,\ f + b + 1 - t\}.$$

If p and q are nonnegative integers satisfying

$$p \equiv 0 \pmod 2, \quad 0 \leq p \leq 2(s-r),$$
$$q \equiv 1 \pmod 2, \quad q \geq 2fb + 12s - 1 - 10p,$$

then the symmetric bilinear form $pE_8 \oplus qH$ has ∞^2-property (cf. Definition 1) and

$$\Lambda_p \leq 2fb + 12s - 1 - 10p.$$

Proof. Let $Z = X \#_{\Sigma_{f+b} = \Sigma'_{f+b}} E(2r)_K$ be the simply connected spin symplectic 4-manifold in Section 2. Recall from the proof of Lemma 6 that Z contains two copies of Gompf nucleus denoted by N_1 and N_2. Let T be a symplectic torus fiber in the second nucleus N_2. Since $\pi_1(Z \setminus \nu T) = 1$, we can apply Theorem 9 to Z, and obtain a closed simply connected spin symplectic 4-manifold Y with intersection form

$$\left(2(s-r) + \frac{c - 8\chi}{8}\right) E_8 \oplus (2fb + 20r - 8s - 1 + 10\chi - c)\, H. \tag{7}$$

If $p = 2(s-r)$ and $q = 2fb+20r-8s-1$, then our claim follows from (5) and Lemma 6. Otherwise we set

$$\chi = \frac{1}{2}(q - 2fb - 4r - 8s + 1 + 8p),$$
$$c = 4(q - 2fb - 12s + 1 + 10p),$$

which turns (7) into $pE_8 \oplus qH$.

By performing a knot surgery inside the nucleus $N_1 \subset Y$, we obtain an irreducible 4-manifold $Y_{K'}$ that is homeomorphic to Y. By varying our choice of the knot K', we can again realize infinitely many pairwise nondiffeomorphic 4-manifolds, either symplectic or nonsymplectic. □

Finally we are ready to prove Theorem 5 from the introduction.

Proof of Theorem 5. We apply Corollary 10 to surface bundles $X_{g,n}$ in Example 7 with $g = 2$, $r = 1$ and odd $n \geq 3$. Since $f = t = 2n$, $b = 2n^2 + 1$ and $s = \frac{1}{6}n(n^2-1)$, we conclude that $pE_8 \oplus qH$ has ∞^2-property when

$$p \equiv 0 \pmod 2, \quad 0 \leq p \leq \frac{1}{3}n(n^2-1) - 2,$$
$$q \equiv 1 \pmod 2, \quad q \geq 10n^3 + 2n - 1 - 10p.$$ □

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