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LOW FREQUENCY RESOLVENT ESTIMATES FOR LONG RANGE PERTURBATIONS OF THE EUCLIDEAN LAPLACIAN

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Abstract. Let P be a long range metric perturbation of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geq 3$. We prove that the following resolvent estimate holds:

$$\| \langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta} \| \lesssim 1 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, \, |z| < 1,$$

if $\alpha, \beta > 1/2$ and $\alpha + \beta > 2$. The above estimate is false for the Euclidean Laplacian in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.

1. Introduction

There are now many results dealing with the low frequency behavior of the resolvent of Schrödinger type operators. The methods used to obtain these results are various: one can apply the Fredholm theory to study perturbations by a potential (see *e.g.* [7]) or a short range metric (see *e.g.* [9]). The resonance theory is also useful to treat compactly supported perturbations of the flat case (see *e.g.* [3]). Using the general Mourre theory, one can obtain limiting absorption principles at the thresholds (see *e.g.* [5] or [8]). The pseudo-differential calculus of Melrose allows to describe the kernel of the resolvent at low energies for compactifiable manifolds (see *e.g.* [6]). Concerning the long range case, Bouclet [2] has obtained a uniform control of the resolvent for perturbations in divergence form. We refer to his article and to [4] for a quite exhaustive list of previous results for perturbations of the Euclidean Laplacian.

On $\mathbb{R}^d$ with $d \geq 3$, we consider the following operator

$$P = -b \operatorname{div}(G \nabla b) = - \sum_{i,j=1}^{d} b(x) \frac{\partial}{\partial x_i} G_{i,j}(x) \frac{\partial}{\partial x_j} b(x), \tag{1}$$

where $b(x) \in C^\infty(\mathbb{R}^d)$ and $G(x) \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ is a real symmetric $d \times d$ matrix. The C^∞ hypothesis is made mostly for convenience, much weaker regularity could actually be considered. We make an ellipticity assumption:

$$\exists C > 0, \ \forall x \in \mathbb{R}^d \qquad G(x) \geq C I_d \ \text{ and } \ b(x) \geq C, \tag{H1}$$

I_d being the identity matrix. We also assume that P is a long range perturbation of the Euclidean Laplacian:

$$\exists \rho > 0, \ \forall \alpha \in \mathbb{N}^d \qquad |\partial_x^\alpha (G(x) - I_d)| + |\partial_x^\alpha (b(x) - 1)| \lesssim \langle x \rangle^{-\rho - |\alpha|}. \tag{H2}$$

In particular, if $b = 1$, we are concerned with an elliptic operator in divergence form $P = -\operatorname{div}(G \nabla)$. On the other hand, if $G = (g^2 g^{i,j}(x))_{i,j}$, $b = (\det g^{i,j})^{1/4}$, $g =$

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$\frac{1}{b}$, then the above operator is unitarily equivalent to the Laplace–Beltrami $-\Delta_{\mathfrak{g}}$ on $(\mathbb{R}^d, \mathfrak{g})$ with metric

$$\mathfrak{g} = \sum_{i,j=1}^{d} g_{i,j}(x)\, dx^i\, dx^j,$$

where $(g_{i,j})_{i,j}$ is inverse to $(g^{i,j})_{i,j}$ and the unitary transform is just multiplication by g.

Theorem 1. *Let P be of the form* (1) *in $\mathbb{R}^d$ with $d \geq 3$. Assume* (H1) *and* (H2).

i) For all $\varepsilon > 0$, we have

$$\big\| \langle x \rangle^{-1/2-\varepsilon} (\sqrt{P} - z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \big\| \lesssim 1, \tag{2}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

ii) For all $\varepsilon > 0$, we have

$$\big\| \langle x \rangle^{-1/2-\varepsilon} (P - z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \big\| \lesssim |z|^{-1/2}, \tag{3}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

iii) For all $\alpha, \beta > 1/2$ with $\alpha + \beta > 2$, we have

$$\big\| \langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta} \big\| \lesssim 1, \tag{4}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

Remark 2. *i) The estimate* (4) *is not far from optimal. Indeed, this estimate is false for the Euclidean Laplacian $-\Delta$ in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.*

ii) One can interpret (4) *in the following way: one needs a $\langle x \rangle^{-1/2}$ on the left and on the right to assure that the resolvent is continuous on $L^2(\mathbb{R}^d)$ and one needs an additional $\langle x \rangle^{-1}$ (distributed, as we want, among the left and the right) to guarantee that its norm is uniform with respect to z.*

iii) By interpolation of (3) *and* (4), *for $\alpha, \beta > 1/2$ with $\alpha + \beta \leq 2$, one obtains estimates like* (4) *with $|z|^{-1+\frac{\alpha+\beta}{2}-\varepsilon}$ on the right hand side.*

iv) In dimension 1, the kernel of $(-\Delta - z)^{-1}$ is given by $\frac{i e^{i\sqrt{z}|x-y|}}{2\sqrt{z}}$. In particular, this operator satisfies (3) *but not* (4) *(for any α, β). Therefore it seems that* (3) *is more general than* (4). *It could perhaps be possible to prove* (3) *in lower dimensions (at least, in dimension 2 and when P is of divergence form $P = -\operatorname{div}(G\nabla)$).*

v) For large z, the estimate (3) *coincides with the high energy estimate in the non-trapping case. In particular, if we suppose in addition a non trapping condition for P, then* (2) *and* (3) *hold uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$.*

The proof of the above theorem is based on the low frequency estimates of [1]. Concerning the square root of P, they are used to treat the wave equation. Note that in [1] they are formulated for the Laplace–Beltrami operator $-\Delta_{\mathfrak{g}}$, but they obviously hold for the operators studied in the present paper. Essentially, we will show that (2)⇒(3)⇒(4).

2. Proof of the results

We begin by recalling some results of [1]. For $\lambda \geq 1$, we set

$$\mathcal{A}_\lambda = \varphi(\lambda P) A_0 \varphi(\lambda P),$$

where

$$A_0 = \frac{1}{2}(xD + Dx), \quad D(A_0) = \big\{u \in L^2(\mathbb{R}^d);\ A_0 u \in L^2(\mathbb{R}^d)\big\},$$

is the generator of dilations and $\varphi \in C_0^\infty(]0,+\infty[;[0,+\infty[)$ satisfies $\varphi(x) > 1$ on some open bounded interval $I = [1-\widetilde{\varepsilon}, 1+\widetilde{\varepsilon}]$, $0 < \widetilde{\varepsilon} < 1$ sufficiently small. As usual, we define the multi-commutators $\operatorname{ad}_A^j B$ inductively by $\operatorname{ad}_A^0 B = B$ and $\operatorname{ad}_A^{j+1} B = [A, \operatorname{ad}_A^j B]$. We recall [1, Proposition 3.1]:

Proposition 3. *i) We have $(\lambda P)^{1/2} \in C^2(\mathcal{A}_\lambda)$. The commutators $\operatorname{ad}^j_{\mathcal{A}_\lambda}(\lambda P)^{1/2}$, $j = 1, 2$, can be extended to bounded operators and we have, uniformly in $\lambda \geq 1$,*

$$\big\| [\mathcal{A}_\lambda, (\lambda P)^{1/2}] \big\| \lesssim 1,$$

$$\big\| \operatorname{ad}^2_{\mathcal{A}_\lambda}(\lambda P)^{1/2} \big\| \lesssim \begin{cases} 1 & \rho > 1, \\ \lambda^\delta & \rho \leq 1, \end{cases}$$

where $\delta > 0$ can be chosen arbitrary small.

ii) For λ large enough, we have the following Mourre estimate:

$$\mathbb{1}_I(\lambda P)\big[i(\lambda P)^{1/2}, \mathcal{A}_\lambda\big]\mathbb{1}_I(\lambda P) \geq \frac{\sqrt{\inf I}}{2}\mathbb{1}_I(\lambda P).$$

iii) For $0 \leq \mu \leq 1$ and $\psi \in C_0^\infty(]0,+\infty[)$, we have

$$\big\| \langle \mathcal{A}_\lambda \rangle^\mu \psi(\lambda P) \langle x \rangle^{-\mu} \big\| \lesssim \lambda^{-\mu/2+\delta},$$

for all $\delta > 0$.

We will also need [1, Lemma B.12]:

Lemma 4. *Let $\chi \in C_0^\infty(\mathbb{R})$ and $\beta, \gamma \geq 0$ with $\gamma + \beta/2 \leq d/4$. Then, for all $\delta > 0$, we have*

$$\big\| \langle x \rangle^\beta \chi(\lambda P) u \big\| \lesssim \lambda^{-\gamma+\delta} \big\| \langle x \rangle^{\beta+2\gamma} u \big\|,$$

uniformly in $\lambda \geq 1$.

By Mourre theory (see Theorem 2.2 and Remark 2.3 of [1] for example) and Proposition 3, we obtain the following limiting absorption principle:

$$\sup_{\operatorname{Re} z \in I,\, \operatorname{Im} z \neq 0} \big\| \langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon} ((\lambda P)^{1/2} - z)^{-1} \langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon} \big\| \lesssim \lambda^\delta, \tag{5}$$

for all $\varepsilon, \delta > 0$. This entails the following

Lemma 5. *For $\Psi \in C_0^\infty(]0,+\infty[)$ and $\varepsilon > 0$, we have*

$$\big\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(\sqrt{P} - \lambda^{-1/2} z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \big\| \lesssim 1, \tag{6}$$

$$\big\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(P - \lambda^{-1} z^2)^{-1} \langle x \rangle^{-1/2-\varepsilon} \big\| \lesssim \frac{\sqrt{\lambda}}{|z|}, \tag{7}$$

uniformly in $\lambda \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Re} z \in I$.

Proof. Let $\widetilde{\Psi} \in C_0^\infty(]0,+\infty[)$ be such that $\Psi\widetilde{\Psi} = \Psi$.

To prove the first identity, we write

$$\begin{aligned}
&\big\| \langle x\rangle^{-1/2-\varepsilon}\Psi(\lambda P)(P^{1/2}-\lambda^{-1/2}z)^{-1}\langle x\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad \lesssim \big\|\langle x\rangle^{-1/2-\varepsilon}\Psi(\lambda P)\langle \mathcal{A}_\lambda\rangle^{1/2+\varepsilon}\big\| \big\|\langle \mathcal{A}_\lambda\rangle^{-1/2-\varepsilon}(P^{1/2}-\lambda^{-1/2}z)^{-1}\langle \mathcal{A}_\lambda\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad\quad \times \big\|\langle \mathcal{A}_\lambda\rangle^{1/2+\varepsilon}\widetilde{\Psi}(\lambda P)\langle x\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad \lesssim \lambda^{-\frac14-\frac{\varepsilon}{2}+\delta}\lambda^{\frac12+\delta}\lambda^{-\frac14-\frac{\varepsilon}{2}+\delta} \lesssim 1.
\end{aligned}$$

Here we have used Proposition 3 *iii)*, Lemma 4 as well as the fact that δ can be chosen arbitrary small.

To obtain (7), it is sufficient to write

$$\begin{aligned}
&\big\|\langle x\rangle^{-1/2-\varepsilon}\Psi(\lambda P)(P-\lambda^{-1}z^2)^{-1}\langle x\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad \lesssim \lambda^{1/2}\big\|\langle x\rangle^{-1/2-\varepsilon}\Psi(\lambda P)((\lambda P)^{1/2}+z)^{-1}\langle x\rangle^{1/2+\varepsilon/2}\big\| \\
&\qquad\quad \times \big\|\langle x\rangle^{-1/2-\varepsilon/2}\widetilde{\Psi}(\lambda P)(P^{1/2}-\lambda^{-1/2}z)^{-1}\langle x\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad \lesssim \frac{\lambda^{1/2}}{|z|}.
\end{aligned}$$

Here we have used (6) and Lemma 4. It is clear from the proof of Lemma 4 in [1] that we can apply it to $\Psi(\lambda P)((\lambda P)^{1/2}+z)^{-1}$ and that we gain $\frac{1}{|z|}$. Indeed, as an almost analytic extension, we can just take the almost analytic extension of Ψ multiplied by the analytic function $\frac{1}{\sqrt{x}+z}$. □

Proof of Theorem 1. We only show the third part of the theorem, the proof of the other parts is analogous. Also it is clearly sufficient to replace z by $\lambda^{-1}\widetilde{z}^2$ with $\operatorname{Re}\widetilde{z} = 1 \in I$ and $\lambda \geq 1$ (for instance, $\lambda = (\operatorname{Re}\sqrt{z})^{-2}$ and $\widetilde{z} = \sqrt{z}/(\operatorname{Re}\sqrt{z})$). Let $\varphi, \widetilde{\varphi} \in C_0^\infty([\frac13, 3])$ and $f \in C^\infty(\mathbb{R})$ be such that $\widetilde{\varphi} = 1$ on the support of φ, $f(x) = 0$ for $x < 2$ and

$$f(x) + \sum_{\mu = 2^n,\, n\geq 0} \varphi(\mu x) = 1,$$

for all $x > 0$. Since 0 is not an eigenvalue of P, we can write

$$\begin{aligned}
\langle x\rangle^{-\alpha}(P-z)^{-1}\langle x\rangle^{-\beta} &= \langle x\rangle^{-\alpha} f(P)(P-z)^{-1}\langle x\rangle^{-\beta} \\
&\quad + \sum_{\mu=2^n,\, n\geq 0} \langle x\rangle^{-\alpha}\varphi(\mu P)(P-\lambda^{-1}\widetilde{z}^2)^{-1}\langle x\rangle^{-\beta}.
\end{aligned}$$

Of course, since $|z| < 1$, the functional calculus gives

$$\big\|\langle x\rangle^{-\alpha} f(P)(P-z)^{-1}\langle x\rangle^{-\beta}\big\| \lesssim 1.$$

Let $\widetilde{\alpha} = \min(\alpha, \frac{d}{2})$ and $\widetilde{\beta} = \min(\beta, \frac{d}{2})$. Note that $\widetilde{\alpha} + \widetilde{\beta} > 2$ since $d \geq 3$. Let $\Psi \in C_0^\infty(]0,+\infty[)$ be such that $\Psi = 1$ near $[\frac{1}{12}, 12]$. Then, for $\frac{\mu}{4} \leq \lambda \leq 4\mu$, we have

$$\begin{aligned}
&\big\|\langle x\rangle^{-\alpha}\varphi(\mu P)(P-\lambda^{-1}\widetilde{z}^2)^{-1}\langle x\rangle^{-\beta}\big\| \\
&\qquad \lesssim \big\|\langle x\rangle^{-\alpha}\varphi(\mu P)\langle x\rangle^{1/2+\varepsilon}\big\| \big\|\langle x\rangle^{-1/2-\varepsilon}\Psi(\lambda P)(P-\lambda^{-1}\widetilde{z}^2)^{-1}\langle x\rangle^{-1/2-\varepsilon}\big\| \\
&\qquad\quad \times \big\|\langle x\rangle^{1/2+\varepsilon}\widetilde{\varphi}(\mu P)\langle x\rangle^{-\beta}\big\| \\
&\qquad \lesssim \lambda^{\frac14+\frac{\varepsilon}{2}-\frac{\widetilde{\alpha}}{2}+\delta}\lambda^{\frac12}|\widetilde{z}|^{-1}\lambda^{\frac14+\frac{\varepsilon}{2}-\frac{\widetilde{\beta}}{2}+\delta} \lesssim \lambda^{1+\varepsilon+2\delta-\frac{\widetilde{\alpha}+\widetilde{\beta}}{2}} \lesssim 1,
\end{aligned}$$

for all $\varepsilon, \delta > 0$ small enough. Here we have used (7) and two times Lemma 4. On the other hand, for $\lambda \notin [\frac{\mu}{4}, 4\mu]$, the functional calculus and Lemma 4 yield

$$\begin{aligned}\big\| \langle x \rangle^{-\alpha} \varphi(\mu P)(P - \lambda^{-1} \widetilde{z}^2)^{-1} \langle x \rangle^{-\beta} \big\| &\lesssim |\mu^{-1} - \lambda^{-1}|^{-1} \big\| \langle x \rangle^{-\alpha} \varphi(\mu P) \big\| \big\| \widetilde{\varphi}(\mu P) \langle x \rangle^{-\beta} \big\| \\ &\lesssim |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \varepsilon},\end{aligned}$$

for all $\varepsilon > 0$. Splitting the sum into two, we get

$$\sum_{4\mu < \lambda} |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \varepsilon} \lesssim \sum_{4\mu < \lambda} \mu \mu^{-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \varepsilon} \lesssim 1,$$

$$\sum_{\mu > 4\lambda} |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \varepsilon} \lesssim \sum_{\mu > 4\lambda} \lambda \mu^{-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \varepsilon} \lesssim 1.$$

This finishes the proof of the theorem. Note that the contribution of the energies less than $(4\lambda)^{-1}$ can be estimated without dyadic decomposition. □

Proof of Remark 2 i). Let us recall that the kernel of the resolvent of the flat Laplacian in $\mathbb{R}^3$ at $z = 0$ is given by

$$K(x, y, 0) = \frac{1}{4\pi |x - y|}.$$

Assume that $\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta}$ is bounded on $L^2(\mathbb{R}^3)$. Applying to $\chi \in C_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, we find

$$\big(\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta} \chi \big)(x) = \int \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \chi(y) dy \gtrsim \langle x \rangle^{-\alpha - 1},$$

for $|x| \gg 1$. But $\langle x \rangle^{-1-\alpha} \in L^2(\mathbb{R}^3)$ if and only if $\alpha > 1/2$. The condition $\beta > 1/2$ is checked in the same way. We now apply the resolvent to $f(x) = \langle x \rangle^{-3/2 - \varepsilon} \in L^2(\mathbb{R}^3)$ and find

$$\begin{aligned}\big(\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta} f \big)(x) &= \int \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \langle y \rangle^{-3/2 - \varepsilon} dy \\ &\geq \int_{|y| \leq \frac{|x|}{2}} \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \langle y \rangle^{-3/2 - \varepsilon} dy \\ &\gtrsim \langle x \rangle^{-\alpha - 1} \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{-3/2 - \varepsilon - \beta} dy \gtrsim \langle x \rangle^{3/2 - \alpha - \beta - 1 - \varepsilon}.\end{aligned}$$

This leads to the condition $2(3/2 - \alpha - \beta - 1) \leq -3$ which implies $\alpha + \beta \geq 2$. □

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