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ESSENTIAL DIMENSIONS OF A_7 AND S_7

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Abstract. We show that Y. Prokhorov's "Simple Finite Subgroups of the Cremona Group of Rank 3" implies that, over any field of characteristic 0, the essential dimensions of the alternating group, A_7, and the symmetric group, S_7, are 4.

1. Introduction

Let k be a field of characteristic 0. In this note, a *variety* is an integral separated scheme of finite type over k. We assume all actions and maps are defined over k.

Let G be a finite group. A *compression* is a dominant rational G-equivariant map of faithful G-varieties. Let V be a faithful linear representation of G viewed as a G-variety. We define the *essential dimension of* G, denoted $\mathrm{ed}_k(G)$, to be the minimal value of $\dim(X)$, where X is taken from the set of all faithful G-varieties sitting under a compression $V \dashrightarrow X$. From [3, Theorem 3.1], we see that the essential dimension depends only on k and G — the choice of linear representation V does not matter.

The purpose of this note is to show that the essential dimension of the alternating group A_7 and the symmetric group S_7 can be computed using the recent work of Prokhorov [12] on the classification of rationally connected threefolds with faithful actions of non-abelian simple groups. Our main result is the following:

Theorem 1. $\mathrm{ed}_k(A_7) = \mathrm{ed}_k(S_7) = 4$.

The essential dimension of a finite group was introduced by Buhler and Reichstein in [3]. The concept has since been extended to much broader contexts (see [13] and [1]).

The results of this paragraph hold when k contains all roots of unity. If G is an abelian group then $\mathrm{ed}_k(G) = \mathrm{rank}(G)$ [3, Theorem 6.1]. We have $\mathrm{ed}_k(G) = 1$ if and only if G is cyclic or odd dihedral [3, Theorem 6.2]; see also [10] and [5]. If G is a p-group then $\mathrm{ed}_k(G)$ is equal to the minimal dimension of a faithful linear representation of G; this is a deep result of Karpenko and Merkurjev [9].

The values of $\mathrm{ed}_k(S_n)$ are of special interest because they relate to classical questions of simplifying degree n polynomials via Tschirnhaus transformations. In particular, the degree 7 case features prominently in algebraic variants of Hilbert's 13th problem. In this language, several results for small n were established by Hermite, Joubert and Klein in the 1800s. For more information, see the discussion in [3] or [4].

The values of $\mathrm{ed}_k(S_n)$ and $\mathrm{ed}_k(A_n)$ are known for all $n \leq 6$. Buhler and Reichstein [3] establish bounds for symmetric groups when $n \geq 5$:

$$n - 3 \geq \mathrm{ed}_k(S_n) \geq \lfloor n/2 \rfloor . \tag{1}$$

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We note that these bounds tell us that $\mathrm{ed}_k(S_7)$ is either 3 or 4. For the alternating groups A_n, they found the following bounds when $n \geq 5$:

$$n - 3 \geq \mathrm{ed}_k(A_n) \geq 2\lfloor n/4 \rfloor\,. \tag{2}$$

From this $\mathrm{ed}_k(A_6)$ is either 2 or 3. Recently, Serre found the exact value:

Theorem 2 (Serre [15, Proposition 3.6])**.** $\mathrm{ed}_k(A_6) = 3$.

Taking Theorems 1 and 2 into account we can improve some of the known bounds in higher dimensions. From [3, Theorem 6.5], we have that $\mathrm{ed}_k(S_{n+2}) \geq \mathrm{ed}_k(S_n) + 1$ for any $n \geq 1$. Similarly, from [3, Theorem 6.7], we have $\mathrm{ed}_k(A_{n+4}) \geq \mathrm{ed}_k(A_n) + 2$ for any $n \geq 4$. We have the following for $n \geq 6$:

$$n - 3 \geq \mathrm{ed}_k(S_n) \geq \left\lfloor \frac{n+1}{2} \right\rfloor\,, \tag{3}$$

$$n - 3 \geq \mathrm{ed}_k(A_n) \geq \begin{cases} \frac{n}{2} & \text{for } n \text{ even} \\ \frac{n-1}{2} & \text{for } n \equiv 1 \bmod 4 \\ \frac{n+1}{2} & \text{for } n \equiv 3 \bmod 4 \end{cases}. \tag{4}$$

2. Proof of the main theorem

We first consider the case where the base field k is $\mathbb{C}$. The general case will be deduced from this specific case.

Our proof of Theorem 1 is in the same spirit as Serre's proof of Theorem 2. For Serre's argument, it suffices to show $\mathrm{ed}_k(A_6) \neq 2$ by the bounds (2). One must show no A_6-surface sits under a compression from a linear A_6-variety. Serre uses the Enriques-Manin-Iskovskikh classification of minimal rational G-surfaces (see [11] and [7]) to reduce the problem to one surface with an A_6-action ($\mathbb{P}^2$ with the linear action). It is then shown that the group acting on this remaining surface has an abelian subgroup without fixed points. This eliminates this last surface in view of the following proposition of Reichstein and Youssin [14, Proposition 5.3] (a short proof, due to Kollár and Szabó, can be found in [14, Proposition A.2]).

Proposition 3 (Going Down)**.** *Let A be a finite abelian group and $\psi : V \dashrightarrow X$ be an A-equivariant rational map of A-varieties over $\mathbb{C}$. If V has a smooth A-fixed point and X is proper then X has an A-fixed point.*

For our proof, will need to show that $\mathrm{ed}_{\mathbb{C}}(A_7) \neq 3$. Serre looked at rational surfaces; we consider unirational threefolds. Our analog of Serre's reduction to $\mathbb{P}^2$ is Prokhorov's classification for the group A_7:

Theorem 4 (Prokhorov [12, Theorem 1.5])**.** *Let X be a rationally connected threefold over $\mathbb{C}$ with a faithful action of A_7. Then X is equivariantly birationally equivalent to one of the following:*

(i) *The subvariety of $\mathbb{P}^6$, with the standard permutation A_7 action, cut out by symmetric polynomials of degrees* 1, 2 *and* 3.
(ii) $\mathbb{P}^3$ *with a linear action of A_7.*

Proof of Theorem 1. First, we prove the theorem in the case where $k = \mathbb{C}$.

We have the following string of inequalities:

$$4 \geq \mathrm{ed}_{\mathbb{C}}(S_7) \geq \mathrm{ed}_{\mathbb{C}}(A_7) \geq \mathrm{ed}_{\mathbb{C}}(A_6) = 3\,.$$

Indeed, the first inequality follows from the bound (1). The second and third inequalities follow from the standard fact that $\mathrm{ed}_k(G) \geq \mathrm{ed}_k(H)$ for any subgroup H of a finite group G. The last inequality follows from the Theorem 2. Thus it suffices to prove that $\mathrm{ed}_{\mathbb{C}}(A_7) \neq 3$.

Suppose $\mathrm{ed}_{\mathbb{C}}(A_7) = 3$. Then there exists a dominant rational A_7-equivariant map $\psi : V \dashrightarrow X$ from a linear A_7-variety V to a 3-dimensional A_7-variety X. From this, X is unirational and, thus, rationally connected. We may assume that X is one of the threefolds from Prokhorov's Theorem.

Note that V has an A_7-fixed point (the origin) and X is proper. Thus all abelian subgroups of A_7 have fixed points by Proposition 3. For each threefold, we will exhibit an abelian subgroup of A_7 without fixed points on X. This leads to a contradiction and, so, $\mathrm{ed}_{\mathbb{C}}(A_7) \neq 3$ as desired.

Case (i): Consider $A = \langle (1\ 2\ 3), (4\ 5\ 6) \rangle$, an abelian subgroup of A_7. Let ζ be a third root of unity. Consider the following points in $\mathbb{P}^6$:

$$\begin{gathered}(\lambda_1 : \lambda_1 : \lambda_1 : \lambda_2 : \lambda_2 : \lambda_2 : \lambda_3)\\ (1 : \zeta : \zeta^2 : 0 : 0 : 0 : 0)\\ (1 : \zeta^2 : \zeta : 0 : 0 : 0 : 0)\\ (0 : 0 : 0 : 1 : \zeta : \zeta^2 : 0)\\ (0 : 0 : 0 : 1 : \zeta^2 : \zeta : 0)\end{gathered}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are not all 0. These correspond to the eigenspaces of a lift of A acting on $\mathbb{C}^7$. Thus these are all the A-fixed points on $\mathbb{P}^6$.

We claim that none of these points lie on X. For points of the first form, there are only two solutions of $x_1 + \ldots + x_7 = 0$ and $x_1^2 + \ldots + x_7^2 = 0$:

$$\lambda_1 = -1 \pm \sqrt{-7}, \quad \lambda_2 = -1 \mp \sqrt{-7}, \quad \lambda_3 = 6$$

One then checks that $x_1^3 + \ldots + x_7^3 \neq 0$ for these two points and for the remaining points. We have an abelian subgroup without fixed points — a contradiction.

Case (ii): In this case A_7 acts linearly on $\mathbb{P}^3$ and can be viewed as a subgroup of $\mathrm{PGL}_4(\mathbb{C})$. Let

$$A = \langle (1\ 2)(3\ 4), (1\ 2)(5\ 6) \rangle$$

be an abelian subgroup of A_7. Let B be the inverse image of A in $\mathrm{GL}_4(\mathbb{C})$. We have the following exact sequence of groups:

$$1 \to \mathbb{C}^\times \to B \to A \to 1$$

where $\mathbb{C}^\times$ is the set of scalar matrices in $\mathrm{GL}_4(\mathbb{C})$. Recall that A has a fixed point on $\mathbb{P}^3$. This is equivalent to saying that the action of B (viewed as a 4-dimensional linear representation) has a 1-dimensional subrepresentation $\chi : B \to \mathbb{C}^\times$. This gives us a splitting $B \simeq A \times \mathbb{C}^\times$. In particular, B is abelian.

From [6, page 10], there are two distinct projective representations of A_7 inside $\mathrm{PGL}_4(\mathbb{C})$ which are quotients of representations of the double cover $2.A_7$ in $\mathrm{GL}_4(\mathbb{C})$. There is only one element of order 2 in $2.A_7$ (namely the generator of the center). Thus any lift in B of the abelian subgroup $A \simeq C_2 \times C_2$ cannot be abelian. Thus B is not abelian — a contradiction.

We have proved the theorem in the case where $k = \mathbb{C}$. Now we use this to show the general case where k is any field of characteristic 0.

First, note that $\mathrm{ed}_k(G) \geq \mathrm{ed}_K(G)$ for K an algebraic closure of k (see [1, Proposition 1.5]). Next, we have $\mathrm{ed}_K(G) = \mathrm{ed}_{\mathbb{C}}(G)$ since K and $\mathbb{C}$ both contain an algebraic closure of $\mathbb{Q}$ (see [2, Proposition 2.14(1)]). Recalling the bounds (1) and (2) we have the general theorem. □

Remark 1. The bounds (1) and (2) hold for any field of characteristic $\neq 2$; in characteristic 2 only the upper bound is known to hold (see [1] and [8]). We note also that Proposition 3 holds for algebraically closed fields of arbitrary characteristic. Thus, for some fields of positive characteristic, an analog of Prokhorov's theorem may suffice to prove a generalisation of Theorem 1 using the same techniques as this paper.

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