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BILL-NOETHER THEORY OF BINARY CURVES

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Abstract. The theorems of Riemann, Clifford and Martens are proved for every line bundle parametrized by the compactified Jacobian of every binary curve. The Clifford index is used to characterize hyperelliptic and trigonal binary curves. The Brill-Noether theorem for $r \leq 2$ is proved for a general binary curve.

1. Introduction

The purpose of this paper is to contribute to the Brill-Noether theory of stable curves, about which very little is known. We work over an algebraically closed field, and consider the compactified universal Picard variety, $\overline{P}_{d,\,g} \to \overline{M}_g$, parametrizing degree-$d$ balanced line bundles on semistable curves of genus g (or, which is equivalent, semistable torsion-free sheaves of rank one on stable curves). The moduli properties of $\overline{P}_{d,\,g}$ are nowadays quite well understood, both from the scheme theoretic point of view and the stack theoretic one; moreover it has several equivalent geometric descriptions [Al04], [M07], [C08]. In this paper, the Brill-Noether varieties of stable curves are defined inside $\overline{P}_{d,\,g}$.

In older times, lacking a thorough understanding of how to compactify the Picard functor, or, later on, in the presence of different, seemingly unrelated, solutions of this problem, research about such topics followed different approaches. As examples, let us recall two famous constructions, which have had several important applications. The first is the theory of admissible covers, due to J. Harris and D. Mumford [HM82], studying degenerations of linear series of dimension one. The second is the theory of limit linear series, created by D. Eisenbud and J. Harris [EH86]; this theory, valid for linear series of any dimension, makes no use of compactified Jacobians, and works best for curves of compact type, whose Jacobian is projective; see also [B99], [EM02] and [O06] for more recent developements.

The subsequent progress on compactified moduli spaces of line bundles followed different directions. This led to the construction of moduli spaces (the compactified Jacobians, or Picard schemes, mentioned at the beginning) which are natural ambient spaces where studying Brill-Noether type questions.

In this field there are many open problems, some of which appear almost intractable, owing to the combinatorial complexity of stable curves. As a consequence, much of the previous work on the subject deals only with certain types of stable curves: ([EH86], [O06] dealing with curves of compact type, or [B99], [EM02] dealing with curves with two components). In the present paper also, only a certain type of curve is studied: the so-called "binary curves", namely, nodal curves made of two

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smooth rational components, intersecting at $g+1$ points. Their moduli scheme is irreducible of dimension $2g-4$.

Binary curves arise naturally in a variety of situations, sometimes with a different name, such as "split curves". Their canonical model (for non-hyperelliptic ones) is the union of two rational normal curves meeting transversally at $g+1$ points, a remarkable curve, useful as a test case and as a limit case. Also, canonical binary curves specialize to rational ribbons, another particularly interesting type of curve.

Although binary curves are reducible, many numerical and combinatorial difficulties tremendously simplify for them. Moreover, as they are made of rational components, moduli spaces of marked rational curves, and of their maps to projective spaces, provide a powerful tool.

We begin the paper with some preliminary results about compactified Jacobians and Brill-Noether varieties. Then we proceed to extend some among the fundamental theorems on which the classical Brill-Noether theory of Riemann surfaces is based: the theorems of Riemann, of Clifford, of Martens, and of Brill-Noether. Notice that none of them is known for all stable curves.

The first three of them are here proved to hold for the line bundles parametrized by the compactified Picard scheme. The analogue of Riemann's theorem is not difficult; see Proposition 11. In Section 3 we establish Clifford's theorem and study the Clifford index (Theorem 16), characterizing binary curves having Clifford index 0 or 1 in terms of their gonality. We extend Martens theorem in Proposition 22. We never use that such theorems hold for smooth curves.

While the rest of the paper deals with every binary curve, Section 4 focuses on the general one and is devoted to the Brill-Noether theorem (on the dimension of Brill-Noether varieties) for $r \leq 2$; see Theorem 24. The proof is independent from the theorem for smooth curves, which can hence be re-obtained as a consequence.

Finally, a few words about further developments. For binary curves, there are several appealing questions remaining, such as a Brill-Noether theorem for higher r. Another direction is to consider all stable curves: how do our results generalize? In both cases the situation is considerably more complex; in fact, our preliminary investigation (to appear in a forthcoming paper) has shown that the Clifford theorem and the Brill-Noether theorem do fail in some cases.

2. Set-up

2.1. Binary curves and balanced line bundles. A reduced nodal curve X is called a *binary curve* if $X = C_1 \cup C_2$ with $C_i \cong \mathbb{P}^1$; let g be the arithmetic genus of X, then $g \geq -1$ and $\#C_1 \cap C_2 = g+1$.

As binary curves are union of smooth rational components, certain moduli spaces come naturally into the picture. For any $n \geq 4$ consider $M_{0,n}$, the moduli space of n-marked smooth rational curves. $M_{0,n}$ is irreducible of dimension $n-3$.

We denote by $M_0(\mathbb{P}^r, d)$ the moduli space of maps of degree $d \geq 1$ from $\mathbb{P}^1$ to $\mathbb{P}^r$. More generally, for any $n \geq 0$ consider the moduli space $M_{0,n}(\mathbb{P}^r, d)$ of degree d maps from n-marked, smooth, rational curves to $\mathbb{P}^r$. It is irreducible of dimension

$$\dim M_{0,n}(\mathbb{P}^r, d) = \dim M_0(\mathbb{P}^r, d) + n = (r+1)d + r - 3 + n. \tag{1}$$

Lemma 1. *Let $B_g \subset \overline{M}_g$ be the locus of binary curves of genus $g \geq 2$. Then B_g is irreducible of dimension $2g-4$.*

Proof. There is a surjective morphism, having finite fibers,

$$\gamma_g : M_{0,g+1} \times M_{0,g+1} \longrightarrow B_g \tag{2}$$

mapping $\Big((C_1; p_1, \ldots, p_{g-2}, 0, 1, \infty), (C_2; q_1, \ldots, q_{g-2}, 0, 1, \infty)\Big)$ to the binary curve obtained by gluing p_i with q_i and $0, 1, \infty \in C_1$ with $0, 1, \infty \in C_2$. As $M_{0,g+1}$ is irreducible of dimension $g-2$, the Lemma follows. □

The description of the compactified Picard scheme of a binary curve (see Section 2.2 below) is based on Definition 2, a special case of (for example) 4.6 in [C08].

Definition 2. Let X be a binary curve of genus $g \geq -1$. A multidegree $\underline{d} = (d_1, d_2)$ with $d = |\underline{d}| = d_1 + d_2$ is *balanced* on X if, for either $i \in \{1, 2\}$,

$$m(d,g) := \frac{d-g-1}{2} \leq d_i \leq \frac{d+g+1}{2} =: M(d,g). \tag{3}$$

We say that $L \in \operatorname{Pic}^d X$ is balanced if $\underline{\deg}\, L$ is balanced on X. We say that $\underline{d}$, or L, is *strictly balanced* if (3) holds with strict inequalities. We denote

$$B_d(X) = \{\underline{d} : |\underline{d}| = d,\ \underline{d} \text{ balanced }\} \supset B^*_d(X) = \{\underline{d} \text{ strictly balanced}\}. \tag{4}$$

Clearly $B_d(X)$ and $B^*_d(X)$ depend only on g, so we shall sometimes write

$$B_d(g) := B_d(X), \quad B^*_d(g) := B^*_d(X). \tag{5}$$

Remark 3. The following facts will be used several times.
(a) For every d: $B^*_d(X) \neq \emptyset$ if $g \geq 1$, and $B_d(X) \neq \emptyset$ if $g \geq 0$.
(b) If $g = -1$, then $B_d(X) \neq \emptyset \iff m(d,g) \in \mathbb{Z}$.
(c) $\underline{d} \in B_d(X) \iff d_i \geq m(d,g),\ \forall i = 1,2 \iff d_i \leq M(d,g),\ \forall i = 1,2$.
(d) $\underline{d}$ is balanced $\iff \underline{d} + n\,\underline{\deg}\,\omega_X$ is balanced.

Remark 4. Let d and $g \geq -1$ be integers. Then one easily checks the following.
(A) $m(d,g) = m(d-1, g-1)$ and $M(d,g) = M(d-1,g-1)+1$.
(B) $m(d,g) > m(d, g+n)$ for every $n \geq 1$.
(C) $M(d,g) < M(d, g+n)$ for every $n \geq 1$.
(D) $B_d(g) \subset B_d(g+n)$ for any $n \geq 0$.

As it is well known, there are two common (equivalent) ways of describing the geometric objects parametrized by the compactified Jacobian: via torsion-free sheaves or via line bundles; we choose the second one, introduced in [C94]. In order to describe it, we introduce some terminology. Let X be a nodal curve and S a set of nodes of X. By "the normalization of X at S" we mean the local desingularization (or normalization) of X at every node in S. We say that a nodal curve $\widehat{X}_S$ is the "blow -up" of X at S if there exists $\pi : \widehat{X}_S \to X$ such that $\pi^{-1}(n_i) = E_i \simeq \mathbb{P}^1$ for any $n_i \in S$, and $\pi : \widehat{X}_S \smallsetminus \cup_i E_i \to X \smallsetminus S$ is an isomorphism. Thus $\overline{\widehat{X}_S \smallsetminus \cup_i E_i}$ is the normalization of X at S.

The boundary points of the compactified Jacobian parametrize balanced line bundles on (strictly) semistable curves; a balanced line bundle is defined to be one whose multidegree is balanced. To define this for strictly semistable curves we introduce some notation that will be used throughout the paper. Let X be a binary curve and $S \subset X_{\rm sing}$ be a set of nodes of X, set $e = \#S$; we shall sometimes write $S = S^e$. We denote $\widehat{X}_S$ the blow-up of X at S. We call $E_1, \ldots, E_e$ the exceptional components of $\widehat{X}_S$, and Y_S their complementary curve (the normalization of X at S). Y_S is a binary curve of genus $g - e$, and

$$\widehat{X}_S = Y_S \cup \cup_{i=1}^e E_i = C_1 \cup C_2 \cup E_1 \cup \ldots \cup E_e.$$

We will write a multidegree $\widehat{\underline{d}} = (d_1, d_2, d_3, \ldots, d_{2+e})$ on $\widehat{X}_S$ using the convention that for $i = 1, 2$ we have $d_i = \widehat{\underline{d}}_{C_i}$, and for $i \geq 3$ we have $d_i = \widehat{\underline{d}}_{E_i}$. We also write $\widehat{\underline{d}}_{Y_S} = (d_1, d_2)$, so that $|\widehat{\underline{d}}_{Y_S}| = d - \sum_{i=3}^e d_i$.

Definition 5. A multidegree $\widehat{\underline{d}}$ on $\widehat{X}_S$, with $|\widehat{\underline{d}}| = d$, is balanced if (1) and (2) hold:
(1) $d_i = 1$, $\forall i = 3, \ldots, e$ (i.e. if $\widehat{\underline{d}}_{E_i} = 1$ for every E_i);
(2) $\widehat{\underline{d}}_{Y_S}$ is balanced on Y_S (i.e. if $\widehat{\underline{d}}_{Y_S} \in B_{d-e}(Y_S)$).
$\widehat{\underline{d}}$ is called strictly balanced if $\widehat{\underline{d}}_{Y_S}$ is strictly balanced on Y_S.

We denote $B_d(\widehat{X}_S)$ and $B_d^*(\widehat{X}_S)$ the set of balanced and strictly balanced multidegrees on $\widehat{X}_S$. As we said, $\widehat{L} \in \operatorname{Pic}^{\widehat{\underline{d}}} \widehat{X}_S$ is called balanced if $\widehat{\underline{d}}$ is balanced. Two balanced line bundles $\widehat{L}', \widehat{L} \in \operatorname{Pic}^{\widehat{\underline{d}}} \widehat{X}_S$ are defined to be *equivalent* if their restrictions to Y_S are isomorphic.

2.2. The compactified Picard scheme of binary curves. Let X be a stable binary curve of genus $g \geq 2$, and d a fixed integer. We shall now describe its compactified degree-d Picard variety $\overline{P^d_X}$. As d varies, the structure of $\overline{P^d_X}$ varies between two different types, according to whether or not $m(d, g)$ is an integer. The terminology we will use reflects the relation with Néron models; see [C08] and [M07].

N-type: $m(d, g) \notin \mathbb{Z}$. X is said to be *d-general*, and $\overline{P^d_X}$ of *Néron type*.

In this case every point of $\overline{P^d_X}$ corresponds to an equivalence class of balanced line bundles. We have a natural isomorphism

$$\overline{P^d_X} \cong \coprod_{\underline{d} \in B_d(X)} \operatorname{Pic}^{\underline{d}} X \coprod_{e=1}^{g} \Big(\coprod_{\substack{S^e \subset X_{\rm sing} \\ \#S^e = e}} \; \coprod_{\underline{d}^e \in B_{d-e}(Y_{S^e})} \operatorname{Pic}^{\underline{d}^e} Y_{S^e} \Big). \tag{6}$$

Note that $B^*_{d-e}(Y_{S^e}) = B_{d-e}(Y_{S^e})$ for every $\emptyset \subseteq S^e \subset X_{\rm sing}$.

D-type: $m(d, g) \in \mathbb{Z}$. Now $\overline{P^d_X}$ is called of *Degeneration type*.

In this case there exist balanced multidegrees that are not strictly balanced. More precisely, for every partial normalization Y_{S^e} of X, $e \geq 0$, there exists a unique such multidegree, namely $(m(d, g), M(d, g) - e) \in B_{d-e}(Y_{S^e})$ (cf. Lemma 4). All line bundles having these multidegrees are identified to a unique point $\ell_0 \in \overline{P^d_X}$. Of course, to ℓ_0 there corresponds a unique closed orbit; indeed there exists a unique balanced line bundle on a unique curve parametrized by ℓ_0, namely the line bundle $(\mathcal{O}_{C_1}(m(d, g)), \mathcal{O}_{C_2}(m(d, g)))$ on the normalization of X (the disjoint union of two copies of $\mathbb{P}^1$). We have a description analogous to (6)

$$\overline{P^{\underline{d}}_X} \smallsetminus \{\ell_0\} \cong \coprod_{\underline{d}\in B^*_d(X)} \operatorname{Pic}^{\underline{d}} X \coprod_{e=1}^{g-1}\Big(\coprod_{\substack{S^e\subset X_{\rm sing}\\ \#S^e=e}} \ \coprod_{\underline{d}^e\in B^*_{d-e}(Y_{S^e})} \operatorname{Pic}^{\underline{d}^e} Y_{S^e}\Big). \tag{7}$$

Note that if $e = g$ then $B^*_{d-e}(Y_{S^e})$ is empty.

For any $S^e \subset X_{\rm sing}$ and any $\underline{d}^e \in B_{d-e}(Y_S)$ we shall denote $P^{\underline{d}^e}_{S^e} \subset \overline{P^{\underline{d}}_X}$ the stratum isomorphic to $\operatorname{Pic}^{\underline{d}^e} Y_{S^e}$. Also, for a fixed $S \subset X_{\rm sing}$ we denote P_S the union of all strata $P^{\underline{d}}_S$ ad $\underline{d}$ varies, omitting "e" from the notation, for simplicity. Note that all the strata above are tori: $P^{\underline{d}^e}_{S^e} \cong (k^*)^{g-e}$. Moreover,

$$\overline{P^{\underline{d}}_S} \supset P^{\underline{d}'}_{S'} \iff S \subset S' \text{ and } \underline{d} \geq \underline{d}' \tag{8}$$

where $\underline{d}' \in B_{d-e'}(Y_{S'})$, and $\underline{d} \geq \underline{d}'$ means $d_i \geq d'_i$, $i = 1, 2$.

2.3. Brill-Noether varieties. Given $\underline{d}$ and r we denote

$$W^r_{\underline{d}}(X) := \{L \in \operatorname{Pic}^{\underline{d}} X : h^0(L) \geq r + 1\} \tag{9}$$

if $r = 0$ we usually omit r: $W^0_{\underline{d}}(X) = W_{\underline{d}}(X) = \{L \in \operatorname{Pic}^{\underline{d}} X : h^0(L) \neq 0\}$. $W^r_{\underline{d}}(X)$ is endowed with a natural scheme structure, obtained either as for smooth curves ([ACGH]), or using the GIT construction of $\overline{P^d_X}$. We omit the details as this is irrelevant for our purposes. For any r and d, we denote

$$B^r_{g,d} = \{X \in B_g : \exists \underline{d} \in B_d(X) : \ W^r_{\underline{d}}(X) \neq \emptyset\}, \tag{10}$$

and for any $\underline{d} \in B_d(g)$

$$B^r_{g,\underline{d}} = \{X \in B_g : \ W^r_{\underline{d}}(X) \neq \emptyset\}. \tag{11}$$

By the above description, every point λ of $\overline{P^d_X}$, belongs to a stratum P_S, for some $S \subset X_{\rm sing}$. So λ determines a unique strictly balanced line bundle M_S, of degree $d - \#S$, on a unique curve Y_S, the normalization of X at S. Viewing the isomorhisms of (6) and (7) as identifications, we shall often denote the points of $\overline{P^d_X}$ as follows

$$[M, S] \in \overline{P^d_X}, \quad S \subset X_{\rm sing}, \quad M \in \operatorname{Pic}^{d-\#S} Y_S, \tag{12}$$

where M is strictly balanced. On the other hand, a point of $\overline{P^d_X}$ parametrizes a pair $(\widehat{X}_S, [\widehat{L}])$, where $\widehat{X}_S = Y_S \cup_{i=1}^{\#S} E_i$ is the blow-up of X at S, and $[\widehat{L}]$ is an equivalence class of strictly balanced line bundles on $\operatorname{Pic}^d \widehat{X}_S$, all having restriction M on Y_S. So, we will also denote simply by $[\widehat{L}]$ a point of $\overline{P^d_X}$.

With the above notations, one easily sees (cf. Lemma 4.2.5 in [C07])

$$h^0(Y_S, M) = h^0(\widehat{X}_S, \widehat{L}'), \quad \forall \widehat{L}' \in [\widehat{L}_S]. \tag{13}$$

Now, we define

$$\overline{W^r_{d,X}} = \{[M, S] \in \overline{P^d_X} : h^0(Y_S, M) > r\} = \{[\widehat{L}] \in \overline{P^d_X} : h^0(\widehat{X}_S, \widehat{L}) > r\}. \tag{14}$$

We denote by $M^r_{g,d} \subset M_g$ the locus of smooth curves C such that $W^r_d(C) \neq \emptyset$, and by $\overline{M^r_{g,d}} \subset \overline{M}_g$ its closure in $\overline{M}_g$.

Proposition 6. *Let $r \geq 0$, $g \geq 2$ and $d \leq r + g - 1$.*

(i) There is a natural isomorphism

$$\overline{W^r_{d,X}} \cong \coprod_{\underline{d}\in B^*_d(X)} W^r_{\underline{d}}(X) \coprod_{e=1}^{g}\Big(\coprod_{\substack{S^e\subset X_{sing}\\ \#S^e=e}} \coprod_{\underline{d}^e\in B^*_{d-e}(Y_{S^e})} W^r_{\underline{d}^e}(Y_{S^e})\Big). \tag{15}$$

(ii) Denote by $W_{\underline{d}^e,S^e}\subset \overline{W^r_{d,X}}$ *the stratum isomorphic to* $W^r_{\underline{d}^e}(Y_{S^e})$ *under (15). If* $\overline{W_{\underline{d}^e,S^e}} \supset W_{\underline{d}^{e'},S^{e'}}$ *then* $S^e\subset S^{e'}$ *and* $\underline{d}^e \geq \underline{d}^{e'}$.

(iii) If $X\in \overline{M^r_{g,d}}$ *then* $\overline{W^r_{d,X}}\neq\emptyset$.

Proof. We earlier gave a description of $\overline{P^d_X}$ by a natural isomorphism analogous to (15). We explained that there are two possibilities, (6) and (7), according to whether $m(d,g)$ is an integer or not. If $m(d,g)\notin\mathbb{Z}$, then (6) holds and $B^*_{d-e}(Y_{S^e}) = B_{d-e}(Y_{S^e})$ for every e and S. Therefore (15) follows immediately from (6).

Suppose $m(d,g)\in\mathbb{Z}$, and consider the line bundle

$$M := \Big(\mathcal{O}_{C_1}(m(d,g)), \mathcal{O}_{C_2}(m(d,g))\Big) \in \mathrm{Pic}(C_1 \coprod C_2)$$

corresponding to the point $\ell_0 \in \overline{P^d_X}$. Now, as $d\leq r+g-1$, we have

$$m(d,g) = \frac{d-g-1}{2} \leq \frac{r+g-1-g-1}{2} = \frac{r}{2}-1.$$

Therefore $h^0(M) = 2h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m(d,g))) = r$ hence $\ell_0\notin\overline{W^r_{d,X}}$. This implies that (15) follows from (7), as in the previous case.

Part (ii) follows from the previous one and from (8).

Now part (iii). Let $X \in \overline{M^r_{g,d}}$. Then there exists a family of smooth curves specializing to X such that the general fiber, C, of the family has a non empty $W^r_d(C)$. Up to replacing the family by some base change, we may assume that the family has a section. This enables us to apply a construction of E. Arbarello and M. Cornalba (see Section 2 of [AC81]) yielding that the $W^r_d(C)$ form a family contained in the relative Picard scheme. Therefore, as C specializes to X, $W^r_d(C)$ specializes to some non-empty subset W_0 of $\overline{P^d_X}$. By uppersemicontinuity of h^0, W_0 lies in $\overline{W^r_{d,X}}$, which is thus non empty. ☐

For every $d\geq 1$ and $\underline{d}=(d_1,d_2)$, denote $X^{\underline{d}} := C_1^{d_1}\times C_2^{d_2}$. Consider the Abel map of multidegree $\underline{d}$

$$\alpha^{\underline{d}}_X : C_1^{d_1}\times C_2^{d_2} \dashrightarrow W_{\underline{d}}(X); \quad (p_1,\dots,p_d)\mapsto \mathcal{O}_X(\sum_{i=1}^d p_i). \tag{16}$$

$\alpha^{\underline{d}}_X$ is regular away from the points lying over $C_1\cap C_2$. We denote $A_{\underline{d}}(X)\subset W_{\underline{d}}(X)$ the closure of the image of $\alpha^{\underline{d}}_X$. It is clear that $A_{\underline{d}}(X)$ is irreducible.

Lemma 7. *Let* $1\leq d\leq g$ *and* $\underline{d}\in B_d(X)$. *Then* $h^0(X,L)=1$ *for the general* $L\in A_{\underline{d}}(X)$, *and* $\dim A_{\underline{d}}(X) = d$.

Proof. We have $\dim A_{\underline{d}}(X)\leq d$, of course. The fiber of $\alpha^{\underline{d}}_X$ over a general $L\in A_{\underline{d}}(X)$ has dimension $h^0(L)-1$, hence it suffices to prove that $h^0(L)\leq 1$ for some $L\in A_{\underline{d}}(X)$.

Pick $S \subset X_{\text{sing}}$ such that $\#S = d$. As $d < g+1 = \#X_{\text{sing}}$ we can consider the normalization of X at S, $Y_S \to X$, and the curve $\widehat{X}_S$, the blow-up of X at S.

Consider $M_S \in \operatorname{Pic}^{(0,0)} Y_S$, note that, since Y_S is connected, $h^0(Y_S, M_S) \leq 1$, and equality holds if and only if $M_S = \mathcal{O}_{Y_S}$. Therefore, as already observed in (13), for every balanced line bundle $\widehat{L}$ on $\widehat{X}_S$ restricting to M_S on Y_S, we have

$$h^0(\widehat{X}_S, \widehat{L}_S) = h^0(Y_S, M_S) \leq 1 \tag{17}$$

with equality if and only if $M_S = \mathcal{O}_{Y_S}$ (by Corollary 2.2.5 of [C07]). Fix $M_S = \mathcal{O}_{Y_S}$ and $\widehat{L}_S$ as above. The point of $\overline{P_X^d}$ parametrizing $\widehat{L}_S$ is in the closure of $A_{\underline{d}}(X)$. Indeed, we can simultaneously specialize d distinct nonsingular points of X to the d nodes of S. By (17) we get $h^0(X, L) \leq h^0(\widehat{X}_S, \widehat{L}_S) \leq 1$ for L general in $A_{\underline{d}}(X)$, as wanted. □

Let $\nu : Y \to X$ be the normalization of X at s nodes, $n_1, \ldots, n_s$, and $\nu^{-1}(n_s) = \{p_s, q_s\}$. In symbols:

$$\nu : Y \longrightarrow X = Y/_{\{p_i = q_i,\ i=1\ldots s\}}. \tag{18}$$

Let M be a line bundle on Y such that $h^0(Y, M) \neq \emptyset$. Denote by $F_M(X)$ the fiber over M of $\nu^* : \operatorname{Pic} X \to \operatorname{Pic} Y$, i.e. $F_M(X) := \{L \in \operatorname{Pic} X : \nu^* L = M\}$. We ask under what conditions there exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(Y, M)$. We introduce the following terminology.

Definition 8. Let p, q be nonsingular points of a curve Y; pick $M \in \operatorname{Pic} Y$. We say that p and q are *equivalent, or neutral, with respect to* M, and write $p \sim_M q$, if $h^0(Y, M-p) = h^0(Y, M-q) = h^0(Y, M-p-q)$.

The following is a straightforward consequence of Lemmas 2.2.3 and 2.2.4 in [C07].

Lemma 9. *Let $Y \to X = Y/_{\{p_1=q_1,\ldots,p_s=q_s\}}$; pick $M \in \operatorname{Pic} Y$ with $h^0(Y, M) \neq 0$. There exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(Y, M)$ if and only if $p_i \sim_M q_i$ for every $i = 1, \ldots, s$.*

Such an L is unique (if it exists) if p_i and q_i are not base points for M for all i.

This implies the following useful result.

Lemma 10. *Let $\underline{d} = (d_1, d_2)$ be a multidegree on a binary curve X of genus g; assume $d_2 \geq d_1 \geq -1$. Then for every $L \in \operatorname{Pic}^{\underline{d}} X$*

$$h^0(X, L) \leq d_1 + d_2 + 1 - \min\{d_2, g\}. \tag{19}$$

(i) If $d_2 \geq g$, equality holds for every $L \in \operatorname{Pic}^{\underline{d}} X$.
(ii) If $d_2 < g$, equality holds for at most one $L \in \operatorname{Pic}^{\underline{d}} X$.

Proof. Set $d = d_1 + d_2$. If $g = -1$ then $\min\{d_2, g\} = -1$, hence (as $d_i \geq -1$)

$$h^0(X, L) = h^0(C_1, L_1) + h^0(C_2, L_2) = d_1 + 1 + d_2 + 1 = d + 1 - \min\{d_2, g\}.$$

We can assume $g \geq 0$, i.e. X is connected. For every $0 \leq e \leq \min\{d_2, g\}$, denote

$$X_e := \frac{C_1 \coprod C_2}{(p_i = q_i,\ i = 1, \ldots, e)} \xrightarrow{\nu_e} X$$

so that ν_e is a normalization at $g + 1 - e$ nodes. Set $M_e = \nu_e^* L$; we have, of course, $h^0(X_e, M_e) \geq h^0(X, L)$.

If $e = 0$ then $h^0(X_0, M_0) = d_1 + d_2 + 2 = d + 2$. More generally, we claim that

$$h^0(X_e, M_e) = d + 2 - e$$

for every e. By induction on e. Notice that $\deg L_2(-\sum_{i=1}^{e} q_i) \geq 0$, therefore, as $C_2 \cong \mathbb{P}^1$, there exists a section $s_2 \in H^0(C_2, L_2(-\sum_{i=1}^{e} q_i))$ not vanishing at q_{e+1}. This implies that M_e has a section vanishing at p_{e+1} but not at q_{e+1}; indeed, just glue s_2 to the zero section on C_1, which we can do as s_2 vanishes at every p_i with $i \leq e$. Therefore $p_{e+1} \not\sim_{M_e} q_{e+1}$. Lemma 9 now yields

$$h^0(X_{e+1}, M_{e+1}) = h^0(X_e, M_e) - 1 = d + 2 - e - 1 = d + 1 - e.$$

Applying this to $e = \min\{d_2, g\}$ we obtain

$$h^0(X, L) \leq h^0(X_{e+1}, M_{e+1}) = d + 1 - \min\{d_2, g\}.$$

We have thus shown that (19) holds, with equality if $d_2 \geq g$.

Part (ii) follows from the uniqueness part in Lemma 9. □

Using Lemma 10 we can now extend Riemann's theorem:

Proposition 11. *Let X be a binary curve of genus g, and let $d \geq 2g - 1$.*

(i) For every balanced $L \in \operatorname{Pic}^d X$ we have $h^0(L) = d - g + 1$.

(ii) For every $[\widehat{L}] \in \overline{P^d_X}$ we have $h^0(\widehat{L}) = d - g + 1$.

Proof. Let $\underline{\deg}\, L = (d_1, d_2)$ and assume $d_1 \leq d_2$. Then $d_2 \geq g$, for otherwise $d_1 + d_2 \leq 2(g-1)$ which is ruled out, by hypothesis. As $\underline{\deg}\, L$ balanced, we have

$$d_i \geq m(d, g) = \frac{d - g - 1}{2} \geq \frac{2g - 1 - g - 1}{2} = \frac{g - 2}{2} \geq -\frac{3}{2}.$$

Therefore $d_i \geq -1$ for $i = 1, 2$, so that Lemma 10 applies. We obtain that (19) holds, with equality, as $d_2 \geq g$. Hence

$$h^0(X, L) = d_1 + d_2 + 1 - \min\{d_2, g\} = d + 1 - g,$$

as stated in part (i). Now, to prove part (ii) it suffices to consider $\widehat{L} \in \operatorname{Pic} \widehat{X}_S$ with $\#S = e \geq 1$ (notation as in Subsection 2.3). By (13) we have

$$h^0(\widehat{X}_S, \widehat{L}) = h^0(Y_S, M) = (d - \#S) - (g - \#S) + 1 = d - g + 1$$

where the second equality follows from part (i) applied to the binary curve Y_S (of course Y_S has genus $g - e$, so that $d - e \geq 2g - 1 - e \geq 2g - 1 - 2e = 2g_{Y_S} - 1$). □

Proposition 12. *Let $\underline{d} = (d_1, d_2)$ be a balanced multidegree on a binary curve X. Assume $d_1 \leq d_2$ and set $d = |\underline{d}|$. Then $W^r_{\underline{d}}(X) = \emptyset$ in the following cases.*

(i) $d_1 < 0$ and $d \leq g + r$.

(ii) $0 \leq d_1 \leq r - 1$ and $d \leq g + r - 1$.

Proof. We must prove that $h^0(X, L) \leq r$ for every $L \in \operatorname{Pic}^{\underline{d}} X$. In case (i)

$$h^0(X, L) = h^0(C_2, L_2(-C_1 \cap C_2)) = \max\{0, d_2 - g\}.$$

$\underline{d}$ is balanced, hence

$$d_2 - g \leq (d + g + 1)/2 - g = (d - g + 1)/2 \leq (r + 1)/2.$$

We obtain $h^0(X, L) \leq (r + 1)/2 < r + 1$ and we are done.

In case (ii), as $d_1 \leq d_2$, we have, by Lemma 10, $h^0(X, L) \leq d + 1 - \min\{d_2, g\}$.

If $d_2 \leq g$ we obtain $h^0(X,L) \leq d+1-d_2 = d_1+1 \leq r-1+1 = r$ and we are done. If $d_2 > g$ we have $h^0(X,L) \leq d+1-g \leq r$. The proof is complete. □

3. Clifford theory

3.1. Clifford's inequality and hyperelliptic binary curves. The main result of this Section is Theorem 16, extending Clifford's theorem. Its first part, the Clifford inequality, is the subsequent Proposition 13.

Proposition 13 (Clifford's inequality)**.** *Let X be a binary curve of genus $g \geq 1$, and let d be such that $0 \leq d \leq 2g$.*

1. *For every $\underline{d} \in B_d(X)$, and every $L \in \operatorname{Pic}^{\underline{d}} X$, we have $h^0(L) \leq d/2+1$.*
   If $d = 0$ and $h^0(L) = 1$ then $L = \mathcal{O}_X$; if $d = 2g-2$ and $h^0(L) = g$ then $L = \omega_X$.
2. *For every $[\widehat{L}] \in \overline{P^d_X}$ we have $h^0(\widehat{L}) \leq d/2+1$.*

Proof. We may assume $d_1 \leq d_2$. If $d_1 < 0$ then

$$h^0(X,L) = h^0(C_2, L_2(-C_1 \cap C_2)) = d_2 - g \leq M(d,g) - g = \frac{d-g+1}{2}$$

($\underline{d}$ is balanced). Now, as $g \geq d/2$ we obtain $h^0(X,L) \leq d/4+1/2$, so we are done. If $d_1 \geq 0$, by Lemma 10 we have

$$h^0(X,L) \leq d+1-\min\{d_2, g\}.$$

If $d_2 < g$, we obtain

$$h^0(X,L) \leq d+1-d_2 = d_1+1 \leq d/2+1$$

(as $d_1 \leq d/2$); so we are done. If $d_2 > g$, then

$$h^0(X,L) \leq d+1-g \leq d/2+1$$

(as $g \geq d/2$). If $d = 0$ and $h^0(L) = 1$, by Proposition 12 we need to have $\underline{\deg}\, L \geq 0$. By Corollary 2.2.5 in [C07] we get $L = \mathcal{O}_X$. Finally, suppose $d = 2g-2$ and let L be balanced, such that $h^0(L) = g$; by Serre duality $h^0(\omega_X \otimes L^{-1}) = 1$. By the previous case and Remark 3 (d). $\omega_X \otimes L^{-1} = \mathcal{O}_X$, so the proof of Proposition 1 is done.

For part 2 let $\widehat{L} \in \operatorname{Pic} \widehat{X}_S$ with $\#S = e \geq 1$ (notation in Subsection 2.3). We have $h^0(\widehat{X}_S, \widehat{L}) = h^0(Y_S, M)$ (by (13)), where $M = \widehat{L}_{|Y_S}$ has degree $d-e < d$. If $e \leq g-1$ then Y_S has genus at least 1 so the result follows from (Proposition 1) applied to Y_S, which we can do because Y_S is a binary curve and M is balanced (cf. Definition 5). Otherwise Y_S has genus 0 in case $e = g$, or -1 if $e = g+1$. In both cases we get $h^0(Y_S, M) \leq d-g+1 \leq d/2+1$. □

Let now $0 < d < 2g-2$, recall that for a smooth curve C, there exists $L \in \operatorname{Pic}^d C$ with $h^0(L) = d/2+1$ if and only if C is hyperelliptic and L is a multiple of the hyperelliptic class. The analogous fact holds for binary curves, as we shall see in Theorem 16. First we need to define and study hyperelliptic binary curves.

Let X be a binary curve of genus $g \geq 2$. X (like all stable curves, cf. [HM82]) is called *hyperelliptic*, if X lies in the closure, $\overline{H}_g \subset \overline{M}_g$, of the locus, H_g, of smooth hyperelliptic curves. We say that X is *weakly hyperelliptic* if $W^1_{\underline{d}}(X) \neq \emptyset$ for some balanced $\underline{d}$ with $|\underline{d}| = 2$. If $g \leq 1$ we say that every binary curve is hyperelliptic (and weakly hyperelliptic), for simplicity.

Remark 14. By Proposition 12, X is weakly hyperelliptic if and only if $W^1_{(1,1)}(X) \neq \emptyset$.

Lemma 15. *Let X be a binary curve of genus $g \geq 2$.*

(i) X is weakly hyperelliptic if and only if it is hyperelliptic.
(ii) If X is hyperelliptic, then $W^1_{(1,1)}(X) = \{H_X\}$; H_X will be called the hyperelliptic class *of X.*
(iii) If X is hyperelliptic, every normalization of X is hyperelliptic. If $g \geq 4$ and X is not hyperelliptic, there exists a node $n \in X_{sing}$ such that the normalization of X at n is not hyperelliptic.

Proof. Suppose X hyperelliptic, then $\overline{W^1_{2,X}} \neq \emptyset$, by Proposition 6 (iii). To show that X is weakly hyperelliptic, we need to prove $W^1_{\underline{d}}(X) \neq \emptyset$, for some $\underline{d} \in B_d(X)$. Pick $[M, S] \in \overline{P^2_X}$ with $S \neq \emptyset$; it suffices to show that $h^0(Y_S, M) \leq 1$.

As $\#S = e \geq 1$ we get $\deg M = 2 - e \leq 1$. We also know that $\underline{\deg}\, M$ is balanced, by Definition 5. By Proposition 13, we have

$$h^0(Y_S, M) \leq \deg M/2 + 1 \leq 3/2$$

hence $h^0(Y_S, M) \leq 1$.

Conversely, let X be weakly hyperelliptic. By Remark 14 this is equivalent to saying that $W^1_{(1,1)}(X) \neq \emptyset$, so $X \in B^1_{g,2}$ (notation in (10)). On the other hand, every $X' \in B^1_{g,2}$ has $W^1_{(1,1)}(X') \neq \emptyset$. Therefore $B^1_{g,2} = B^1_{g,(1,1)}$; now $B^1_{g,(1,1)}$ is easily seen to be irreducible of dimension $g - 2$.

Consider $\overline{H}_g \subset \overline{M}_g$, the locus of hyperelliptic stable curves. By the previous part $\overline{H}_g \cap B_g \subset B^1_{g,2}$, hence

$$\dim \overline{H}_g \cap B_g \leq \dim B^1_{g,2} = g - 2. \tag{20}$$

On the other hand, as B_g is irreducible of codimension $g + 1$ in $\overline{M}_g$ (cf. Lemma 1) we have

$$\dim \overline{H}_g \cap B_g \geq \dim \overline{H}_g - (g + 1) = g - 2.$$

Combining this with (20) we obtain $\dim \overline{H}_g \cap B_g = g - 2 = \dim B^1_{g,2}$. Since $B^1_{g,2}$ is irreducible and contains $\overline{H}_g \cap B_g$, we conclude $\overline{H}_g \cap B_g = B^1_{g,2}$, proving (i).

Now, suppose X hyperelliptic, so that $W^1_{(1,1)}(X) \neq \emptyset$. To prove (ii) we use induction on g: if $g = 2$, by Proposition 13, $W^1_{(1,1)}(X)$ contains a unique element: $\omega_X = H_X$. Now, let $g \geq 3$ and $Y \to X$ the normalization of one node of X, so that $g_Y = g - 1 \geq 2$. As $W^1_{(1,1)}(X) \neq \emptyset$ the pull-back map

$$\rho : W^1_{(1,1)}(X) \longrightarrow W^1_{(1,1)}(Y); \quad L \mapsto \nu^* L$$

shows that Y is also weakly hyperelliptic, hence hyperelliptic.

By induction $W^1_{(1,1)}(Y) = \{H_Y\}$; by Proposition 13, $h^0(Y, H_Y) = 2$, and H_Y has no base points. Therefore, by Lemma 9, $\rho^{-1}(H_Y)$ is a point, so we are done.

For the final part, it remains to show that if X is non-hyperelliptic and $g \geq 4$, there exists $n \in X_{\text{sing}}$ such that the normalization at n is not hyperelliptic. By contradiction, suppose this is not the case. Let $Z \to X$ be the normalization of X at two nodes, n_1, n_2, and call Y_i the normalization of X at n_i; so Y_i is hyperelliptic, for

$i = 1, 2$. Therefore Z is hyperelliptic (by the previous part) and has genus at least 2. Hence $W^1_{(1,1)}(Z) = \{H_Z\}$ and, as Y_i is hyperelliptic,

$$p_i \sim_{H_Z} q_i, \quad i = 1, 2,$$

where $p_i, q_i \in Z$ are the branches over n_i. But then, by Lemma 9, there exists $L \in \operatorname{Pic}^{(1,1)} X$ which pulls back to H_Z and such that $h^0(X, L) = 2$. Hence X is weakly hyperelliptic, and hence hyperelliptic (by the previous part), a contradiction. □

3.2. Clifford index. Recall that the *Clifford index* of a line bundle L on a curve X is $\operatorname{Cliff} L := \deg L - 2h^0(L) + 2$. Let us define the Clifford index of X:

$$\operatorname{Cliff} X := \min\{\operatorname{Cliff} L | L \in \operatorname{Pic} X,\ \underline{\deg} L \in B_d(X),\ h^0(L) \geq 2,\ h^1(L) \geq 2\}. \tag{21}$$

For a smooth curve C, $\operatorname{Cliff} C \geq 0$, and $\operatorname{Cliff} C = 0$ if and only if C is hyperelliptic (Clifford's theorem). If C is non-hyperelliptic, then $\operatorname{Cliff} C = 1$ if and only if C is trigonal or bielliptic or a plane quintic (Mumford's theorem, see [ACGH] IV (5.2)).

Theorem 16. *Let X be a binary curve.*

(I) $\operatorname{Cliff} X \geq 0$.
(II) $\operatorname{Cliff} X = 0$ *if and only if X is hyperelliptic (i.e. weakly hyperelliptic).*
(III) Assume $\operatorname{Cliff} X \neq 0$. *Then* $\operatorname{Cliff} X = 1$ *if and only if $W^1_{\underline{d}}(X) \neq \emptyset$ for some balanced $\underline{d}$ with $|\underline{d}| = 3$.*

Part (I) is Proposition 13. To prove the rest we need some auxiliary results.

Lemma 17. *Let X be a binary curve of genus $g \geq 1$; let $\underline{d} = (d_1, d_2) \in B_d(X)$, with $0 \leq d \leq 2g - 2$. Assume $d_1 \leq d/2 - 1$. Then $W^{[\frac{d}{2}]}_{\underline{d}}(X) = \emptyset$.*

Proof. Let $L \in \operatorname{Pic}^{\underline{d}} X$ and $l = h^0(X, L)$; it suffices to prove that $l \leq d/2$.

If $d_1 < 0$ we have

$$l = h^0(C_2, L_2(-C_1 \cap C_2)) = \max\{0, d_2 - g\}.$$

As $\underline{d}$ is balanced, by (3) we have

$$d_2 - g \leq (d - g + 1)/2 \leq d/4$$

(as $g \geq d/2 + 1$). So we are done.

Let $d_1 \geq 0$; Lemma 10 yields $l \leq d + 1 - \min\{d_2, g\}$. If $d_2 \geq g$ we get

$$l \leq d + 1 - g \leq d + 1 - d/2 - 1 = d/2$$

(again, as $g \geq d/2 + 1$). So we are done. Finally, if $d_2 < g$,

$$l \leq d + 1 - d_2 = d_1 + 1.$$

By hypothesis, if d is even, $d_1 \leq d/2 - 1$, hence $l \leq d/2$ and we are done. If d is odd, $d_1 \leq (d-3)/2$, so that $l \leq (d-1)/2$, so we are done. □

Corollary 18. *Let X be a binary curve of genus g.* $\operatorname{Cliff} X = 0$ *if and only if there exists an integer h, $1 \leq h \leq g - 2$, such that $W^h_{(h,h)}(X) \neq \emptyset$.*

Assume $\operatorname{Cliff} X > 0$*; then* $\operatorname{Cliff} X = 1$ *if and only if there exists an integer h, $1 \leq h \leq g - 2$, such that $W^h_{(h,h+1)}(X) \neq \emptyset$.*

Proposition 19. *Let X be a binary curve; its dualizing sheaf, ω_X, is very ample if and only if $W^1_{(1,1)}(X) = \emptyset$ (if and only if X is not hyperelliptic).*

Proof. The part in parentheses follows from Remark 14 and Lemma 15. Assume $W^1_{(1,1)}(X)=\emptyset$. We denote $\dot{X}:=X\smallsetminus X_{\text{sing}}$ the smooth locus of $\dot{X}$. For every (not necessarily distinct) $p,q\in X$ we have

$$h^0(\omega_X(-p-q))=g-3+h^0(X,p+q)=g-2 \tag{22}$$

($h^0(X,p+q)=1$ by hypothesis and by Lemma 17).

Now, for every node $n\in X_{\text{sing}}$, denote $\nu:Y\to X$ the normalization at n, and $\nu^{-1}(n)=\{r,s\}$; note that $\omega_Y=\nu^*\omega_X(-r-s)$. Calling $\mathcal{I}_n$ the ideal sheaf of n in X, we have

$$h^0(X,\omega_X\otimes\mathcal{I}_n)=h^0(Y,\nu^*\omega_X(-r-s))=h^0(Y,\omega_Y)=g-1. \tag{23}$$

Formulas (22) and (23) yield that ω_X is globally generated and induces a morphism $\phi:X\to\mathbb{P}^{g-1}$ whose restriction to $\dot{X}$ is an immersion. It remains to prove that ϕ is injective, and an immersion locally at the singular points of X. Notice that for every nonsingular point $y\in Y$ we have

$$h^0(Y,\omega_Y(-y))=g-2 \tag{24}$$

(as $h^0(Y,y)=1$). Now, for every $p\in\dot{X}$ and $n\in X_{\text{sing}}$ we have, with the same notation as above (calling again $p\in Y$ the point over $p\in X$)

$$h^0(X,\omega_X(-p)\otimes\mathcal{I}_n)=h^0(Y,\nu^*\omega_X(-p-r-s))=h^0(Y,\omega_Y(-p))=g-2$$

by (24). Hence $\phi(n)\neq\phi(p)$. Now let $n_1,n_2\in X_{\text{sing}}$, denote $\nu':Y'\to X$ the normalization at n_1 and n_2, and $(\nu')^{-1}(n_i)=\{r_i,s_i\}$. We have

$$h^0(X,\omega_X\otimes\mathcal{I}_{n_1}\otimes\mathcal{I}_{n_2})=h^0(Y,\nu^*\omega_X(-r_1-s_1-r_2-s_2))=h^0(Y',\omega_{Y'})=g-2.$$

Therefore ϕ is injective. To show that ϕ is an immersion at every $n\in X_{\text{sing}}$ it suffices to show that $H^0(Y,\nu^*\omega_X(-2r-2s))\neq H^0(Y,\nu^*\omega_X(-2r-s))$ and that $H^0(Y,\nu^*\omega_X(-3r-s))\neq H^0(Y,\nu^*\omega_X(-2r-s))$ (notation as above). By (24),

$$h^0(Y,\nu^*\omega_X(-2r-s))=g-2.$$

On the other hand

$$h^0(Y,\nu^*\omega_X(-2r-2s))=h^0(Y,\omega_Y(-r-s))=g-4+h^0(Y,r+s)=g-3,$$

indeed, if we had $h^0(Y,r+s)=2$ then, by Lemma 9, $W^1_{(1,1)}(X)$ would be non empty, which is impossible. Similarly, $h^0(Y,\nu^*\omega_X(-3r-s))=g-4+h^0(Y,2r)=g-3$ by Proposition 12. This finishes the first half of the proof.

The opposite implication is easy; let ω_X be very ample. By contradiction, let $L\in W^1_{(1,1)}(X)$. For any $p\in\dot{X}$ we have $h^0(L(-p))=1$. So, $L=\mathcal{O}_X(p+q)$ for some $p,q\in\dot{X}$. Hence $h^0(\omega_X(-p-q))=g-1$, contradicting the very ampleness of ω_X. □

Lemma 20. *Let X be a binary curve of genus $g\geq 3$ with ω_X very ample. Then*

(i) $W^h_{(h,h)}(X)=\emptyset$ *for every* $2\leq h\leq g-2$.
(ii) $W^h_{(h,h+1)}(X)=\emptyset$ *for every* $2\leq h\leq g-4$.

Proof. As ω_X is very ample, we identify X with its canonical model in $\mathbb{P}^{g-1}$, which is a union of two rational normal curves, C_1 and C_2 meeting transversally at $g+1$ points. By contradiction, let $L\in W^r_{\underline{d}}(X)$, with (r,d) as in the statement.

We claim that there exists $D \in \mathrm{Div}\, X$, $D \geq 0$, D supported on the smooth locus of X, such that $L = \mathcal{O}_X(D)$. By contradiction, assume there is a node $n \in X_{\mathrm{sing}}$ such that $s(n) = 0$ for every $s \in H^0(X, L)$. Denote $\nu : Y \to X$ the normalization of X at n, so that Y is a binary curve of genus $g - 1 \geq 2$. Set $\nu^{-1}(n) = \{p, q\}$, and $M = \nu^* L$. By assumption, $h^0(Y, M(-p-q)) \geq h^0(X, L)$, therefore $h^0(Y, M(-p-q)) \geq h + 1$. On the other hand, $\underline{\deg}\, M(-p-q) = (h-1, h-1)$ is obviously balanced; furthemore $\underline{\deg}\, M \geq 0$, hence Clifford's inequality yields $h^0(Y, M(-p-q)) \leq h$, a contradiction. The claim is proved.

Fix such a D, and denote by $\Lambda \subset \mathbb{P}^{g-1}$ the linear subspace spanned by D (if D is reduced Λ is the ordinary linear span of the points of D, otherwise Λ is the linear span of the appropriate osculating spaces of X at the points of $\mathrm{Supp}\, D$). The geometric version of the Riemann-Roch Theorem ([ACGH] p. 12) yields

$$h^0(X, L) = \deg L - \dim \Lambda. \tag{25}$$

In case (i), since $h^0(X, L) \geq h + 1$ and $\deg L = 2h$, we get

$$\dim \Lambda \leq h - 1. \tag{26}$$

If $h = 1$, then $\dim \Lambda = 0$, which is impossible, as Λ is spanned by two distinct points (as $\underline{\deg}\, D = (1, 1)$). So we can assume $h \geq 2$. We denote $D = \sum_{i=1}^{h}(r_i + s_i)$ with $r_i \in C_1$ and $s_i \in C_2$. We have $h^0(X, D - r_1) \geq h + 1 - 1 \geq 2$, hence there exists an effective divisor $D' \neq D$, with $D \sim D'$, $\mathrm{Supp}\, D' \subset \dot{X}$, and such that r_1 is in the support of D'. Let Λ' the linear subspace spanned by D' and $\Gamma = < \Lambda, \Lambda' >$. We have $\dim \Lambda' \leq h - 1$ and

$$\dim \Gamma \leq 2h - 1 - c$$

where c is the degree of the greatest common (effective) divisor of D and D'; thus $c \geq 1$, by construction. Now we have, as $r_1 \notin C_2$,

$$\deg \Gamma \cdot C_2 \geq h + h - c + 1 = 2h - c + 1,$$

and this is impossible: C_2 is a rational normal curve, so Γ cuts on it a divisor of degree at most $\dim \Gamma + 1 = 2h - c$.

For part (ii) the method is essentially the same. By (25) we have $\dim \Lambda \leq h$ and Λ is an $(h, h+1)$-secant space of X. Set $D = \sum_{i=1}^{h}(r_i + s_i) + s_{h+1}$ with $r_i \in C_1$ and $s_i \in C_2$. We have $h^0(X, D - r_1) \geq 2$, hence there is an effective $D' \neq D$, $D \sim D'$, with $D' - r_1 \geq 0$. With the same notation as above, $\dim \Lambda' \leq h$ and $\dim \Gamma \leq 2h + 1 - c$, where $c \geq 1$ was defined above.

Now, $\deg \Gamma \cdot C_2 \geq 2h + 2 - c + 1 = 2h - c + 3$, a contradiction. □

End of the proof of Theorem 16. Part (II). By Lemma 15, X is hyperelliptic if and only if it is weakly hyperelliptic. If X is weakly hyperelliptic, then $\mathrm{Cliff}\, X = 0$. We prove the converse by showing that if X is not weakly hyperelliptic, then $\mathrm{Cliff}\, X > 0$. By Corollary 18, it is enough to prove that $W^h_{(h,h)}(X) = \emptyset$ for every h with $1 \leq h \leq g - 2$.

To say that X is not weakly hyperelliptic is to say that $W^1_{(1,1)}(X) = \emptyset$. By Proposition 19, this implies that ω_X is very ample. Lemma 20 yields $W^h_{(h,h)}(X) = \emptyset$, as wanted. The proof of part (II) is complete.

For part (III), one direction is obvious. For the converse, suppose $W^1_{\underline{d}}(X) = \emptyset$ for every $\underline{d} \in B_3(X)$ and let us prove that $\mathrm{Cliff}\, X > 1$. As we are also assuming

Cliff $X \neq 0$ we have $W^1_{(1,1)}(X) = \emptyset$, hence ω_X is very ample. Lemma 20 (ii) yields $W^h_{(h,h+1)}(X) = \emptyset$ for every $2 \leq h \leq g-4$. By Lemma 17 it remains to show that $W^1_{(1,2)}(X)$ and $W^{g-3}_{(g-3,g-2)}(X)$ are empty. The former is empty by assumption; the latter is empty because the former is (by Serre duality). Theorem 16 is proved. □

3.3. Extension of Martens theorem.

Lemma 21. *Let X be a hyperelliptic binary curve of genus $g \geq 2$, and $L \in \operatorname{Pic}^d X$ be balanced, with $0 \leq d \leq 2g-2$. Then $\operatorname{Cliff} L = 0$ if and only if $L = H_X^{\otimes \frac{d}{2}}$ (H_X as in Lemma 15).*

Proof. By the base-point-free-pencil trick we have $h^0(X, H_X^{\otimes \frac{d}{2}}) = d/2+1$, so that $\operatorname{Cliff} H_X^{\otimes \frac{d}{2}} = 0$. If $g=2$ the statement was proved in Proposition 13. We continue by induction on g. If $d = 2g-2$ then $L = \omega_X$, hence $\omega_X = H_X^{g-1}$. So we can further assume $d \leq 2g-4$. Let $\underline{d} = \underline{\deg} L$, so that $\underline{d} \in B_d(X)$. By Proposition 12 we must have $\underline{d} = (d/2, d/2)$; set $r = d/2$. Let $\nu : Y \to X$ be the normalization of X at one node, then $\nu^* L \in W^r_{(r,r)}(Y)$. Obviously (r,r) is balanced on Y. By induction $W^r_{(r,r)}(Y) = \{H^r_Y\}$, and $h^0(Y, H^r_Y) = r+1$ by Clifford. By Lemma 9, $W^r_{(r,r)}(X)$ contains at most one element, hence $L = H^r_X$. □

Martens Theorem holds for binary curves, by the following Proposition.

Proposition 22. *Let X be a binary curve of genus $g \geq 3$. Fix d, r such that $2 \leq d \leq g-1$ and $0 < 2r \leq d$. Let $\underline{d} = (d_1, d_2) \in B_d(X)$ and assume $r \leq d_i$ for $i = 1,2$ (otherwise $\dim W^r_{\underline{d}}(X) = \emptyset$, by Prop. 12).*

If X is not hyperelliptic, then $\dim W^r_{\underline{d}}(X) \leq d - 2r - 1$.

If X is hyperelliptic, then $\dim W^r_{\underline{d}}(X) = d - 2r$.

Proof. Recall that if X is hyperelliptic then $W^1_{(1,1)}(X) = \{H_X\}$; if X is not hyperelliptic, then $W^1_{(1,1)}(X)$ is empty. We use induction on g.

If $g = 3$ then $d = 2$ and $r = 1$, so the only case to consider is $\underline{d} = (1,1)$. If X is hyperelliptic, $W^1_{(1,1)}(X) = \{H_X\}$ so it is irreducible of dimension 0, as claimed. If X is not hyperelliptic, then $W^1_{(1,1)}(X) = \emptyset$, so we are done.

Let $g \geq 4$. If X is not hyperelliptic, by Lemma 15 there exists a node $n \in X_{\text{sing}}$ such that the normalization $\nu : Y \to X$ of X at n is not hyperelliptic. Suppose $W^r_{\underline{d}}(X) \neq \emptyset$; consider the pull-back map

$$\rho : W^r_{\underline{d}}(X) \longrightarrow W^r_{\underline{d}}(Y); \quad L \mapsto \nu^* L.$$

Notice that $\underline{d} \in B_d(Y)$; indeed if (say)

$$d_1 < m(d, g-1) = \frac{d-(g-1)-1}{2} = \frac{d-g}{2} \leq \frac{g-1-g}{2}$$

(as $d \leq g-1$). So $d_1 < 0$, hence $W^r_{\underline{d}}(X) = \emptyset$. A contradiction.

If $d \leq g-2 = g_Y - 1$ we use induction to get $\dim W^r_{\underline{d}}(Y) \leq d - 2r - 1$ and $\dim W^{r+1}_{\underline{d}}(Y) \leq d - 2r - 3$. Now, suppose $W^r_{\underline{d}}(Y)$ does not have the two points $\nu^{-1}(n)$ as fixed base points. Then the fibers of ρ over $W^r_{\underline{d}}(Y) \smallsetminus W^{r+1}_{\underline{d}}(Y)$ have dimension 0 (by Lemma 9), and over $W^{r+1}_{\underline{d}}(Y)$ have dimension at most 1. Therefore $\dim W^r_{\underline{d}}(X) \leq d - 2r - 1$.

If instead $\nu^{-1}(n)$ are base points of every element of $W^r_{\underline{d}}(Y)$, then, by induction, $\dim W^r_{\underline{d}}(Y) \leq (d-2) - 2r - 1 = d - 2r - 3$ and hence $\dim W^r_{\underline{d}}(X) \leq d - 2r - 2$. The case $d \leq g - 2$ is settled.

Now let $d = g - 1$; then, by Serre duality, $W^r_{\underline{d}}(Y) \cong W^{r-1}_{\underline{e}}(Y)$ where

$$\underline{e} = (g_Y - 1, g_Y - 1) - \underline{d} \in B_{g_Y-2}(Y),$$

by Remark 3 (d). Therefore, by induction,

$$\dim W^r_{\underline{d}}(Y) = \dim W^{r-1}_{\underline{e}}(Y) \leq g_Y - 2 - 2r + 2 - 1 = (g-1) - 2r - 1$$

and $\dim W^{r+1}_{\underline{d}}(Y) = \dim W^r_{\underline{e}}(Y) \leq (g-1) - 2r - 3$. Arguing as before we are done.

Let now X be hyperelliptic. Then $W^r_{(r,r)}(X) = \{H^r_X\}$ by Lemma 21. Therefore the statement holds if $d = 2r$, and we can assume $d > 2r$. An induction argument, analogous to the previous one, shows that $\dim W^r_{\underline{d}}(X) \leq d - 2r$ (now Y is hyperelliptic). To prove that equality holds, pick $x_1, \ldots, x_{d-2r} \in \dot{X}$ such that $\underline{\deg}\, H^r_X(\sum x_i) = \underline{d}$. It is clear that $H^r_X(\sum_1^{d-2r} x_i) \in W^r_{\underline{d}}(X)$. Moreover, by Lemma 7,

$$H^r_X(\sum_{i=1}^{d-2r} x_i) \ncong H^r_X(\sum_{i=1}^{d-2r} x'_i),$$

for x_i and x'_i generic. This shows that $\dim W^r_{\underline{d}}(X) \geq d - 2r$, finishing the proof. □

Suppose $d = g - 1$, then

$$\overline{W^0_{g-1,X}} = \Theta(X)$$

where $\Theta(X)$ is the Theta divisor, known to be Cartier and ample ([Al04]). It is thus worth pointing out the following special case of Proposition 22.

Remark 23. Let X be a binary curve of genus $g \geq 3$. For every multidegree $\underline{g-1} \in B_{g-1}(X)$ with $\underline{g-1} > 0$ we have

$$\dim W^1_{\underline{g-1}}(X) = \begin{cases} g-3 & \text{if } X \text{ is hyperelliptic} \\ g-4 & \text{otherwise.} \end{cases}$$

If X is an irreducible curve the same holds ([C07] Thm. 5.2.4).

4. Dimension of Brill-Noether varieties.

The Brill-Noether number $\rho^r_d(g)$ is defined as follows

$$\rho^r_d(g) = g - (r+1)(g-d+r) = (r+1)d - rg - (r+1)r. \tag{27}$$

By the famous Brill-Noether theorem, $\dim W^r_d(C) = \rho^r_d(g)$ for a general smooth curve C. The proof of this theorem has an interesting history, as many mathematicians have contributed to it: Arbarello, Cornalba, Eisenbud, Gieseker, Griffiths, Harris, Kempf, Kleiman, Laksov, Lazarsfeld, Martens, among others. We refer to Chapter 5 of [ACGH] for details and references.

The goal of this section is to prove it for binary curves, assuming $r \leq 2$. More precisely, we shall prove that for a general binary curve X of genus g and every balanced multidegree $\underline{d} \in B_d(g)$ we have $\dim W^r_{\underline{d}}(X) \leq \rho^r_d(g)$, with equality holding for certain $\underline{d}$. As a by-product we have a new proof for smooth curves. More generally, Theorem 24 implies that the Brill-Noether theorem holds on every stratum of $\overline{M}_g$ containing B_g in its closure.

Theorem 24. *Let X be a general binary curve of genus g; fix $r \leq 2$ and $d \in \mathbb{Z}$; let $\underline{d} \in B_d(X)$. Then*

(i) $\dim W^r_{\underline{d}}(X) \leq \rho^r_d(g)$ *and equality holds for some* $\underline{d}$.

(ii) $\dim \overline{W^r_{d,X}} = \rho^r_d(g)$.

Proof. We have $\dim \overline{W^r_{d,X}} \geq \rho^r_d(g)$ (by Theorem V (1.1) in [ACGH], which is independently due to [K71] or [KL72]); also, $\rho^r_{d-1}(g-1) = \rho^r_d(g) - 1$. Therefore, by (15), part (ii) follows from part (i). So it suffices to prove $\dim W^r_{\underline{d}}(X) \leq \rho^r_d(g)$.

If $d \geq r+g$ then $\rho^r_d(g) \geq g$, so the statement is trivial. We shall thus assume $d \leq r+g-1$. If $d \leq 0$, then $W^r_{\underline{d}}(X) = \emptyset$ (by Proposition 12), unless $\underline{d} = (0,0)$, in which case $W^r_{\underline{d}}(X) = \{\mathcal{O}_X\} = W^0_{\underline{d}}(X)$ (by Corollary 2.2.5 in [C07]), so the theorem holds.

We can thus use induction on d. We set $d_1 \leq d_2$. By Lemma 12, $W^r_{\underline{d}}(X) = \emptyset$ if $d_1 \leq r-1$; therefore we can assume $d_i \geq r$.

We begin with $r = 0$, in which case a more precise result holds. Recall that we called $A_{\underline{d}}(X) \subset W_{\underline{d}}(X)$ the closure of the image of the $\underline{d}$-th Abel map; see Lemma 7.

Proposition 25. *Let X be any binary curve of genus g, and $d \leq g-1$. For every $\underline{d} \in B_d(X)$ with $\underline{d} \geq 0$, $\dim W_{\underline{d}}(X) = d$. Moreover $W_{\underline{d}}(X)$ has a unique irreducible component of dimension d, namely $A_{\underline{d}}(X)$.*

Proof. Suppose $d = g-1$. By Theorem 3.1.2 of [C07], if $\underline{d}$ is strictly balanced the proposition holds. For a binary curve, the only balanced, non strictly balanced, multidegree is $(-1, g)$, which is ruled out by hypothesis. The case $d = g-1$ is done.

We continue by induction on $g-d$. Let $d \leq g-2$ and consider the normalization $\nu : Y \to X$ of X at one node, n; set $\nu^{-1}(n) = \{p, q\}$. Thus Y is a binary curve of genus $g-1$. Consider

$$\rho : W_{\underline{d}}(X) \longrightarrow W_{\underline{d}}(Y); \quad L \mapsto \nu^* L.$$

If $\underline{d}$ is not balanced for Y, we may assume (up to switching C_1 and C_2)

$$d_1 < \frac{d-(g-1)-1}{2} \leq \frac{g-2-g}{2} = -1;$$

impossible. So $\underline{d}$ is balanced for Y. Thus, by induction, $W_{\underline{d}}(Y)$ has a unique component of dimension d, namely $A_{\underline{d}}(Y)$.

Call $B \subset A_{\underline{d}}(Y)$ the locus of M such that $h^0(M) = h^0(M(-p)) = h^0(M(-q)) = 1$. Then one easily checks that $\dim B \leq d-2$, hence $\dim \rho^{-1}(B) \leq d-1$ (since the fibers of ρ have dimension at most 1, of course).

By Lemma 7, there exists a dense open subset $U \subset A_{\underline{d}}(Y) \smallsetminus B$ such that for every $M \in U$ we have $h^0(Y, M) = 1$. By Lemma 9 the fibers of ρ over such and M is a unique point. Therefore $\rho^{-1}(U)$ is irreducible of dimension d

Now, by Proposition 22, $\dim W^1_{\underline{d}}(Y) \leq d-2$, therefore any other component (if it exists) of $W_{\underline{d}}(X)$ has dimension at most $d-1$. This proves that $W_{\underline{d}}(X)$ has a unique component, W, of dimension d; by Lemma 7, $W = A_{\underline{d}}(X)$. □

We point out a simple consequence.

Corollary 26. *Let X be any binary curve of genus g, and $d = g + r - 1$. Then $\dim W_{\underline{d}}(X) = \rho^r_d(g) = g - r - 1$ for every $\underline{d} \in B_d(X)$ with $d_i \geq r$ for $i = 1, 2$. Moreover $W_{\underline{d}}(X)$ has a unique irreducible component, W, of dimension $\rho^r_d(g)$, and for the general $L \in W$ we have $h^0(X, L) = r + 1$.*

Proof. Set $(d'_1, d'_2) = \underline{d}' = \underline{\deg}\,\omega_X - \underline{d} = (g - 1 - d_1, g - 1 - d_2)$. We have $d'_2 = g-1-d+d_1 \geq g-1-d+r = 0$; similarly $d'_1 \geq 0$, hence $\underline{d}' \geq 0$. Also, $\underline{d}'$ is balanced, because $\underline{d}$ is (by Remark 3 (d). By Serre duality, $W^r_{\underline{d}}(X) \cong W^0_{\underline{d}'}(X)$, and the corollary follows from Proposition 25 and Lemma 7. □

We now go on with the proof of the theorem.

Claim 27. *Assume $r \geq 1$. Let W be an irreducible component of $W^r_{\underline{d}}(X)$ having maximal dimension. Then the general $L \in W$ is globally generated.*

We have $\dim W \geq \rho^r_d(g)$. By contradiction, suppose that every section of L vanishes at $p \in X$. If X is smooth at p, then $L(-p)$ is a line bundle of multidegree $\underline{d}' = (d_1 - 1, d_2)$ (say). We claim $\underline{d}'$ is balanced. Indeed if $\underline{d}' \notin B_{d-1}(X)$ we must have (since $m(d, g) > m(d - 1, g)$)

$$d_1 - 1 < m(d-1, g) = \frac{d-g-2}{2} \leq \frac{r-3}{2}$$

(using $d \leq g + r - 1$). Therefore $d_1 < (r-1)/2$, hence $d_1 \leq r - 1$, a contradiction. So, $\underline{d}'$ is balanced; induction yields

$$\dim W^r_{\underline{d}'}(X) \leq \rho^r_{d-1}(g) = \rho^r_d(g) - r - 1.$$

Therefore the set of line bundles in $W^r_{\underline{d}}(X)$ admitting a base point has dimension at most $\rho^r_{d-1}(g) + 1 = \rho^r_d(g) - r \leq \rho^r_d(g)$ (consider the rational map $W^r_{\underline{d}}(X) \times X \dashrightarrow \operatorname{Pic}^{\underline{d}'} X$ mapping (L, p) to $L(-p)$). So, $\dim W < \rho^r_d(g)$, a contradiction.

Now assume that every section of L vanishes at a node n of X. Since X has finitely many nodes, we may assume that the node n is the same for the general L. Let $\nu : Y \to X$ be the normalization of X at n, so that Y is a binary curve of genus $g - 1$. Denote $\nu^{-1}(n) = \{p, q\}$. Then $\nu^* L(-p-q) \in W^r_{d_1-1,d_2-1}(Y)$. We claim that $(d_1 - 1, d_2 - 1)$ is balanced on Y. If that were not the case, then (say)

$$d_1 - 1 < m(d-2, g-1) = \frac{d-2-g+1-1}{2} = \frac{d-2-g}{2} \leq \frac{r-3}{2}.$$

Therefore (as before) $d_1 < (r-1)/2$, hence $d_1 \leq r - 1$; a contradiction. As $(d_1 - 1, d_2 - 1)$ is balanced we get (by induction)

$$\dim W^r_{(d_1-1,d_2-1)}(Y) \leq \rho^r_{d-2}(g-1) = g - 1 - (r+1)(g + r - d + 1) = \rho^r_d(g) - r - 2.$$

Now consider the map

$$W^r_{\underline{d}}(X) \to \operatorname{Pic}^{(d_1-1,d_2-1)} Y; \quad L \mapsto \nu^* L(-p-q).$$

Its fibers have dimension at most 1, of course. The restriction of the above map to W maps the general element of W in $W^r_{d_1-1,d_2-1}(Y)$, hence

$$\dim W \leq \dim W^r_{d_1-1,d_2-1}(Y) + 1 \leq \rho^r_d(g) - r - 2 + 1 < \rho^r_d(g),$$

which is impossible.The claim is proved.

Proof of Theorem 24 for $r = 1$.

For $i = 1, 2$ consider the moduli spaces $M_{0,g+1}(\mathbb{P}^1, d_i)$; there are natural maps

$$\begin{array}{rccc} \epsilon_i : & M_{0,g+1}(\mathbb{P}^1, d_i) & \longrightarrow & (\mathbb{P}^1)^{g+1} \\ & (\phi_i, (p_1, \dots, p_{g+1})) & \mapsto & (\phi_i(p_1), \dots, \phi_i(p_{g+1})). \end{array} \tag{28}$$

We thus obtain the cartesian diagram:

$$\begin{array}{ccc} V := M_{0,g+1}(\mathbb{P}^1, d_1) \times_{(\mathbb{P}^1)^{g+1}} M_{0,g+1}(\mathbb{P}^1, d_2) & \xrightarrow{\pi_1} & M_{0,g+1}(\mathbb{P}^1, d_1) \\ \pi_2 \downarrow & & \downarrow \epsilon_1 \\ M_{0,g+1}(\mathbb{P}^1, d_2) & \xrightarrow{\epsilon_2} & (\mathbb{P}^1)^{g+1}. \end{array}$$

As ϵ_1 and ϵ_2 are dominant, we get

$$\begin{aligned} &\dim V = \dim M_{0,g+1}(\mathbb{P}^1, d_1) + \dim M_{0,g+1}(\mathbb{P}^1, d_2) - \dim(\mathbb{P}^1)^{g+1} = \\ &2(d_1 + d_2) - 4 + 2(g + 1) - g - 1 = 2d + g - 3 = \rho^1_d(g) + 2g - 1 \end{aligned} \tag{29}$$

($\rho^1_d(g) = 2d-g-2$). Moreover, V is irreducible, as every scheme in the above diagram is so. Now, V has a natural $PGL(2)$-invariant map to B_g

$$\alpha_{\underline{d}} : V \xrightarrow{\psi} M_{0,g+1} \times M_{0,g+1} \xrightarrow{\gamma_g} B_g$$

(where ψ forgets the maps to $\mathbb{P}^1$). By Claim 27, $\alpha_{\underline{d}}$ dominates $B^1_{g,\underline{d}}$. Furthermore, for every curve $X \in \alpha_{\underline{d}}(V)$ there is a natural, $PGL(2)$-invariant map

$$\alpha^{-1}_{\underline{d}}(X) \longrightarrow W^1_{\underline{d}}(X); \tag{30}$$

this, together with Claim 27, yields

$$\dim W^1_{\underline{d}}(X) \leq \dim \alpha^{-1}_{\underline{d}}(X) - 3. \tag{31}$$

Also, $B^1_{g,\underline{d}}$ is irreducible and

$$\dim B^1_{g,\underline{d}} \leq \min\{\dim V - 3, \dim B_g\} = \min\{\rho^1_d(g) + 2g - 4, \dim B_g\}. \tag{32}$$

Recall that $\dim B_g = 2g - 4$. If $\alpha_{\underline{d}}$ dominates B_g, (31) yields, for X general,

$$\dim W^1_{\underline{d}}(X) \leq \dim V - \dim B_g - 3 = \rho^1_d(g).$$

On the other hand if $\alpha_{\underline{d}}$ does not dominate B_g, $W^1_{\underline{d}}(X)$ is empty.

If $\rho^1_d(g) < 0$, by (32) $\alpha_{\underline{d}}$ is not dominant, hence $W^1_{\underline{d}}(X) = \emptyset$ for X general in B_g.

If $\rho^1_d(g) \geq 0$ then, by [KL72] or [K71], there exists a $\underline{d}$ such that $\alpha_{\underline{d}}$ dominates B_g. By what we said above, for every such $\underline{d}$, $\dim W^1_{\underline{d}}(X) \leq \rho^1_d(g)$ for the general binary curve X. The proof for $r = 1$ is complete.

Proof of Theorem 24 for $r = 2$.

By Proposition 13 we can assume $g \geq 3$. Define $J \subset M_{0,g+1}(\mathbb{P}^2, d_1) \times M_{0,g+1}(\mathbb{P}^2, d_2)$ as follows

$$J = \{((\phi_1; p_1, \dots, p_{g+1}); (\phi_2; q_1, \dots, q_{g+1}))|\ \phi_1(p_i) = \phi_2(q_i)\ \ \forall i = 1, \dots, g + 1\}.$$

Consider the map $\Psi : J \longrightarrow M_0(\mathbb{P}^2, d_1)$, where Ψ is the projection to the first factor composed with the map forgetting $(p_1, \ldots, p_{g+1})$.

Pick $\phi_1 \in M_0(\mathbb{P}^2, d_1)$. For every $\phi_2 \in M_0(\mathbb{P}^2, d_2)$, either $\operatorname{Im}\phi_2 \cap \operatorname{Im}\phi_1$ is a finite set, or $\operatorname{Im}\phi_1 \subseteq \operatorname{Im}\phi_2$ (recall that $d_1 \leq d_2$); this second case occurs only if $d_2 = cd_1$ for some $c \geq 1$. We partition $J = J_a \cup J_b$, where J_a parametrizes points such that $\operatorname{Im}\phi_1 \not\subset \operatorname{Im}\phi_2$, and $J_b = J \smallsetminus J_a$. So, $J_b = \emptyset$ if and only if d_1 does not divide d_2, and $J_a = \emptyset$ if and only if $d_1 d_2 < g+1$.

Assume $d_1 d_2 \geq g+1$. The restriction of Ψ to J_a is dominant and $\operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2$ is made of $d_1 d_2$ distinct points, for ϕ_1 and ϕ_2 general. Hence there are finitely many choices for the $g+1$ marked points, $(p_1, \ldots, p_{g+1})$ and $(q_1, \ldots, q_{g+1})$. We conclude

$$\dim J_a = \dim M_0(\mathbb{P}^2, d_1) + \dim M_0(\mathbb{P}^2, d_2) = 3d - 2 \tag{33}$$

(cf. (1)). On the other hand, if $d_2 = cd_1$, the fiber of J_b over ϕ_1 is the set of all $(\phi_2; q_1, \ldots, q_{g+1})$ such that $\phi_2 = \psi \circ \phi_1$ with $\psi \in M_0(\mathbb{P}^1, c)$. Hence

$$\dim J_b \leq \dim M_0(\mathbb{P}^2, d_1) + \dim M_0(\mathbb{P}^1, c) + g + 1 = 3d_1 + 2c - 2 + g. \tag{34}$$

Now consider $B^2_{g,\underline{d}} \subset B^2_{g,d} \subset \overline{M}_g$. J has a natural map, $\beta_{\underline{d}}$, to $B^2_{g,\underline{d}}$, obtained by restricting to J the composition of the forgetful map (disregarding the maps), with the map γ_g

$$\beta_{\underline{d}} : J \longrightarrow M_{0,g+1} \times M_{0,g+1} \xrightarrow{\gamma_g} B_g.$$

We claim that $\beta_{\underline{d}}(J_b)$ is never dense in B_g (also when $d_1 d_2 < g+1$). Notice that the restriction of $\beta_{\underline{d}}$ to J_b forgets both ϕ_1 and $\psi \in M_0(\mathbb{P}^1, c)$, and it is invariant with respect to the $PGL(2)$ diagonal action on J_b. Therefore by (34), and recalling that $g > 2$, we have

$$\dim \beta_{\underline{d}}(J_b) \leq \dim J_b - (3d_1 - 1) - (2c - 2) - 3 \leq g - 2 < \dim B_g.$$

On the other hand $\beta_{\underline{d}}$ restricted to J_a is $PGL(3)$-invariant, hence, by (33),

$$\dim \beta_{\underline{d}}(J_a) \leq \dim J_a - \dim PGL(3) = 3d - 10 = 2g - 4 + \rho^2_d(g) = \dim B_g + \rho^2_d(g)$$

(as $\rho^2_d(g) = 3d - 2g - 6$).

Now we argue as for $r = 1$; note that $\beta_{\underline{d}}$ dominates $B^2_{g,\underline{d}}$. If $\beta_{\underline{d}}(J_a)$ is not dense in B_g, then, by what we said, $W^2_{\underline{d}}(X)$ is empty for $X \in B_g$ general. By the above inequality, this will always be the case if $\rho^2_d(g) < 0$.

As observed at the beginning of the proof, if $\rho^2_d(g) \geq 0$, then $\beta_{\underline{d}}$ dominates B_g for some $\underline{d}$. For such $\underline{d}$ we derive $\dim W^2_{\underline{d}}(X) = \rho^2_d(g)$ for the general binary curve X.

It remains to handle the case $d_1 d_2 < g+1$, when J_a is empty. We proved above that $\dim B^2_{g,\underline{d}} \leq \dim \beta(J_b) \leq g - 2 < 2g - 4$, hence $W^r_{\underline{d}}(X) = \emptyset$ for X general in B_g.

Theorem 24 is proved. □

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