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THE CALABI FLOW ON TORIC FANO SURFACES

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1. Introduction

This is a continuation of the earlier work by the authors on the Calabi flow [9, 10]. We follow the setup of [10]; in particular we shall use the results concerning the formation of singularities along the Calabi flow on Kähler surfaces which appear in [10]. Readers are encouraged to consult [10] for the setup and for references on this topic. The search for extremal Kähler metrics is a very hot topic in Kähler geometry and many people have been contributing in this effort; we list here a few relevant references [6, 21, 20, 2, 1, 3, 16].

We believe that the Calabi flow is an effective tool for exploring the existence of extremal metrics on compact Kähler manifolds. One of the main problems arising in the study of the Calabi flow is longtime existence. In [9], we proved that the Calabi flow exists as long as the Ricci curvature tensors of the evolving metrics stay bounded. This is the first attempt to understand a conjecture by the first author: starting from any smooth Kähler metric on a compact Kähler manifold (complex dimension of $n \geq 2$), the Calabi flow exists for all positive time. In [10], we focused on the study of the Calabi flow on Kähler surfaces with the assumption that the Sobolev constants of the evolving metrics are uniformly bounded. First we [10] studied the formation of singularities on Kähler surfaces. If the curvature tensor blows up along the Calabi flow, we could then construct a singular model, which is a complete asymptotically locally Euclidean (ALE) scalar flat Kähler surface; as in [11], we call such a singular model a *deepest bubble* (it was called a *maximal bubble* in [10]; but it seems that the word "maximal" is a bit confusing). Then we studied some examples where such a bubble cannot be formed; in particular, we considered a family of Kähler classes on Kähler surfaces of the differential type of $\mathbb{CP}^2\sharp k\overline{\mathbb{CP}^2} (1 \leq k \leq 3)$. These surfaces are known as del Pezzo surfaces with toric symmetry. We then followed the approach in [11] to analyze all possible deepest bubbles. Actually a deepest bubble can only be formed in a fairly restricted way; in particular with the toric symmetry. With the aid of the special geometry of manifolds we considered, in particular the toric symmetry and the discrete symmetry that those Kähler classes admit, we could rule out the formation of deepest bubbles. Hence we [10] could prove longtime existence and convergence of the Calabi flow for those examples. However, the analysis there is quite delicate, complicated and sometimes very challenging. It is also very hard to push these ideas beyond the examples we considered in [10]; for example, for the Kähler classes without discrete symmetry.

In this note we shall adopt a different strategy to rule out possible bubbles; in particular we shall use the toric condition in a more essential way. This allows us

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to prove some longtime existence and convergence results in a fairly large family of Kähler classes on toric Fano surfaces.

Let (M, J) be a compact Kähler surface and let $[\omega]$ be a fixed Kähler class on M. We shall use c_1 to denote the first Chern class of (M, J). We may define a functional

$$\mathcal{B}([\omega]) = 32\pi^2 \left(c_1^2 + \frac{1}{3}\frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \right) + \frac{1}{3}\|\mathcal{F}\|^2,$$

where $\|\mathcal{F}\|^2$ is the norm of Calabi-Futaki invariant [17, 8]. Our main result is

Theorem 1.1. *Let $(M, [\omega], J)$ be a toric Fano surface with positive extremal Hamiltonian potential. If the Calabi flow has a Kähler metric with toric symmetry for initial data which satisfies*

$$\int_M R^2 dV_g < \mathcal{B}([\omega]), \tag{1.1}$$

then the Calabi flow exists for all time and converges in subsequence to an extremal metric in $[\omega]$ in the Cheeger-Gromov sense.

The definition of *extremal Hamiltonian potential* will be given in Section 2. An immediate corollary of Theorem 1.1 is

Corollary 1.2. *Let $(M, [\omega], J)$ be a toric Fano surface with positive extremal Hamiltonian potential. If there is a toric metric $\omega_0 \in [\omega]$ such that the Calabi energy of ω_0 is less than $\mathcal{B}([\omega])$, then there exists an extremal metric in $[\omega]$.*

Remark 1.3. *In* [13, 15]*, Donaldson uses a continuity method to deform metrics to seek extremal metrics on toric surfaces and has made striking progress on the existence of constant scalar curvature metrics. His approach uses convex analysis, which depends on the fact that, one can express a toric metric on a toric surface in terms of a convex function in a convex polytope in $\mathbb{R}^2$.*

2. Sobolev Constant

In this section we shall prove that the Sobolev constants of the evolving metrics along the Calabi flow are uniformly bounded under certain natural geometric conditions. We shall first define an extremal Hamiltonian potential of an *invariant Käher metric* in a fixed Kähler class $(M, [\omega])$, which is essentially defined by [17, 22]. Recall that an extremal vector field for $(M, [\omega])$ is *a priori* determined [17] up to conjugation. Let $\mathcal{X}$ be the extremal vector field and let $\mathcal{X}_{\mathbb{R}}$ be the real part of $\mathcal{X}$. Define $\mathcal{K}_{\mathcal{X}}$ to be the set of all invariant metrics in $[\omega]$ which satisfy

$$\mathcal{K}_{\mathcal{X}} = \{\omega : L_{\mathcal{X}_{\mathbb{R}}}\omega = 0\}.$$

For any Kähler metric $\omega \in \mathcal{K}_{\mathcal{X}}$, one can define the real potential θ_ω [22] by

$$\nabla_\omega \theta_\omega = \mathcal{X}_{\mathbb{R}}$$

which satisfies the normalized condition

$$\int_M \theta_\omega \omega^n = 0.$$

We have the L^2 orthogonal decomposition [19, 8]

$$R_\omega = \underline{R} + \theta_\omega + \theta_\omega^{\perp},$$

where the average of the scalar curvature $\underline{R}$ is determined by $(M, [\omega])$. We can then define the extremal Hamiltonian potential as

Definition 2.1. *Let $\omega \in \mathcal{K}_{\mathcal{X}}$; the extremal Hamiltonian potential of ω is given by*

$$\rho_\omega = \underline{R} + \theta_\omega.$$

An extremal metric ω then satisfies $R_\omega = \rho_\omega$. By definition, the maximum and the minimum of θ_ω are the invariants of $(M, [\omega])$ (c.f. [22]). We may denote

$$\theta_- = \min_{\omega \in \mathcal{K}_{\mathcal{X}}} \theta_\omega, \ \theta_+ = \max_{\omega \in \mathcal{K}_{\mathcal{X}}} \theta_\omega.$$

We can also denote

$$\rho_- = \underline{R} + \theta_-, \ \rho_+ = \underline{R} + \theta_+.$$

It is clear that ρ_- and ρ_+ are the minimum and the maximum of the scalar curvature of an extremal metric respectively if it exists in $[\omega]$. If $(M, [\omega])$ is a Fano surface and the Futaki invariant of $[\omega]$ is zero, then $\theta_- = \theta_+ = 0$ and so $\rho_- = \rho_+ = \underline{R}$ is positive. Hence ρ_- is positive for Fano surfaces of differential type of $\mathbb{CP}^2 \sharp k \overline{\mathbb{CP}^2} (4 \le k \le 8)$. An interesting question is

Question 2.2. *Let $(M, [\omega])$ be a toric Fano surface; is ρ_- positive?*

Note that ρ_- is an invariant of $(M, [\omega])$ and it is computable; in particular when (M, J) is a toric Fano surface. Hence one can check numerically whether ρ_- is positive or not for any given Kähler class on M. However, in general it seems not easy to verify that it is positive since its expression is quite complicated. Without giving a detailed argument, S. Simanca claimed the answer to Question 2.2 yes (cf. [24]). Since no detailed computation is given in [23, 24], we believe that there is a real need for definitive clarification of this issue. Note that the average of the scalar curvature on $(M, [\omega])$ is positive when M is a Fano surface. Intuitively, if there is an extremal metric, then the scalar curvature of the extremal metric should be positive since it minimizes the Calabi energy. To verify this, one needs to consider the case of $M \sim \mathbb{CP}^2 \sharp k \overline{\mathbb{CP}^2}$ $(k = 1, 2, 3)$. When $k = 1$, one can check that the scalar curvatures of all the extremal metrics constructed by E. Calabi [6] are positive. LeBrun-Simanca [21] computed the Futaki invariant and the extremal vector field of a Kähler class explicitly for Kähler surfaces with a semi-free $\mathbb{C}^*$ action. In particular their results can be applied to toric Fano surfaces and one can compute further ρ_-. For example some explicit formulas are given in [28]. However, it seems that only when $(M, [\omega])$ admits some additional discrete symmetry, the formula of ρ_- is simple enough and one can check directly that it is actually positive. For example, when $k = 2$, it is proved that ρ_- is positive for the bilaterally symmetric Kähler classes [11].

We shall then show how to bound the Sobolev constants on Fano surfaces under natural geometric conditions. The idea dates back to Tian [26] for Kähler metrics of constant scalar curvature (see [27] also) and it is generalized to extremal metrics in Chen-Weber [12].

Lemma 2.3. *Let $(M, [\omega])$ be a Fano surface such that $\rho_- > 0$ and let g be a Kähler metric in $[\omega]$. If g is invariant ($g \in \mathcal{K}_{\mathcal{X}}$) and if*

$$\int_M R^2 dV_g < \mathcal{B}([\omega]), \tag{2.1}$$

then the Sobolev constant is bounded a priori as in (2.19).

When g is not invariant, Lemma 2.3 still holds with stronger restriction on the Calabi energy. But we shall not need this. We define the Sobolev constant for a compact 4 manifold (M, g) to be the smallest constant C_s such that the estimate

$$\|f\|^2_{L^4} \le C_s \left(\|\nabla f\|^2_{L^2} + V^{-1/2}\|f\|^2_{L^2} \right) \tag{2.2}$$

holds, where V is the volume of the manifold (M, g). Note that the Sobolev inequality (2.2) is scaling-invariant. When the Yamabe constant is positive, the Sobolev constant is essentially bounded by the Yamabe constant [4]. Recall that the Yamabe constant for a conformal class $[g]$ of Riemannian metrics on a compact 4 manifold is given by

$$Y_{[g]} = \inf_{\tilde g \in [g]} \frac{\int_M R_{\tilde g} dV_{\tilde g}}{\sqrt{\int_M dV_{\tilde g}}}.$$

By the celebrated work of Trudinger, Aubin and Schoen [5, 25], for any conformal class $[g]$ the infimum is achieved by the so-called Yamabe minimizer $g_Y \in [g]$ which necessarily has constant scalar curvature. If $\tilde g = u^2 g$, the scalar curvature is given by

$$R_{\tilde g} = u^{-3}(6\triangle_g u + R_g u),$$

so the Yamabe constant is given by the formula

$$Y_{[g]} = \inf_{u \ne 0} \frac{\int_M (6|\nabla u|^2 + R_g u^2) dV_g}{(\int_M u^4 dV_g)^{1/2}}. \tag{2.3}$$

Now we are in the position to prove Lemma 2.3.

Proof. We can rewrite (2.1) as

$$96\pi^2 c_1^2 - 2\int_M R^2 dV_g > \int_M (R - \underline{R})^2 dV_g - \|\mathcal{F}\|^2. \tag{2.4}$$

Following computations in [26, 11] (for example, see Section 5 [11]), we have

$$Y^2_{[g]} \ge 96\pi^2 c_1^2 - 2\int_M R^2 dV_g. \tag{2.5}$$

It then follows from (2.4) and (2.5) that

$$Y^2_{[g]} > \int_M (R - \underline{R})^2 dV_g - \|\mathcal{F}\|^2. \tag{2.6}$$

We shall also need a decomposition formula of the Calabi energy [19, 8],

$$\int_M (R - \underline{R})^2 dV_g - \|\mathcal{F}\|^2 = \int_M (R - \underline{R} - \theta_\omega)^2 dV_g. \tag{2.7}$$

First we show that $Y_{[g]}$ has to be positive. Choose a sequence of functions u_i $(u_i \ne 0)$ which minimizes the expression in (2.3). Hence we have

$$Y_{[g]} + \epsilon_i = \frac{\int_M (6|\nabla u_i|^2 + R_g u_i^2) dV_g}{(\int_M u_i^4 dV_g)^{1/2}}, \tag{2.8}$$

such that $\epsilon_i \to 0$ as $i \to \infty$. We can rewrite (2.8) as

$$(Y_{[g]} + \epsilon_i)\|u_i\|^2_{L^4} = 6\int_M |\nabla u_i|^2 dV_g + \int_M R u_i^2 dV_g, \tag{2.9}$$

where we write $R = R_g$ for simplicity. It then follows from (2.9) that

$$(2.10)\quad (Y_{[g]} + \epsilon_i)\|u_i\|^2_{L^4} - \int_M (R - \underline{R} - \theta_\omega)u_i^2 dV_g = 6\|\nabla u_i\|^2_{L^2} + (\underline{R} + \theta_\omega)\|u_i\|^2_{L^2}.$$

By the Cauchy-Schwarz inequality, we compute

$$(2.11)\quad \left|\int_M (R - \underline{R} - \theta_\omega)u_i^2 dV_g\right| \le \left(\int_M (R - \underline{R} - \theta_\omega)^2 dV_g\right)^{1/2} \left(\int_M u_i^4 dV_g\right)^{1/2}.$$

Then we compute, by (2.11),

$$(2.12)\quad (Y_{[g]} + \epsilon_i)\|u_i\|^2_{L^4} - \int_M (R - \underline{R} - \theta_\omega)u_i^2 dV_g \le (Y_{[g]} + \epsilon_i + \|R - \underline{R} - \theta_\omega\|_{L^2})\|u_i\|^2_{L^4}.$$

If $Y_{[g]} < 0$, then by (2.6) and (2.7), we know that

$$(2.13)\quad Y_{[g]} + \|R - \underline{R} - \theta_\omega\|_{L^2} < 0.$$

Since g is fixed, then by (2.13), $Y_{[g]} + \|R - \underline{R} - \theta_\omega\|_{L^2} + \epsilon_i$ is less than zero for sufficiently large i; hence by (2.12), we can get that for i large enough,

$$(2.14)\quad (Y_{[g]} + \epsilon_i)\|u_i\|^2_{L^4} - \int_M (R - \underline{R} - \theta_\omega)u_i^2 dV_g < 0.$$

However $\underline{R} + \theta_\omega \ge \rho_- > 0$, the right hand side of (2.10) is then positive, which contradicts (2.14). Hence $Y_{[g]} > 0$; it then follows from (2.6) that

$$(2.15)\quad Y_{[g]} > \|R - \underline{R} - \theta_\omega\|_{L^2}.$$

We can then rewrite (2.3) as, for $u > 0$,

$$(2.16)\quad \|u\|^2_{L^4} \le \frac{6}{Y_{[g]}}\|\nabla u\|^2_{L^2} + \frac{1}{Y_{[g]}}\int_M R u^2 dV_g.$$

It is easy to see that (2.16) holds for any u since $|\nabla |u|| \le |\nabla u|$ at $u \neq 0$. Now we rewrite (2.16) as

$$(2.17)\quad \|u\|^2_{L^4} - \frac{1}{Y_{[g]}}\int_M (R - \underline{R} - \theta_\omega)u^2 dV_g \le \frac{6}{Y_{[g]}}\|\nabla u\|^2_{L^2} + \frac{1}{Y_{[g]}}\int_M (\underline{R} + \theta_\omega)u^2 dV_g.$$

Note that $\underline{R} + \theta_\omega \le \rho_+$. It follows from (2.17) and the Cauchy-Schwarz inequality that

$$(2.18)\quad \left(1 - \frac{1}{Y_{[g]}}\|R - \underline{R} - \theta_\omega\|_{L^2}\right)\|u\|^2_{L^4} \le \frac{6}{Y_{[g]}}\|\nabla u\|^2_{L^2} + \frac{\rho_+}{Y_{[g]}}\|u\|^2_{L^2}.$$

It then follows from (2.18) that the Sobolev constant of g is bounded a priori. In other words, we have

$$(2.19)\quad C_s \le \max\left\{\frac{6}{Y_{[g]} - \|R - \underline{R} - \theta_\omega\|_{L^2}},\ \frac{\sqrt{V}\rho_+}{Y_{[g]} - \|R - \underline{R} - \theta_\omega\|_{L^2}}\right\}.$$

□

3. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. First let us recall the formation of singularities along the Calabi flow on Kähler surfaces. Let $(M, [\omega])$ be a toric Fano surface as in Theorem 1.1. Suppose that the Calabi flow exists on $[0, T)$, $0 < T \leq \infty$ and that the curvature tensor blows up when $t \to T$. Note that under the assumption in Theorem 1.1, the Sobolev constants of the evolving metrics are uniformly bounded by Lemma 2.3, since the Calabi energy is decreasing along the flow. Hence the result (Theorem 1.1, [10]) is applicable. Since the blowing up process is required in the following argument, we shall state the result as follows.

Proposition 3.1. *Assume the hypotheses in Theorem 1.1. If the curvature blows up when $t \to T$, there exists a sequence of points $(x_i, t_i) \in (M, [0, T))$ where $t_i \to T$ and $Q_i = \max_{t \leq t_i} |Rm| = |Rm(x_i, t_i)| \to \infty$ such that the pointed manifolds*

$$(M, x_i, Q_i g(t_i + t/Q_i^2))$$

converge locally smoothly to an ancient solution of the Calabi flow

$$(M_\infty, x_\infty, g_\infty(t)), t \in (-\infty, 0].$$

Moreover, $g_\infty(t) \equiv g_\infty(0)$ and $g_\infty := g_\infty(0)$ is a complete scalar flat ALE Kähler metric on M_∞.

One of the key points in [11] is that (M_∞, g_∞), as a limit of pointed manifolds (M, g_i), is toric since $g_i := Q_i g(t_i)$ is toric. Moreover (M_∞, g_∞) contains holomorphic cycles. The result (Proposition 16, [11]) is only stated for $M \sim \mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$, but the result and the proof hold for all toric Fano surfaces without any change. We shall state the result as follows.

Proposition 3.2. *Assume the hypotheses in Theorem 1.1. Suppose the curvature tensor blows up along the Calabi flow and let (M_∞, g_∞) be a deepest bubble. Then (M_∞, g_∞) is toric and $H_2(M_\infty, \mathbb{Z})$ is generated by holomorphically embedded $\mathbb{CP}^1$s in M_∞.*

On the other hand, we show that a holomorphic cycle cannot be formed in such a blowup process. The idea is more lucid when the cohomology class $[\omega]$ is rational.

Proposition 3.3. *Assume the hypotheses in Theorem 1.1. Let $[\omega] \in H^2(M, \mathbb{Q})$. Then (M_∞, g_∞) cannot contain a holomorphic $\mathbb{CP}^1$.*

Proof. (M_∞, g_∞) is the limit of pointed manifolds (M, g_i). Hence there is a sequence of compact sets K_i, $K_i \subset K_{i+1}$, $\cup K_i = M_\infty$, and a sequence of diffeomorphisms $\Phi_i : K_i \to \Phi_i(K_i) \subset M$,

$$\Phi_i^*(g_i) \to g_\infty,$$

where the convergence is smooth in K_{i-1}. Let S be an embedded holomorphic $\mathbb{CP}^1$ in M_∞. There is a sequence of compact two spheres, which are denoted as $S_i = \Phi_i(S)$ and $S_i \subset \{M, Q_i g(t_i)\}$. Let ω_∞ be the Kähler form of g_∞ and let $\omega_i = Q_i \omega(t_i)$ be the Kähler form of g_i. Since $\Phi_i^* g_i$ converges to g_∞ smoothly, then for any fixed positive constant ϵ we have

$$\left| \int_{S_i} \omega_i - \int_S \omega_\infty \right| = \left| \int_S \Phi_i^* \omega_i - \int_S \omega_\infty \right| < \epsilon \tag{3.1}$$

when i is sufficiently large. Hence $\int_{S_i} \omega_i$ is uniformly bounded and then

$$\int_{S_i} \omega(t_i) = \frac{1}{Q_i} \int_{S_i} \omega_i \to 0. \tag{3.2}$$

On the other hand, we know that

$$\int_{S_i} \omega(t_i) = \int_{S_i} \omega = [\omega][S_i] = a_i$$

is a constant depending only on $[\omega], [S_i]$. Since $[\omega] \in H^2(M, \mathbb{Q})$, there exists some $k \in \mathbb{N}$ such that $[k\omega] \in H^2(M, \mathbb{Z})$. It then follows that $\int_{S_i} k\omega$ is an integer, hence ka_i is an integer for any i. By (3.2), $a_i \to 0$, hence ka_i has to be zero when i large enough. It then follows that $a_i = 0$ when i is sufficiently large. If $a_i = 0$, by (3.1), it follows that

$$\int_S \omega_\infty = 0.$$

This contradicts the fact that S is a holomorphic embedded $\mathbb{CP}^1$ in M_∞. □

When $[\omega]$ is not a rational class, the proof is more involved. The key is then to show that $\{[S_i]\}$ can only contain finitely many homology classes, which rely on (3.1), (3.2) and on positivity of a Kähler class.

Proposition 3.4. *Assume the hypotheses in Theorem 1.1.* (M_∞, g_∞) *cannot contain a holomorphic* $\mathbb{CP}^1$*.*

Proof. We use the same notations as in Proposition 3.3. It is clear that we can still get (3.1) and (3.2) and when $i \to \infty$,

$$[\omega][S_i] = a_i \to 0. \tag{3.3}$$

We show that any such sequence $\{[S_i]\}$ contains only finite homology classes in $H_2(M, \mathbb{Z})$. Recall that the self-intersection of $S \in H_2(M_\infty, \mathbb{Z})$ is a negative integer [11]. Let $[S][S] = -k$, for some fixed integer $k \geq 1$. Since the self-intersection is invariant under diffeomorphism, hence for any i,

$$[S_i][S_i] = -k. \tag{3.4}$$

The toric Fano surfaces are described as $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$ ($\mathbb{CP}^2$ blowup at two distinct points), $\mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}^2}$ ($\mathbb{CP}^2$ blowup at three non-linear points). We only exhibit the example when $M \sim \mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}^2}$, all other examples are similar (and simpler). Let H be a hyperplane in $\mathbb{CP}^2$. M can be obtained by blowing up at three generic points on $\mathbb{CP}^2$. After blowup, we still use H to denote the corresponding hypersurface on M and $E_i, i = 1, 2, 3$ to denote the exceptional divisors. For simplicity, we use $[H], [E_i]$ to denote the homology classes and their Poincaré duals—the cohomology classes. The Kähler classes on M can be expressed as

$$[\omega]_{x,y,z} = 3[H] - x[E_1] - y[E]_2 - z[E_3].$$

Since $[\omega]$ is a positive class, then x, y, z have to satisfy that

$$0 < x, y, z; \text{ and } x + y, y + z, x + z < 3. \tag{3.5}$$

We can see (3.5) as follows: for example, $x = [E_1][\omega]_{x,y,z} > 0$ and $H - E_1 - E_2$ is a holomorphic curve which has area $3 - x - y$ with respect to $[\omega]_{x,y,z}$, hence $x + y < 3$.

And $H_2(M, \mathbb{Z})$ can be generated by $\{[H], [E_i], i = 1, 2, 3\}$. We can then express $[S_i]$ as

$$[S_i] = m[H] + n[E_1] + j[E_2] + l[E_3],$$

for some integers m, n, j, l. We can write (3.3) and (3.4) as, when $i \to \infty$,

$$3m - nx - jy - lz \to 0 \tag{3.6}$$

and

$$m^2 - n^2 - j^2 - l^2 = -k. \tag{3.7}$$

We can compute, by (3.6),

$$n^2 + j^2 + l^2 \geq \frac{(nx + jy + lz)^2}{x^2 + y^2 + z^2} \to \frac{9m^2}{x^2 + y^2 + z^2}. \tag{3.8}$$

Hence, by (3.7) and (3.8),

$$m^2 + k + 1 = n^2 + j^2 + l^2 + 1 \geq \frac{9m^2}{x^2 + y^2 + z^2}.$$

But by (3.5), it is easy to see that

$$x^2 + y^2 + z^2 < 9.$$

For any fixed x, y, z, it then follows that

$$m^2 \left(\frac{9}{x^2 + y^2 + z^2} - 1 \right) \leq k + 1.$$

It follows that m has at most finitely many solutions. So there are at most finitely many m, n, j, l such that (3.6) and (3.7) are satisfied. It then follows that the homology classes of $[S_i]$ are finite. Hence we can find a subsequence $S_{\bar{i}}$ of S_i, such that $[S_{\bar{i}}] \in H_2(M, \mathbb{Z})$ has the same homology class for any $\bar{i}$. Hence $a_{\bar{i}} = [\omega][S_{\bar{i}}]$ is a constant independent of $\bar{i}$. By (3.3), $a_{\bar{i}} \equiv 0$. It then follows that $[S][\omega_\infty] = 0$ by (3.1). This contradicts the fact that S is a holomorphic cycle in M_∞. □

Remark 3.5. *Similar idea can be applied to the Calabi flow on toric surfaces, if one assumes that the Sobolev constants of the evolving metrics are uniformly bounded.*

Now we shall state a convergence result for the Calabi flow.

Proposition 3.6. *Let (M, J) be a Kähler manifold. Suppose $(M, g(t), J), 0 \leq t < \infty$ is a solution of the Calabi flow such that the Sobolev constants and the curvature tensors of the evolving metrics are uniformly bounded. Then for every sequence $t_i \to \infty$, there is a subsequence t_{i_k} and a sequence of diffeomorphisms $\Phi_{i_k} : M \to M$ such that,*

$$\Phi_{i_k}^* g(t_{i_k}) \to g_\infty, \Phi_{i_k\,*}^{-1} \circ J \circ \Phi_{i_k *} \to J_\infty,$$

under a fixed gauge, where the convergence is in C^∞ topology and (M, g_∞, J_∞) is an extremal Kähler manifold with complex structure J_∞.

Proof. Since we have assumed that the Sobolev constants and curvature tensors are bounded, then it is clear that all higher derivatives of curvature tensors are uniformly bounded; for example, see Lemma 4.2 in [10]. We then use the standard ideas in Ricci flow (see Hamilton [18]) to get similar compactness results for the Calabi flow. For a sequence $t_i \to \infty$, there is a subsequence $t_{i_k} \to \infty$ such that

$$\{M, g(t+t_{i_k}), -t_{i_k} \le t \le 0\} \to \{M_\infty, g_\infty(t), -\infty \le t \le 0\}$$

in Cheeger-Gromov sense. The argument is well-known in geometric flows and we shall skip the details. Let $g_\infty = g_\infty(0), g_{i_k} = g(t_{i_k})$. In particular, $(M, g_{i_k}) \to (M_\infty, g_\infty)$. Namely, there exists a sequence of diffeomorphisms $\Phi_{i_k} : M \to M_\infty$ such that

$$\Phi_{i_k}^* g_{i_k} \to g_\infty.$$

If necessary, by taking a subsequence, we can get that $J_{i_k} = \Phi_{i_k}^{-1}{}_* \circ J \circ \Phi_{i_k *} \to J_\infty$. Since $\nabla_{g_{i_k}} J_{i_k} = 0$, it follows that $\nabla_{g_\infty} J_\infty = 0$. Hence J_∞ is still a complex structure which is compatible with g_∞. We then show g_∞ is an extremal metric. This follows from the fact that the Calabi flow is the gradient flow of the Calabi energy. For any $t_0 \in (-\infty, 0]$, we choose a sequence $\{t_{i_k}\}$ such that $t_{i_k} < t_{i_{k+1}} + t_0$. Let $\mathcal{C}(g)$ be the Calabi energy of g. Since the Calabi energy is decreasing along the Calabi flow, we have

$$\mathcal{C}(g_\infty) = \lim_{t_{i_k} \to \infty} \mathcal{C}(g(t_{i_k})) \ge \lim_{t_{i_{k+1}} \to \infty} \mathcal{C}(g(t_0 + t_{i_{k+1}})) = \mathcal{C}(g_\infty(t_0)).$$

It then follows that $g_\infty(t)$ is an extremal metric for any $t \in (-\infty, 0]$. □

Remark 3.7. *In general J_∞ does not have to be the same as J.*

Now we are in the position to prove Theorem 1.1. We argue by contradiction.

Proof. By Lemma 2.3, the Sobolev constants of the evolving metrics are uniformly bounded under the assumption in Theorem 1.1. If the curvature tensors are not uniformly bounded, there is a contradiction by Proposition 3.1, 3.2 and 3.4. Hence the curvature tensors have to be uniformly bounded and the Calabi flow exists for all time. It then follows that $(M, g(t), J)$ converges to an extremal metric (M, g_∞, J_∞) in subsequence in the Cheeger-Gromov sense by Proposition 3.6. We then finish the proof by showing that (M, J_∞) is biholomorphic to (M, J). The proof follows from [11] (Theorem 27) by using the toric condition carefully and the classification of complex surfaces. Theorem 27 in [11] is only stated for $M \sim \mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$ but the proof holds for all toric Fano surfaces. The key is that in the limiting process, the torus action converges and (M, g_∞, J_∞) is still toric. Moreover, the 2-torus action for (M, g_∞, J_∞) is holomorphic with respect to J_∞. We shall sketch the argument for $M \sim \mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}^2}$. Readers can refer to [11] for details. When $M \sim \mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}^2}$, each of holomorphic curves H, E_1, E_2, E_3 is the fixed point set of the isometric action of some circle action of 2-torus, and so each is totally geodesic with respect to the metrics along the Calabi flow. By looking at the corresponding fixed points set of the limit action of circle subgroups, we can find corresponding totally geodesic 2-spheres in (M, g_∞, J_∞) which are the limits of the images of these submanifolds. Moreover, these limit 2-spheres are holomorphic with respect to J_∞ and the homological intersection numbers of these holomorphic spheres do not vary. Namely, we have still three holomorphic $\mathbb{CP}^1$s with self-intersection -1 as the images of the original exceptional divisors E_1, E_2, E_3. Thus, by blowing down the images of E_1, E_2, E_3 and applying classification of the complex surfaces, we conclude that (M, J_∞) is biholomorphic to $\mathbb{CP}^2$ blown up at three generic points. So there exists a diffeomorphism Ψ such that $\Psi_* J = J_\infty$. So $\Psi^* g_\infty$ is an extremal metric in the class $[\omega]$ for (M, J). □

Remark 3.8. *We may define a functional*

$$\mathcal{A}[\omega] = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}\|^2.$$

This functional has the important property [8, 14] *that any Kähler metric g in the class $[\omega]$ satisifies the curvature inequality*

$$\int_M R^2 dV_g \geq 32\pi^2 \mathcal{A}([\omega])$$

with equality if and only if g is an extremal metric. A necessary condition for (1.1) *to hold is that $(M, [\omega])$ satisfies the generalized Tian condition in* [12]*,*

$$c_1^2 > \frac{2}{3}\mathcal{A}([\omega]).$$

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