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A SIMPLE CRITERION OF TRANSVERSE LINEAR INSTABILITY FOR SOLITARY WAVES

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Abstract. We prove an abstract instability result for an eigenvalue problem with parameter. We apply this criterion to show the transverse linear instability of solitary waves on various examples from mathematical physics.

1. Introduction

We shall study a generalized eigenvalue problem under the form

$$\sigma A(k)U = L(k)U \tag{1.1}$$

where $L(k)$, $A(k)$ are operators (possibly unbounded) which depend smoothly on the real parameter k on some Hilbert space H with moreover $L(k)$ symmetric. Our aim is to give an elementary criterion which ensures the existence of $\sigma > 0$ and $k \neq 0$ such that (1.1) has a nontrivial solution U. Our motivation for this problem is the study of transverse instability of solitary waves in Hamiltonian partial differential equations. Indeed, let us consider a formally Hamiltonian PDE, say in $\mathbb{R}^2$, under the form

$$\partial_t \mathcal{U} = \mathcal{J}\nabla H(\mathcal{U}), \quad \mathcal{J}^* = -\mathcal{J} \tag{1.2}$$

and assume that there is a critical point of the Hamiltonian (hence a stationary solution) $\mathcal{U}(x, y) = Q(x)$ which depends only on one variable. Note that many equations of mathematical physics have one-dimensional solitary waves solutions which can be seen as critical points of a modified Hamiltonian after a suitable change of frame. We shall consider a few examples below. An interesting question is the stability of the one-dimensional state when it is submitted to general two-dimensional perturbations. There are many examples where the one-dimensional state even if it is stable when submitted to one-dimensional perturbations is destabilized by transverse oscillations, we refer for example to [24], [1], [16]. In our previous works [21], [22], [23], we have developed a framework which allows to pass from spectral instability to nonlinear instability. The aim of this note is to state a general criterion which allows to get spectral instability. Note that the linearization of (1.2) about Q reads

$$\partial_t V = \mathcal{J}\mathcal{L}V \tag{1.3}$$

where $\mathcal{L} = D\nabla H(Q)$ is a symmetric operator. Since Q does not depend on the transverse variable y, if $\mathcal{J}$ and H are invariant by translations in y, we can look for a solution of (1.3) under the form

$$V(t, x, y) = e^{\sigma t}e^{iky}U(x). \tag{1.4}$$

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This yields an eigenvalue problem for U under the form

$$\sigma U = (JM)(k)U \tag{1.5}$$

where $M(k)$, $J(k)$ defined by

$$M(k)U = e^{-iky}\mathcal{L}(e^{iky}U), \quad J(k)U = e^{-iky}\mathcal{J}(e^{iky}U)$$

are operators acting only in the x variable. Consequently, if $J(k)$ is invertible, we can set the problem under the form (1.1) with $A(k) = J(k)^{-1}$. As we shall see on the examples, it may happen that the skew symmetric operator $J(k)$ (which very often does not depend on k) is not invertible. In these cases, we can also recast the problem under the form (1.1). For example, we can look for solutions of (1.5) under the form $U = J(k)^*V$ and thus get a problem under the form (1.1) with $A(k) = J(k)^*$, $L(k) = J(k)M(k)J(k)^*$.

For the sake of simplicity, we shall work within a real framework but our result can be easily generalized to complex Hilbert spaces. We shall also study (1.1) only for $k > 0$. A similar instability criterion for $k < 0$ can be obtained by setting $\tilde{A}(k) = A(-k)$, $\tilde{L}(k) = L(-k)$ and by studying the problem (1.1) for $\tilde{A}$ and $\tilde{L}$.

Let us fix the functional framework. We consider a (real) Hilbert space H with scalar product $(\cdot,\cdot)$. We assume that $L(k)$ is a self-adjoint unbounded operator with domain $\mathcal{D}$ continuously imbedded in H and independent of the real parameter k. Moreover, $L(k)$ as an operator from $\mathcal{D}$ to H is assumed to depend smoothly on k. Finally, we also assume that $A(k) \in \mathcal{L}(\mathcal{D}, H)$ and depends smoothly on k. A $\mathcal{C}^1$ dependence is actually sufficient for our purpose.

Our aim here is to present a criterion which allows to prove transverse instability in solitary waves stability problems. This amounts to prove the existence of a nontrivial solution of (1.1) with $k \neq 0$ and σ with positive real part. In solitary wave stability problem, 0 is very often (when the problem is translation invariant in x) an eigenvalue of $L(0)$ with eigenvector Q'. Consequently, since we know that (1.1) has a nontrivial solution for $\sigma = 0$, $k = 0$, we can look for a solution (σ, U, k) of (1.1) in the vicinity of this particular solution. The main difficulty to implement this strategy is that also very often in solitary waves stability problems, 0 is in the essential spectrum of $JM(0)$, therefore the standard Lyapounov-Schmidt reduction cannot be used. One way to solve this problem is to introduce an Evans function with parameter $D(\sigma, k)$ (we refer for example to [2], [9], [20], [14] for the definition of the Evans function) for the operator $JM(k)$ and then to study the zeroes of D in the vicinity of $(0,0)$ (after having proven that D has in a suitable sense a smooth continuation in the vicinity of $(0,0)$). We refer for example to [6], [3], [25] for the study of various examples. Let us also mention [8], [19], [12], for other approaches, where the eigenvalue problem is not reformulated as an ODE with parameters.

Here we shall present a simple approach which relies only on the properties of $L(k)$ which are rather easy to check (mostly since it is a self-adjoint operator) and does not rely in principle on the reformulation of the problem as an ODE.

Our main assumptions are the following:

- **(H1)** There exists $K > 0$ and $\alpha > 0$ such that $L(k) \geq \alpha \,\mathrm{Id}$ for $|k| \geq K$;
- **(H2)** The essential spectrum $Sp_{ess}(L(k))$ of $L(k)$ is included in $[c_k, +\infty)$ with $c_k > 0$ for $k \neq 0$;

(H3) For every $k_1 \geq k_2 \geq 0$, we have $L(k_1) \geq L(k_2)$. In addition, if for some $k > 0$ and $U \neq 0$, we have $L(k)U = 0$, then $(L'(k)U, U) > 0$ (with $L'(k)$ the derivative of L with respect to k);

(H4) The spectrum $Sp(L(0))$ of $L(0)$ is under the form $\{-\lambda\} \cup I$ where $-\lambda < 0$ is an isolated simple eigenvalue and I is included in $[0, +\infty)$.

Let us point out that the structure of the spectrum of $L(0)$ assumed in (H4) is one of the assumption needed to have the one-dimensional stability of the wave (at least when there is a one-dimensional group of invariance in the problem), we refer to [11]. Note that 0 may be embedded in the essential spectrum of $L(0)$.

Our main result is the following:

Theorem 1.1. *Assuming (H1-4), there exists $\sigma > 0$, $k \neq 0$ and $U \in \mathcal{D}\backslash\{0\}$ solutions of* (1.1).

Note that we get an instability with σ real and positive. Once the spectral instability is established, one may use the general framework developed in [22] to prove the nonlinear instability of the wave.

The assumption (H3) is clearly matched if $L'(k)$ is positive for every $k > 0$. This last property is verified for all the examples that we shall discuss in this paper. Moreover if $L'(k)$ is positive for $k > 0$, the proof of Theorem 1.1 can be slightly simplified (see Remark 2.1 below).

The paper is organized as follows. In the following section, we shall give the proof of Theorem 1.1. Next, in order to show how our abstract result can be applied, we shall study various examples: the KP-I equation, the Euler-Korteweg system and the Gross-Pitaevskii equation. Note that we have already used similar arguments to prove the instability of capillary-gravity solitary water-waves in [23]. We hope that our approach can be useful for other examples, we also believe that this approach can be adapted to many situations with slight modifications.

2. Proof of Theorem 1.1

The first step is to prove that there exists $k_0 > 0$ such that $L(k_0)$ has a one-dimensional kernel.

Let us set

$$f(k) = \inf_{\|U\|=1} (L(k)U, U).$$

Note that by (H4) $L(0)$ has a negative eigenvalue, hence we have on the one hand that $f(0) < 0$. On the other hand by assumption (H1), we have that $f(k) > 0$ for $k \geq K$. Since f is continuous, this implies that there exists a minimal $k_0 > 0$ such that $f(k_0) = 0$. For every $k < k_0$, we get that $L(k)$ has a negative eigenvalue (since $f(k)$ is negative and $L(k)$ self-adjoint, this is a direct consequence of the variational characterization of the bottom of the spectrum and of (H2) which gives that the essential spectrum of $L(k)$ is in $(0, +\infty)$). Actually, there is a unique negative simple eigenvalue. Indeed, if we assume that $L(k)$ has two (with multiplicity) negative eigenvalues, then $L(k)$ is negative on a two-dimensional subspace. By (H3), this yields that $L(0) \leq L(k)$ is also negative on this two-dimensional subspace. This contradicts (H4) which contains that $L(0)$ is nonnegative on a codimension one subspace.

By the choice of k_0 and (H2), we also have that the kernel of $L(k_0)$ is non-trivial.

To conclude, we first note that if for every $k \in (0, k_0)$ the kernel of $L(k)$ is trivial, then the kernel of $L(k_0)$ is exactly one-dimensional. Indeed, let us pick $k < k_0$, then, since $L(k)$ has a unique simple negative eigenvalue and a trivial kernel, we get that $L(k)$ is positive on a codimension one subspace. Since $L(k_0) \geq L(k)$ by (H3), this implies that the kernel of $L(k_0)$ is exactly one-dimensional.

Next, we consider the case that there exists $k_1 \in (0, k_0)$ such that $L(k_1)$ has a nontrivial kernel. Since $L(k_1)$ has a unique simple negative eigenvalue, we get that $L(k_1)$ is nonnegative on a codimension 1 subspace $\mathcal{V} = (\varphi)^\perp \equiv \{V \in \mathcal{D} : (V, \varphi) = 0\}$ φ being an eigenvector associated to the negative eigenvalue. Moreover, thanks to (H2), we have an orthogonal decomposition of $\mathcal{V}$,

$$\mathcal{V} = \text{Ker } L(k_1) \oplus_\perp \mathcal{P} \tag{2.1}$$

with $\mathcal{P}$ stable for $L(k_1)$ and $L(k_1)$ restricted to $\mathcal{P}$ coercive. Note that moreover Ker $L(k_1)$ is of finite dimension. For every $U \in \mathcal{S}$ where $\mathcal{S}$ is the unit sphere of Ker $L(k_1)$ i.e. $\mathcal{S} = \{U \in \text{Ker } L(k_1), \|U\| = 1\}$, we have by (H3) that $(L'(k_1)U, U) > 0$. From the compactness of $\mathcal{S}$, we get that $c_0 = \inf_{U \in \mathcal{S}} (L'(k_1)U, U)$ is positive. This yields that for every $k \geq k_1$ close to k_1 and U in $\mathcal{S}$,

$$(L(k)U, U) \geq \frac{c_0}{2}(k - k_1)$$

and hence by homogeneity that

$$(L(k)U, U) \geq \frac{c_0}{2}(k - k_1)\|U\|^2, \quad \forall U \in \text{Ker } L(k_1). \tag{2.2}$$

Now according to the decomposition (2.1) of $\mathcal{V}$, we can write $L(k)$ with the block structure

$$L(k) = \begin{pmatrix} L_1(k) & A(k) \\ A^*(k) & L_2(k) \end{pmatrix}.$$

By the choice of $\mathcal{P}$, $L_2(k_1)$ is coercive, therefore, there exists $\alpha > 0$ such that for every k close to k_1, we have

$$(L_2(k)U, U) \geq \alpha \|U\|^2, \quad \forall U \in \mathcal{P}. \tag{2.3}$$

Moreover, we also have that $A(k_1) = 0$ (since $\mathcal{P}$ is a stable subspace for $L(k_1)$). By the assumed regularity with respect to k, we thus get that

$$\|A(k)\|_{\mathcal{L}(\mathcal{P}, Ker\, L(k_1))} \leq M\,|k - k_1|, \quad \forall k \in [k_1/2, 2k_1] \tag{2.4}$$

for some $M > 0$.

Consequently, by using (2.2), (2.3) and (2.4), we get that for every $U = (U_1, U_2) \in \mathcal{V}$ and every $k > k_1$ close to k_1, we have

$$(L(k)U, U) \geq \frac{c_0}{2}(k - k_1)\|U_1\|^2 + \alpha\|U_2\|^2 - 2M(k - k_1)\|U_1\|\,\|U_2\|.$$

From the Young inequality, we can write

$$2M(k - k_1)\|U_1\|\,\|U_2\| \leq \frac{c_0}{4}\|U_1\|^2(k - k_1) + \tilde{M}(k - k_1)\|U_2\|^2$$

with $\tilde{M} = 4M^2/c_0$ and hence, we obtain

$$(L(k)U, U) \geq \frac{c_0}{4}(k - k_1)\|U_1\|^2 + (\alpha - \tilde{M}(k - k_1))\|U_2\|^2.$$

In particular, we get that for every $k > k_1$ close to k_1, $L(k)$ is coercive on $\mathcal{V}$ and hence positive. Let us take some $k < k_0$ with this last property. Since by (H3), $L(k_0) \geq L(k)$, we get that $L(k_0)$ is also positive on $\mathcal{V}$ which has codimension 1. Therefore the kernel of $L(k_0)$ is exactly one-dimensional.

We have thus obtained as claimed that there exists $k_0 > 0$ such that $L(k_0)$ has a one-dimensional kernel. Thanks to (H2), we also have that $L(k_0)$ is a Fredholm operator with zero index. We can therefore use the Lyapounov-Schmidt method to study the eigenvalue problem (1.1) in the vicinity of $\sigma = 0$, $k = k_0$ and $U = \varphi$ where φ is in the kernel of $L(k_0)$ and such that $\|\varphi\| = 1$.

We look for U under the form $U = \varphi + V$, where

$$V \in \varphi^{\perp} \equiv \{V \in \mathcal{D} \,:\, (V, \varphi) = 0\}.$$

Therefore we need to solve $G(V, k, \sigma) = 0$ with $\sigma > 0$, where

$$G(V, k, \sigma) = L(k)\varphi + L(k)V - \sigma A(k)\varphi - \sigma A(k)V, \quad V \in \varphi^{\perp}\,.$$

We shall use the implicit function theorem to look for V and k as functions of σ. Note that the same approach is for example used in [12]. We have that

$$D_{V,k}G(0, k_0, 0)[w, \mu] = \mu\Big[\frac{d}{dk}L(k)\Big]_{k=k_0}\varphi + L(k_0)w\,. \tag{2.5}$$

By using (H3), we obtain that $D_{V,k}G(0, k_0, 0)$ is a bijection from $\varphi^{\perp} \times \mathbb{R}$ to H. We can thus apply the implicit function theorem to get that for σ in a neighborhood of zero there exists $k(\sigma)$ and $V(\sigma)$ such that $G(V(\sigma), k(\sigma), \sigma) = 0$. This ends the proof of Theorem 1.1.

Remark 2.1. *Let us remark that if we assume that $L'(k)$ is positive for $k > 0$ in place of (H3), then we can simplify the argument giving a $k_0 \neq 0$ such that $L(k_0)$ has a one-dimensional kernel. Namely, in this case by using (H4), we have that $L(0)$ is nonnegative on a codimension 1 subspace $\mathcal{V}$ (given by $\mathcal{V} = \pi_{[0,+\infty)}(L(0)) \cap \mathcal{D}$, where $\pi_{[0,+\infty)}(L(0))$ is the spectral projection on the nonnegative spectrum of $L(0)$). Next, using that $L'(s)$ is positive for $s > 0$, we get for every $k > 0$ that*

$$(L(k)U, U) = \int_0^k (L'(s)U, U)\, ds + (L(0)U, U) \geq 0, \quad \forall U \in \mathcal{V}.$$

Moreover, if $(L(k)U, U) = 0$ for $U \in \mathcal{V}$ then the above identity yields

$$\int_0^k (L'(s)U, U)\, ds = 0$$

and hence by using again that $L'(k)$ is positive for $k > 0$, we obtain that $U = 0$. Consequently, we get that for $k > 0$, $L(k)$ is positive on a codimension 1 subspace. This yields that the dimension of the kernel of $L(k_0)$ is exactly one.

3. Examples

In this section we shall study various physical examples where Theorems 1.1 can be used to prove the instability of line solitary waves.

3.1. KP-I equation. We shall first see that the instability argument given in [22] can be interpreted in the framework of (1.1). Let us consider the generalized KP-I equation where

$$\partial_t u = \partial_x\big(- \partial_{xx} u - u^p\big) + \partial_x^{-1}\partial_{yy} u, \quad p = 2,\, 3,\, 4 \tag{3.1}$$

and $u(t, x, y)$ is real valued. There is an explicit one-dimensional solitary wave (which thus solves the generalized KdV equation):

$$u(t, x) = Q(x - t) = \Big(\frac{p+1}{2}\Big)^{\frac{1}{p-1}} \Big(\mathrm{sech}^2\big(\frac{(p-1)(x-t)}{2}\big)\Big)^{\frac{1}{p-1}}.$$

Note that in this problem, it suffices to study the stability of the speed one solitary wave since the solitary wave with speed $c > 0$ can be deduced from it by scaling: the solitary wave with speed $c > 0$ is given by

$$Q_c(\xi) = c^{\frac{1}{p-1}} Q(\sqrt{c}\,\xi).$$

After changing x into $x - t$ (and keeping the notation x) and linearizing about Q, we can seek for solution under the form

$$e^{\sigma t} e^{iky} V(x)$$

to get the equation

$$\sigma V = \partial_x\big(- \partial_{xx} - k^2\partial_x^{-2} + 1 - pQ^{p-1}\big)V.$$

We can seek for a solution V under the form $V = \partial_x U$ to get that U solves

$$-\sigma\partial_x U = \Big(- \partial_x(-\partial_{xx} + 1 - pQ^{p-1})\partial_x + k^2\Big)U.$$

Therefore, this eigenvalue problem is under the form (1.1) with

$$A(k) = -\partial_x, \quad L(k) = -\partial_x(-\partial_{xx} + 1 - pQ^{p-1})\partial_x + k^2.$$

By choosing $H = L^2(\mathbb{R})$ and $\mathcal{D} = H^4(\mathbb{R})$, we are in an appropriate functional framework. Note that $L(k)$ has a self-adjoint realization.

Let us check the assumptions (H1-4).

Since we have

$$(L(k)U, U) \geq \|\partial_{xx} U\|_{L^2}^2 + k^2\|U\|_{L^2}^2 + \|\partial_x U\|_{L^2}^2 - p\|Q^{p-1}\|_{L^\infty}\|\partial_x U\|_{L^2}^2$$

and that for every $\delta > 0$, there exists $C(\delta) > 0$ such that

$$\|\partial_x U\|_{L^2}^2 \leq \delta\|\partial_{xx} U\|_{L^2}^2 + C(\delta)\|U\|_{L^2}^2,$$

we immediately get that (H1) is verified.

Next, we note that $L(k)$ is a compact perturbation of

$$L_\infty(k) = -\partial_x(-\partial_{xx} + 1)\partial_x + k^2,$$

we thus get from the Weyl Lemma and the explicit knowledge of the spectrum of $L_\infty(k)$ that the essential spectrum of $L(k)$ is included in $[k^2, +\infty)$ and thus that (H2) is verified.

Assumption (H3) is obviously verified since $L'(k)$ is positive for every $k > 0$.

Finally, let us check (H4). Note that $L(0) = -\partial_x C\partial_x$ where C is a second order differential operator. We notice that $CQ' = 0$ and that by the same argument as above, the essential spectrum of C is contained in $[1, +\infty)$. Since Q' vanishes only once, we get by Sturm Liouville theory (we refer for example to [7], chapter XIII) that C has a unique negative eigenvalue with associated eigenvector ψ. Moreover, we also have that

$$(Cu, u) \geq 0 \quad \forall u \in (\psi)^{\perp} \tag{3.2}$$

After these preliminary remarks, we can get that $L(0)$ has a negative eigenvalue. Indeed by an approximation argument, we can construct a sequence u_n in $\mathcal{D}$ such that $\partial_x u_n$ tends to ψ in $\mathcal{D}$ then, for n sufficiently large $(L(0)u_n, u_n) = (C\partial_x u_n, \partial_x u_n)$ is negative. By the variational characterization of the lowest eigenvalue, we get that $L(0)$ has a negative eigenvalue. Moreover, for every U such that $(\partial_x U, \psi) = 0$, we have that

$$\big(L(0)U, U\big) = \big(C\partial_x U, \partial_x U\big) \geq 0$$

This proves that $L(0)$ is nonnegative on a codimension one subspace and hence that there is at most one negative eigenvalue. We have thus proven that (H4) is verified.

Consequently, we get from Theorem 1.1 that the solitary wave is transversally unstable.

3.2. Euler-Korteweg models. We consider a general class of models describing the isothermal motion of compressible fluids and taking into account internal capillarity. The main feature of these models is that the free energy F depends both on ρ and $\nabla\rho$. In the isentropic case, we have:

$$F(\rho, \nabla\rho) = F_0(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2$$

where $F_0(\rho)$ is the standard part and $K(\rho) > 0$ is a capillarity coefficient. The pressure which is defined by $p = \rho\frac{\partial F}{\partial \rho} - F$ reads

$$p(\rho, \nabla\rho) = p_0(\rho) + \frac{1}{2}\big(\rho K'(\rho) - K(\rho)\big)|\nabla\rho|^2$$

The equations of motion read

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{3.3}$$

$$\partial_t u + u \cdot \nabla u + \nabla(g_0(\rho)) = \nabla\big(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\big). \tag{3.4}$$

In this model, $\rho > 0$ is the density of the fluid and u the velocity, g_0 (which is linked to p_0 by $\rho g_0'(\rho) = p_0'(\rho)$) and $K(\rho) > 0$ are smooth functions of ρ for $\rho > 0$.

We shall consider a one-dimensional solitary wave of (3.3), (3.4) under the form

$$(\rho(t,x,y), u(t,x,y)) = (\rho_c(x - ct), u_c(x - ct)) = Q_c(x - ct)$$

such that

$$\lim_{x\to\pm\infty} Q_c = Q_\infty = (\rho_\infty, u_\infty), \quad \rho_\infty > 0. \tag{3.5}$$

We shall assume that

$$\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2. \tag{3.6}$$

This condition ensures that Q_∞ is a saddle point in the ordinary differential equations satisfied by the profile. Under this condition, one can find solitary waves, moreover, they have the interesting property that ρ_c' vanishes only once. We refer for example to [4], for the study of the existence of solitary waves for this system.

Here we shall study the (linear) transverse instability of these solitary waves. We shall restrict our study to potential solutions of (3.3), (3.4) that is to say solutions such that $u = \nabla\varphi$. Note that this will give a better instability result, this means that we are able to find instabilities even in the framework of potential solutions.

This yields the system

$$\partial_t \rho + \nabla\varphi \cdot \nabla\rho + \rho\Delta\varphi = 0, \tag{3.7}$$

$$\partial_t \varphi + \frac{1}{2}|\nabla\varphi|^2 + g_0(\rho) = K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2. \tag{3.8}$$

Changing x into $x - ct$ (and keeping the notation x) to make the wave stationary, linearizing (3.7), (3.8) about a solitary wave $Q_c = (\rho_c, u_c)$ and looking for solutions (η, φ) under the form

$$(\eta, \varphi) = e^{\sigma t} e^{iky} U(x),$$

we find an eigenvalue problem under the form (1.1) with $A(k) = J^{-1}$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$L(k) = \begin{pmatrix} -\partial_x\big(K(\rho_c)\partial_x\big) + k^2 K(\rho_c) - m & -c\partial_x + u_c\partial_x \\ c\partial_x - \partial_x\big(u_c\,\cdot\big) & -\partial_x\big(\rho_c\partial_x\,\cdot\big) + \rho_c k^2 \end{pmatrix}$$

where the function $m(x)$ is defined by

$$m = K'(\rho_c)\rho_c'' + \frac{1}{2}K''(\rho_c)(\rho_c')^2 - g_0'(\rho_c).$$

By taking $H = L^2(\mathbb{R})\times L^2(\mathbb{R})$, $\mathcal{D} = H^2 \times H^2$, we are in the right functional framework, in particular, $L(k)$ has a self-adjoint realization. Let us now check assumptions (H1-4):

- (H1): with $U = (\rho, \varphi)$, we have

$$\big(L(k)U, U\big) \geq \int_{\mathbb{R}} \Big(K(\rho_c)\big(|\partial_x\rho|^2 + k^2|\rho|^2\big) + \rho_c\big(|\partial_x\varphi|^2 + k^2|\varphi|^2\big) \\ - \mathcal{O}(1)\big(|\rho|(|\rho| + |\varphi|) + |\partial_x\rho|\,|\varphi|\big)\Big)\,dx$$

  where $\mathcal{O}(1)$ is independent of k. Since $K(\rho_c) \geq \alpha > 0$, we get by using the Young inequality:

$$ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2, \quad a,\, b \geq 0, \quad \delta > 0, \tag{3.9}$$

  that (H1) is verified for k sufficiently large.

- (H2) By standard arguments (see [13], for example), to locate the essential spectrum of $L(k)$, we have to study the spectrum of

$$L_\infty(k) = \begin{pmatrix} \big(K(\rho_\infty)(-\partial_{xx} + k^2\big) + g_0'(\rho_\infty) & -c\partial_x + u_\infty\partial_x \\ c\partial_x - u_\infty\partial_x & \rho_\infty\big(-\partial_{xx} + k^2\big) \end{pmatrix}.$$

  By using the Fourier transform, we can compute explicitly the spectrum of this operator, we find that μ is in the spectrum of $L_\infty(k)$ if and only if there exists ξ such that

$$\mu^2 - s(\xi, k)\mu + p(\xi, k) = 0$$

with

$$s = (K(\rho_\infty) + \rho_\infty)(k^2 + \xi^2) + g_0'(\rho_\infty),$$
$$p = \rho_\infty K(\rho_\infty)(k^2 + \xi^2)^2 + \rho_\infty g_0'(\rho_\infty)k^2 + \big(\rho_\infty g_0'(\rho_\infty) - (u_\infty - c)^2\big)\xi^2 \geq 0.$$

By using that ρ_∞ and $K(\rho_\infty)$ are positive and the condition (3.6), we get that the two roots are nonnegative for all k and strictly positive for $k \neq 0$. This proves that (H2) is matched.

- (H3) We have
$$L'(k) = \begin{pmatrix} 2kK(\rho_c) & 0 \\ 0 & 2\rho_c k \end{pmatrix}.$$
Consequently, (H3) is verified since ρ_c and $K(\rho_c)$ are positive.

- (H4) We shall use the following algebraic lemma:

Lemma 3.1. *Consider a symmetric operator on H under the form*
$$L = \begin{pmatrix} L_1 & A \\ A^* & L_2 \end{pmatrix}$$
with L_2 invertible. Then we have
$$(LU, U) = \Big(\big(L_1 - AL_2^{-1}A^*\big)U_1, U_1\Big) + \Big(L_2\big(U_2 + L_2^{-1}A^*U_1\big), U_2 + L_2^{-1}A^*U_1\Big).$$

The proof is a direct calculation. Note that the above lemma remains true as soon as the quadratic form in the right-hand side makes sense (and hence even if L_2^{-1} is not well-defined.)

Let us apply this lemma to $L(0)$. We see that with
$$A = (u_c - c)\partial_x, \quad L_2 = -\partial_x(\rho_c \partial_x \cdot),$$
if $u \in H^2$ solves the equation $L_2 u = -A^* U_1$, then
$$\partial_x u = -\frac{1}{\rho_c}(u_c - c)U_1.$$
Consequently, we get
$$AL_2^{-1}A^*U_1 = (u_c - c)\partial_x u = \frac{(u_c - c)^2}{\rho_c}U_1$$
and hence we have the following factorization:
$$(L(0)U, U) = (MU_1, U_1) + \int_{\mathbb{R}} \rho_c \Big|\partial_x U_2 + \frac{1}{\rho_c}(u_c - c)U_1\Big|^2 \, dx \tag{3.10}$$
where
$$MU_1 = -\partial_x\big(K(\rho_c)\partial_x U_1\big) - m\, U_1 - \frac{(u_c - c)^2}{\rho_c}U_1.$$
Note that M is a second order differential operator and that by using the profile equation satisfied by Q_c, we can check that ρ_c' is in the kernel of M. Since ρ_c' has a unique zero, this proves that M has exactly one negative eigenvalue with corresponding eigenfunction R. From the condition (3.6), we

also get that the essential spectrum of M is included in $[\alpha, +\infty)$ for some $\alpha > 0$. In particular (since M is self-adjoint), we get that

$$(MU_1, U_1) \geq 0, \quad \forall\, U_1 \in (R)^{\perp}. \tag{3.11}$$

We can now use these properties of M to prove that (H4) is matched. We can first get from (3.10) that $L(0)$ has indeed one negative direction. A first try would be to take $U_1 = R$ and

$$\partial_x U_2 = -\frac{1}{\rho_c}(u_c - c)R.$$

The problem is that this equation does not have a solution in L^2. Nevertheless, we can get the result by using an approximation argument. Indeed, again by cutting the low frequencies, we can choose a sequence $U_2^n \in H^2$ such that

$$\partial_x U_2^n \to -\frac{1}{\rho_c}(u_c - c)R$$

in H^2. Then since $(MR, R) < 0$, we get that $(L(0)U^n, U^n) < 0$, with $U^n = (R, U_2^n)$, for n sufficiently large and hence by the variational characterization of the smallest eigenvalue, we get that $L(0)$ has a negative eigenvalue. From (3.10) and (3.11), we then get that this negative eigenvalue is unique. This proves that (H4) is verified.

Consequently, we can use Theorem 1.1 to get the instability of the solitary wave. We have thus proven:

Theorem 3.2. *If a solitary wave satisfies the condition* (3.6)*, then it is unstable with respect to transverse perturbations.*

Note that a similar result has been obtained in [3] by using an Evans function calculation.

3.3. Travelling waves of the Gross-Pitaevskii equation. In this subsection, we consider the Gross-Pitaevskii equation which is a standard model for Bose-Einstein condensates,

$$i\partial_t \psi + \frac{1}{2}\Delta\psi + \psi(1 - |\psi|^2) = 0 \tag{3.12}$$

where the unknown ψ is complex-valued. This equation has well-known explicit one-dimensional travelling waves (the so-called dark solitary waves) whose modulus tend to 1 at infinity, for every $c < 1$ they read :

$$\psi(t, x, y) = \Psi_c(z) = \sqrt{1 - c^2}\tanh\left(z\sqrt{1 - c^2}\right)\big) + ic, \quad z = x - ct. \tag{3.13}$$

In the case of the standard solitary waves of the cubic focusing Schrödinger equation, their transverse instability which was shown by Zakharov and Rubenchik can be studied by a standard bifurcation argument since 0 is not in the essential spectrum of the linearized operator, we refer for example to [21] for the details. This is not the case for the dark solitary waves, 0 is in the essential spectrum of the linearized operator, we shall thus use the criterion given by Theorem 1.1.

Note that for $c \neq 0$, Ψ_c does not vanish. Consequently, we can study the stability of these waves (travelling bubbles) by using the Madelung transform, i.e. by seeking solutions of (3.12) under the form

$$\psi = \sqrt{\rho} e^{i\varphi}$$

with smooth ρ and φ. We then classically find that $(\rho, u = \nabla\varphi)$ is a solution of (3.3), (3.4) with:

$$g_0(\rho) = \rho - 1, \quad K(\rho) = \frac{1}{4\rho}.$$

The dark solitary waves for $c \neq 0$ becomes a solitary wave (ρ_c, u_c) of (3.3), (3.4) with

$$\rho_c(z) = c^2 + (1-c^2)\tanh^2\left(z\sqrt{1-c^2}\right)\Big),$$

$$u_c(z) = -\frac{c(1-c^2)}{\rho_c}\Big(1 - \tanh^2\left(z\sqrt{1-c^2}\right)\Big).$$

In particular, we thus have $\rho_\infty = 1$ and $u_\infty = 0$. Since $g_0'(\rho) = 1$, the condition (3.6) reduces to $c^2 < 1$. Consequently, we have by Theorem 3.2 that all the dark solitary waves with $|c| < 1$, $c \neq 0$ are unstable with respect to transverse perturbation. Note that the one-dimensional stability of these travelling bubbles was shown in [17].

It remains to study the case $c = 0$. Note that Ψ_0 is a stationary solution, the so-called black soliton, which has the very simple expression

$$\Psi_0(x) = \tanh\left(x\right).$$

Its one dimensional orbital stability has been shown in [10], [5]. Here we shall prove that it becomes transversally unstable by using Theorem 1.1. Since the Madelung transform is not appropriate (the solitary wave vanishes at the origin) we shall work directly on the formulation (3.12).

Linearizing (3.12) about Ψ_0, splitting real and imaginary parts and seeking solutions under the form (1.4) yield a problem under the form (1.1) with $A(k) = J^{-1}$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L(k) = \begin{pmatrix} \frac{1}{2}(-\partial_{xx} + k^2) + 3\Psi_0^2 - 1 & 0 \\ 0 & \frac{1}{2}(-\partial_{xx} + k^2) - (1 - \Psi_0^2) \end{pmatrix}.$$

Again, with $H = L^2 \times L^2$ and $\mathcal{D} = H^2 \times H^2$, we can check (H1-4).

(H1) and (H3) are obviously matched. Thanks to the decay of the solitary wave, we have that $L(k)$ is a compact perturbation of

$$L_\infty(k) = \begin{pmatrix} \frac{1}{2}(-\partial_{xx} + k^2) + 2 & 0 \\ 0 & \frac{1}{2}(-\partial_{xx} + k^2) \end{pmatrix}$$

and hence, a simple computation shows that (H2) is also matched. Finally, we can also easily check (H4). Let us set $L(0) = \mathrm{diag}\,(L_1, L_2)$. We first notice that

$$L_1 \Psi_0' = 0, \quad L_2 \Psi_0 = 0.$$

and that the essential spectrum of L_1 is contained in $[2, +\infty)$ and the one of L_2 in $[0, +\infty)$. Since Ψ_0' does not vanish and Ψ_0 vanishes only once, we get by Sturm-Liouville theory that 0 is the first eigenvalue of L_1 and that L_2 has a unique negative eigenvalue. This proves that (H4) is matched.

Consequently, we get from Theorem 1.1 that the black soliton Ψ_0 is transversally unstable.

We have thus proven:

Theorem 3.3. *For every c, $|c| < 1$, the dark solitary waves* (3.13) *are transversally unstable.*

Remark 3.4. *Using arguments as above, we can also prove the transverse instability of the one dimensional localized solitary waves of the nonlinear Schrödinger equation and thus obtain another proof of the classical Zakharov-Rubenchik instability result.*

Remark 3.5. *The most difficult assumption to check is often the assumption (H4). Note that on the above examples this is always a direct consequence of Sturm-Liouville theory which is an ODE result. In the above examples, the eigenvalue problem for $L(0)$ is already itself an ODE. Nevertheless, for the capillary-gravity solitary waves problem studied in* [23]*, there is a nonlocal operator arising in the definition of $L(0)$ and hence the eigenvalue problem for $L(0)$ cannot be formulated as an ODE. Nevertheless, it was proven by Mielke* [18] *that (H4) is matched since in the KdV limit the spectral properties of $L(0)$ are the same as the ones of the linearized KdV hamiltonian about the KdV solitary wave which are known (again thanks to Sturm Liouville theory).*

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