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INVARIANT SUBSPACES OF PARABOLIC SELF-MAPS IN THE HARDY SPACE

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Abstract. We provide a precise description of the lattice of invariant subspaces of composition operators acting on the classical Hardy space, whose inducing symbol is a parabolic non-automorphism. This is achieved with an explicit isomorphism between the Hardy space and the Sobolev Banach algebra $W^{1,2}[0,\infty)$ that induces a bijection between the lattice of the composition operator and the closed ideals of $W^{1,2}[0,\infty)$. In particular, each invariant subspace of parabolic non-automorphism composition operator always consists of the closed span of a set of eigenfunctions. As a consequence, such composition operators have no non-trivial reducing subspaces. For the sake of completeness, we also include a characterization of the closed ideals of the Banach algebra $W^{1,2}[0,\infty)$. Although such a characterization is known, the proof we provide here is somehow different.

1. Introduction

The problem of giving a precise description of the lattice of invariant subspaces of a bounded linear operator on Hilbert space is one of the most interesting and difficult in operator theory. Very few operators admit a useful description of the lattice of invariant subspaces. In fact, understanding the lattice of a particular operator can solve the invariant subspace problem. This was done by Nordgren, Rosenthal and Wintrobe, [13] and [14]. They consider the composition operator C_φ acting on the Hardy space, where φ is an automorphism of the disk fixing ± 1. They show that if every invariant subspace of C_φ of infinite dimension has a non-trivial invariant subspace, then the general conjecture is true.

Another instance, Beurling's Theorem provides a complete description for the invariant subspaces for the shift operator acting on $\mathcal{H}^2$. However, the lattice of invariant subspaces of the shift operator acting on the Bergman space is not completely understood, see [3], [4] or [9, Chapters 7 and 8].

In the present work, we will describe the invariant subspaces of the composition operators C_φ acting on the Hardy space $\mathcal{H}^2$, where φ is a parabolic non-automorphism that takes $\mathbb{D}$ into itself, which has the formula

$$(1) \qquad \varphi_a(z) = \frac{(2-a)z+a}{-az+2+a}, \qquad \text{where } \Re a > 0.$$

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Since $\varphi_a(\mathbb{D})$ is contained in $\mathbb{D}$, Littlewood's Subordination Principle implies the composition operator $(C_{\varphi_a} f)(z) = f(\varphi_a(z))$ acts boundedly on $\mathcal{H}^2$, see the book by Cowen and MacCluer [6] for more details.

If T is an operator on a Hilbert space $\mathcal{H}$ and x is a vector in $\mathcal{H}$, then the smallest invariant subspace of T that contains x is the closure of the linear span of the orbit of x under T. If that minimal subspace is $\mathcal{H}$, then x is called a cyclic vector. In the present work we describe all cyclic vectors for C_{φ_a}. In particular, the family of all composition operators induced by parabolic non-automorphism have common dynamics, since they have common cyclic vectors, Corollary 1.2. Moreover, each orbit of any vector under all composition operators induced by parabolic non-automorphisms has a common closure. This is an immediate consequence of Theorem 1.1.

To prove our main result a theorem due to Cowen [5] is essential, see also [6, Theorem 6.1]. He found the spectrum of C_{φ_a}. If $\Re a > 0$, the spectrum $\sigma(C_{\varphi_a})$ is the spiral

$$\sigma(C_{\varphi_a}) = \{0\} \cup \{e^{-at} : t \in [0, \infty)\}.$$

Indeed, C_{φ_a} has a well-known family of inner functions as its eigenfunctions,

$$C_{\varphi_a} e_t = e^{-at} e_t, \quad \text{where } e_t(z) = \exp\left(t \frac{z+1}{z-1}\right) \quad \text{for each } t \geq 0. \tag{2}$$

All invariant subspaces we consider in this work will be closed. Let $\operatorname{Lat} T$ denote the lattice of invariant subspaces of the bounded linear operator T and let $\mathbb{F}[0, \infty)$ denote the set of closed subsets of $[0, \infty)$. As usual, the closed span of the empty set is the trivial subspace consisting of just the zero vector. We will prove

Theorem 1.1. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then*

$$\operatorname{Lat} C_\varphi = \{\overline{\operatorname{span}}\, \{e_t : t \in F\} : F \in \mathbb{F}[0, \infty)\}.$$

In particular, any non-trivial invariant subspace of C_φ contains a non-trivial eigenfunction of C_φ. As an immediate corollary of the above theorem, we have

Corollary 1.2. *Composition operators induced by parabolic non-automorphisms that take the unit disk into itself have the same lattice of invariant subspaces and the same cyclic vectors.*

Recall that a subspace that is invariant for an operator as well as for its adjoint is called a reducing subspace. Using Theorem 1.1, we will prove

Theorem 1.3. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then C_φ has no non-trivial reducing subspace.*

The proof of Theorem 1.1 consists of two steps. First, it is shown that the adjoint operator $C_\varphi^\star$ is similar to the operator of multiplication by a cyclic element in the Sobolev Space $W^{1,2}[0, \infty)$, which is a commutative semisimple regular Banach algebra; the latter idea parallels the one used by Sarason [17] to describe the invariant subspaces of the Volterra operator. Second, the invariant subspaces of such a multiplication operator are precisely the closed ideals of the algebra that can be described by using some elements of the Gelfand Theory.

2. An isomorphism from $\mathcal{H}^2$ onto the Sobolev space $W^{1,2}[0,\infty)$

The Sobolev space $W^{1,2}[0,\infty)$ consists of those functions f in $L^2[0,\infty)$ absolutely continuous on each bounded subinterval of $[0,\infty)$ and whose derivative belongs to $L^2[0,\infty)$. It is well-known and easy to check that the space $W^{1,2}[0,\infty)$ becomes a Hilbert space endowed with the inner product

$$\langle f,g\rangle_{1,2} = \frac{1}{2}\int_0^\infty (f(t)\overline{g(t)} + f'(t)\overline{g'(t)})\,dt.$$

The corresponding norm will be denoted by $\|\cdot\|_{1,2}$. Similarly, we can define $W^{1,2}(\mathbb{R})$.

We will show up an isomorphism, which is closely related to the eigenfunctions of C_φ, between the Hardy space $\mathcal{H}^2$ and the Sobolev space $W^{1,2}[0,\infty)$ that will be crucial to prove Theorem 1.1. The inner functions $e_t(z) = \exp(t(z+1)/(z-1))$, with $t \geq 0$, allow us to consider a complex valued function for each f in $\mathcal{H}^2$ defined by

$$(\Phi f)(t) = \langle f, e_t\rangle_{\mathcal{H}^2}, \qquad t \geq 0.$$

The key point to prove that Φ is an isomorphism from $\mathcal{H}^2$ onto $W^{1,2}[0,\infty)$ is to consider the operator Ψ that to each f in $L^2(\mathbb{T})$, here $\mathbb{T}$ denotes the unit circle, assigns the function defined as

$$(\Psi f)(t) = \langle f, e_t\rangle_{L^2(\mathbb{T})}, \qquad t \in \mathbb{R}.$$

Let $W_0^{1,2}[0,\infty)$ denote the subspace of functions in $W^{1,2}(\mathbb{R})$ that vanish on $(-\infty,0]$. The space $W_0^{1,2}(-\infty,0]$ is defined similarly. Finally, let Π denote the upper half-plane of the complex plane. The Hardy space of the upper half-plane $\mathcal{H}^2(\Pi)$ consists of those functions f analytic on Π for which the norm

$$\|f\|^2_{\mathcal{H}^2(\Pi)} = \sup_{y>0}\int_{-\infty}^\infty |f(x+iy)|^2\,dx$$

is finite, see [16, p. 372]. We will still maintain the symbol $\mathcal{H}^2$ for the Hardy space of the unit disk. We have

Theorem 2.1. *The map Ψ is an isometric isomorphism from $L^2(\mathbb{T})$ onto $W^{1,2}(\mathbb{R})$. In addition, $\Psi(z\mathcal{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi(\bar{z}\overline{\mathcal{H}}^2) = W_0^{1,2}(-\infty,0]$.*

Proof. For each f in $L^2(\mathbb{T})$, we have

$$(\Psi f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\exp\left(t\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)d\theta, \qquad t\in\mathbb{R}.$$

The change of variables $x = i(1+e^{i\theta})/(1-e^{i\theta})$ yields

$$(1)\qquad (\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^\infty f\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{1+x^2}dx, \qquad t\in\mathbb{R}.$$

Therefore, $\Psi = \mathcal{F}MT$, where $\mathcal{F}$ denotes the Fourier transform,

$$(Mg)(y) = \frac{1}{\sqrt{\pi}}\frac{g(y)}{\sqrt{1+y^2}} \quad\text{and}\quad (Tf)(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+x^2}}f\left(\frac{x-i}{x+i}\right).$$

The obvious change of variables shows that T is an isometric isomorphism from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$. In addition, the properties of the Fourier transform along Plancherel's Theorem show that $\mathcal{F}M$ is an isometric isomorphism from $L^2(\mathbb{R})$ onto $W^{1,2}(\mathbb{R})$, which proves the first statement of the proposition.

Now, let f be in $z\mathcal{H}^2$, that is, $f(z) = zg(z)$ with g in $\mathcal{H}^2$. Using (1), we obtain

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{(x+i)^2}\,dx, \qquad \text{for each } t \in \mathbb{R}.$$

Since the map

$$h \to \frac{1}{\sqrt{\pi}(x+i)} h\left(\frac{x-i}{x+i}\right)$$

is an isometric isomorphism from $\mathcal{H}^2$ onto $\mathcal{H}^2(\Pi)$, see [10, p. 106], and multiplication by $(w+i)^{-1}$ is bounded on $\mathcal{H}^2(\Pi)$, we find that Ψf is the Fourier transform of a function of $\mathcal{H}^2(\Pi)$. Thus, the Paley-Wiener Theorem, see [16, p. 372], shows that Ψf, which is continuous, must vanish on $(-\infty, 0]$ and, therefore, $\Psi(z\mathcal{H}^2) \subset W_0^{1,2}[0,\infty)$. Similarly, $\Psi(\bar{z}\overline{\mathcal{H}}^2) \subset W_0^{1,2}(-\infty,0]$. The fact that $\Psi(z\mathcal{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi(\bar{z}\overline{\mathcal{H}}^2) = W_0^{1,2}(-\infty,0]$ follows immediately from the orthogonal decomposition $W^{1,2}(\mathbb{R}) = W_0^{1,2}(-\infty,0] \oplus [e^{-|t|}] \oplus W_0^{1,2}[0,\infty)$, which in turns follows, being Ψ an isometric isomorphism, from the orthogonal decomposition $L^2(\mathbb{T}) = \bar{z}\overline{\mathcal{H}}^2 \oplus [1] \oplus z\mathcal{H}^2$ and the fact that $\Psi 1 = e^{-|t|}$, where $[f]$ denotes the one-dimensional linear space spanned by the vector f. The proof is complete. □

Corollary 2.2. *The operator Φ defines an isomorphism from $\mathcal{H}^2$ onto $W^{1,2}[0,\infty)$. Indeed, $\|\Phi f\|_{1,2}^2 = \|f\|_{\mathcal{H}^2}^2 - |f(0)|^2/2$.*

Proof. Upon applying Theorem 2.1, Φ and Ψ coincide on $z\mathcal{H}^2$. Therefore, Φ defines an isometric isomorphism from $z\mathcal{H}^2$ onto $W_0^{1,2}[0,\infty)$. Since $e^{-|t|}$ is orthogonal to $W_0^{1,2}[0,\infty)$, so is $e^{-t}\chi_{[0,\infty)}$. Thus $W^{1,2}[0,\infty) = [e^{-t}\chi_{[0,\infty)}] \oplus W_0^{1,2}[0,\infty) = \Phi 1 \oplus \Phi(z\mathcal{H}^2) = \Phi(\mathcal{H}^2)$, which proves that Φ is an isomorphism. The formula for the norm is trivial. The proof is complete. □

Remark. In [7, Chaps. IV and V], it is also considered the isomorphism Φ. However, the norm on the space $\Phi(\mathcal{H}^2)$ is defined as $\|\Phi(f)\| = \|f\|_{\mathcal{H}^2}$, without identifying $\Phi(\mathcal{H}^2)$ with $W^{1,2}[0,\infty)$, and, consequently, more difficult to handle.

Now, we shall see that the adjoint of composition operators induced by parabolic non-automorphism can be regarded as a multiplication operator on $W^{1,2}[0,\infty)$.

Proposition 2.3. *Let φ_a, with $\Re a \ge 0$, be as in (1). Then the adjoint of C_{φ_a} acting on $\mathcal{H}^2$ is similar under Φ to the multiplication operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$, acting on $W^{1,2}[0,\infty)$.*

Proof. Using the eigenvalue equation (2), for each $f \in \mathcal{H}^2$, we have

$$(\Phi C_{\varphi_a}^\star f)(t) = \langle C_{\varphi_a}^\star f, e_t\rangle_{\mathcal{H}^2} = \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}\langle f, e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}(\Phi f)(t),$$

for each $t \ge 0$. Thus $M_\psi = \Phi C_{\varphi_a}^\star \Phi^{-1}$. The result is proved. □

The following proposition is another key point to find the description of the Lattice of C_{φ_a}

Proposition 2.4. *The operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$ and $\Re a > 0$, acting on $W^{1,2}[0,\infty)$ is cyclic with cyclic vector ψ.*

Proof. Let $k_\alpha(z) = (1 - \bar{\alpha} z)^{-1}$, where $\alpha = (a-1)/(a+1)$, be the reproducing kernel at $\alpha \in \mathbb{D}$ in the Hardy space $\mathcal{H}^2$. Since $\Phi k_\alpha = \psi$, by Proposition 2.3, it is enough to show k_α is cyclic for $C^\star_{\varphi_a}$. Suppose that f in $\mathcal{H}^2$ is orthogonal to the orbit of k_α under $C^\star_{\varphi_a}$. Then, for each $n \geq 0$, we have

$$0 = \langle C^{\star n}_{\varphi_a} k_\alpha, f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C^n_{\varphi_a} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C_{\varphi_{na}} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, f \circ \varphi_{na}\rangle_{\mathcal{H}^2} = f(\varphi_{na}(\bar{\alpha})).$$

Since $\{\varphi_{na}(\bar{\alpha})\}$ is not a Blaschke sequence, the function f is zero and the result follows. □

An interesting consequence of Corollary 2.2 is a summability theorem for the Laguerre polynomials. Set $u_n(z) = z^n$. Then $\widetilde{u}_n(t) = (\Phi u_n)(t) = L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$, where $L_n^{(-1)}(t)$ is the Laguerre polynomial of degree n and of index -1. Indeed, since $\widetilde{u}_n = \langle z^n, e_t(z)\rangle_{\mathcal{H}^2}$ is the n-th coefficient of the Taylor series of $e_t(z)$, by definition of the Laguerre polynomials see [18, p. 97], we have

$$e_t(z) = e^{-t}\exp\left(-\frac{2tz}{1-z}\right) = \sum_{n=0}^{\infty} e^{-t} L_n^{(-1)}(2t) z^n. \tag{2}$$

Therefore, it follows immediately

Corollary 2.5. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. Then the series $\widetilde{f}(t) = \sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$ converges in $W^{1,2}[0,\infty)$ if and only if $\{a_n\}$ is in the sequence space ℓ^2. Indeed, $\|\widetilde{f}\|^2_{1,2} = -|a_0|^2/2 + \|\{a_n\}_{n\geq 1}\|_2^2$.*

As an application of Corollary 2.5, we show the imbedding theorem for $W^{1,2}[0,\infty)$. This is a well known result, see for instance [2]. However, the proof we provide here is very simple and provides the best constant for the imbedding.

Corollary 2.6. *Each f in $W^{1,2}[0,\infty)$ satisfies $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$ and $\sqrt{2}$ is the best imbedding constant.*

Proof. By Corollary 2.5, we can write $f(t) = \sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}$, where $\{a_n\}$ is in ℓ^2. The Cauchy-Schwarz inequality and Corollary 2.5, for each $t \geq 0$, yields

$$|f(t)| = \left|\sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}\right| \leq \|f\|_{1,2}\left(2e^{-2t} + \sum_{n=1}^\infty (L_n^{(-1)}(2t))^2 e^{-2t}\right)^{1/2}.$$

Since $\|e_t\|_{\mathcal{H}^2} = 1$, using (2), one easily checks that the quantity into the brackets above equals to $1 + e^{-2t} \leq 2$ and, therefore, $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$. The fact that $\sqrt{2}$ is the best imbedding constant is straightforward. □

The following well known result follows immediately from Corollary 2.6.

Proposition 2.7. *The space $W^{1,2}[0,\infty)$ with the pointwise multiplication is a Banach algebra without identity.*

An element a in a Banach algebra $\mathcal{A}$ is called cyclic, if it is cyclic for the bounded multiplication operator M_a that assigns to each b in $\mathcal{A}$ the element ab. The following result is folklore for specialists in Banach algebras, although usually it is stated for commutative Banach algebras. We include a proof, since we have not been able to find a precise reference.

Proposition 2.8. *Let $\mathcal{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are the closed ideals of $\mathcal{A}$.*

Proof. First, since $\mathcal{A}$ has a cyclic element, it is commutative. Let a be a cyclic element of $\mathcal{A}$ and let $\mathcal{L}$ be an invariant subspace of M_a. Clearly,

$$\mathcal{M}_{\mathcal{L}} = \{b \in \mathcal{A} : bx \in \mathcal{L} \text{ for all } x \in \mathcal{L}\}$$

is a closed subalgebra of $\mathcal{A}$. Since $\mathcal{L}$ is an invariant subspace of M_a, we find that $a \in \mathcal{M}_{\mathcal{L}}$ and, therefore, $\mathcal{M}_{\mathcal{L}}$ contains the subalgebra generated by a and, being $\mathcal{M}_{\mathcal{L}}$ closed and a cyclic, it follows that $\mathcal{M}_{\mathcal{L}} = \mathcal{A}$. Hence, $\mathcal{L}$ is a left ideal and thus, being $\mathcal{A}$ commutative, an ideal of $\mathcal{A}$. On the other hand, each ideal of $\mathcal{A}$ is invariant with respect to M_a, which finishes the proof. □

3. Proof of Theorem 1.1

In view of Proposition 2.8, we need to know a description of the closed ideals of $W^{1,2}[0,\infty)$. Such a description is equivalent to the fact that $W^{1,2}[0,\infty)$ has spectral synthesis, see [8] or [1]. From any of these references, we have,

Theorem 3.1. *The closed ideals of $W^{1,2}[0,\infty)$ are*

$$\mathcal{I}_F = \{f \in W^{1,2}[0,\infty) : f \textit{ vanishes on } F\}, \qquad \textit{where } F \in \mathbb{F}[0,\infty).$$

Now, we have all the tools at hand to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.4, the symbol ψ is a cyclic element of the Banach algebra $W^{1,2}[0,\infty)$. Thus, from Proposition 2.8 and Theorem 3.1 it follows that

$$\operatorname{Lat} M_\psi = \{\{\widehat{f} \in W^{1,2}[0,\infty) : \widehat{f} \text{ vanishes on } F\},\ \text{where } F \in \mathbb{F}[0,\infty)\}.$$

Since $M_\psi = \Phi C_\varphi^\star \Phi^{-1}$, we have

$$\operatorname{Lat} C_\psi^\star = \{\{f \in \mathcal{H}^2 : \langle f, e_t\rangle_{\mathcal{H}^2} = 0 \text{ for each } t \in F\},\ \text{where } F \in \mathbb{F}[0,\infty)\}.$$

Since $\operatorname{Lat} C_\varphi$ consists of the orthogonal complements of $\operatorname{Lat} C_\varphi^\star$, the statement of Theorem 1.1 follows immediately. □

Now, the proof of Theorem 1.3 follows easily.

Proof of Theorem 1.3. Let F be in $\mathbb{F}[0,\infty)$ such that $N_F = \overline{\operatorname{span}}\{e_t : t \in F\}$ is nontrivial. We must show that its orthogonal complement $N_F^\perp$ is not invariant under C_φ. We need the following formula, which is easily checked

$$\langle e_t, e_s\rangle = e^{-|t-s|}, \qquad \text{for each } t, s \geq 0. \tag{1}$$

First assume that 0 is not in F. Set $t_0 = \min F$. Since $f_{t_0} = 1 - e^{-t_0} e_{t_0}$ is orthogonal to e_t for each $t \geq t_0$, we find that f_{t_0} is in $N_F^\perp$. If $N_F^\perp$ is invariant under C_φ, then $f_{t_0} - C_\varphi f_{t_0}$ is in $N_F^\perp$. But $f_{t_0} - C_\varphi f_{t_0} = e^{-t_0}(1 - e^{-at_0})e_{t_0}$ is also in N_F, which means that $f_{t_0} - C_\varphi f_{t_0} = 0$. Hence, $f_{t_0} \equiv 1$, a contradiction.

Assume now that 0 is in F. Let M_{e_1} denote the multiplication by e_1. We have

$$M_{e_1}(N_F) = e_1 \overline{\operatorname{span}}\{e_t :\ t \in F\} = \overline{\operatorname{span}}\{e_{1+t} :\ t \in F\} = N_{1+F}. \tag{2}$$

Clearly, M_{e_1} is a Hilbert space isometry preserving inner products. Therefore,

$$M_{e_1}\left(N_F^\perp\right) = M_{e_1}(N_F)^\perp. \tag{3}$$

Proceeding by contradiction, assume that $N_F^\perp$ is also invariant under C_φ. Then

$$M_{e_1}(C_\varphi(N_F^\perp)) \subseteq M_{e_1}(N_F^\perp).$$

Since, for f in $\mathcal{H}^2$, we have $C_\varphi(M_{e_1}f) = C_\varphi(e_1 f) = e^{-a}e_1 C_\varphi f = e^{-a}M_{e_1}(C_\varphi f)$, from the above display, it follows that $C_\varphi(M_{e_1}(N_F^\perp))$ is included in $M_{e_1}(N_F^\perp)$. Therefore, from (2) and (3), we immediately see that $C_\varphi(N_{1+F}^\perp) \subseteq N_{1+F}^\perp$, which is a contradiction because 0 is not in $1+F$. The proof is complete. □

4. Appendix. The closed ideals of $W^{1,2}[0,\infty)$

For the sake of completeness, we end by providing a proof of Theorem 3.1, which describes the closed ideals of $W^{1,2}[0,\infty)$.

A character on a Banach algebra $\mathcal{A}$ is a linear functional $\varkappa : \mathcal{A} \to \mathbb{C}$ such that $\varkappa(ab) = \varkappa(a)\varkappa(b)$ for each a and b in $\mathcal{A}$. We observe that any character on a Banach algebra is continuous [11, p. 201], that is, it belongs to the dual space $\mathcal{A}^\star$. The spectrum of $\mathcal{A}$ is the set $\Omega(\mathcal{A})$ of non-zero characters of $\mathcal{A}$ equipped with the weak-star topology. It is well-known that the spectrum of any Banach algebra is a Hausdorff locally compact topological space and it is compact whenever $\mathcal{A}$ has identity [11, p. 205].

We start by determining $\Omega(W^{1,2}[0,\infty))$. For each $t \geq 0$, let δ_t denote the reproducing kernel at t, that is, $f(t) = \langle f, \delta_t\rangle_{1,2} = \langle \Phi^{-1}f, e_t\rangle_{\mathcal{H}^2}$ for each $f \in W^{1,2}[0,\infty)$ and where Φ is the transform defined in Section 2. Since $W^{1,2}[0,\infty)$ is a Hilbert space, the weak-star topology coincides with the weak topology.

Proposition 4.1. *The spectrum of the Banach algebra $W^{1,2}[0,\infty)$ is*

$$\Omega(W^{1,2}[0,\infty)) = \{\delta_t : t \geq 0\}.$$

Furthermore, the mapping that to each t assigns δ_t is a homeomorphism from $[0,\infty)$ onto $\Omega(W^{1,2}[0,\infty))$.

Proof. Clearly, for each $t \geq 0$, the functional δ_t is a character on $W^{1,2}[0,\infty)$, that is, δ_t is in $\Omega = \Omega(W^{1,2}[0,\infty))$. To prove that each character on $W^{1,2}[0,\infty)$ is one of the δ_t's, we consider the Banach algebra $\mathcal{C}^1[0,1]$, with pointwise multiplication, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Consider also its Banach subalgebra $\mathcal{A}_0 = \{f \in \mathcal{C}^1[0,1] : f(1) = 0\}$. Then, it is easy to check that $(Tf)(x) = f(x/(1+x))$ defines a bounded operator from $\mathcal{A}_0$ into $W^{1,2}[0,\infty)$, which is also an algebra homomorphism. Now, if $\varkappa$ is a character of $W^{1,2}[0,\infty)$, then it is easy to see that the functional $\widetilde{\varkappa}$ on $\mathcal{C}^1[0,1]$ defined by $\widetilde{\varkappa}(f) = \varkappa(T(f - f(1))) + f(1)$ is also a character. Since the characters of $\mathcal{C}^1[0,1]$ are the point evaluations $f \to f(s)$, with $0 \leq s \leq 1$, see [11, p. 204], there is $0 \leq s \leq 1$ such that $\widetilde{\varkappa}(f) = f(s)$ for each f in $\mathcal{C}^1[0,1]$. If $s = 1$, it follows immediately that $\varkappa(Tf) = 0$ for each f in $\mathcal{A}_0$. Hence $\varkappa$ vanishes on the range of T, which is dense because it contains $\mathcal{C}_c^\infty[0,\infty)$. Therefore, $\varkappa$ is the zero functional. If $s \neq 1$, then set $t = s/(1-s) \geq 0$ and observe that $\varkappa(Tf) = (Tf)(t)$ for each $f \in \mathcal{A}_0$. Hence $\varkappa$ and δ_t coincide on a dense set, which implies that $\varkappa = \delta_t$. Thus we have shown that $\Omega = \{\delta_t : t \geq 0\}$.

Next, since each f in $W^{1,2}[0,\infty)$ is continuous, so is the mapping $t \to \delta_t$ from $[0,\infty)$ onto Ω. Since $\|\delta_t\|_{1,2} \leq \|\Phi^{-1}\|\|e_t\|_{\mathcal{H}^2} = \|\Phi^{-1}\|$, we find that Ω is norm bounded on the dual space. Since the weak topology of a separable Hilbert space is metrizable on bounded sets, it follows that Ω is metrizable. Thus, to prove that $t \to \delta_t$ is a homeomorphism, it suffices to show that $t_n \to t_0$ whenever $\delta_{t_n} \to \delta_{t_0}$. Suppose that this is not the case, then there is $\varepsilon > 0$ such that $|t_n - t_0| > \varepsilon$ for each positive integer n. Consider the $W^{1,2}[0,\infty)$-function defined for $t \geq 0$ by

$$f(t) = \begin{cases} \varepsilon - |t_0 - s|, & \text{if } |t_0 - s| \leq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\delta_{t_n}(f) = 0$ and $\delta_{t_0}(f) = \varepsilon$, we find that δ_{t_n} cannot converge to δ_{t_0}. Therefore, the mapping $t \to \delta_t$ is a homeomorphism. The result is proved. □

Now we turn our attention to the structure of the regular ideals of Banach algebras. An ideal $\mathcal{I}$ of a Banach algebra $\mathcal{A}$ is called regular when the quotient algebra $\mathcal{A}/\mathcal{I}$ has identity. In particular, the kernel of any character is a maximal regular ideal. Therefore, the mapping $\varkappa \mapsto \ker \varkappa$ defines a one-to-one correspondence between the spectrum of $\mathcal{A}$ and the set of its maximal regular ideals, which is denoted by $\mathfrak{M}$, see [11, p. 202]. Recall also that a complex algebra is called semisimple if the intersection of all maximal regular ideals, called Jacobson's radical, is zero. Thus a commutative Banach algebra $\mathcal{A}$ is semisimple if and only if the elements of $\Omega(\mathcal{A})$ separate points of $\mathcal{A}$, that is, the intersection of the kernels of the characters is zero.

Given $x \in \mathcal{A}$ and $\mathcal{M} \in \mathfrak{M}$, we denote by $\widehat{x}(\mathcal{M}) = x \bmod \mathcal{M}$ the image of x under the multiplicative linear functional corresponding to $\mathcal{M}$. The mapping $x \mapsto \widehat{x}$ is a homomorphism from $\mathcal{A}$ into $C_0(\mathfrak{M})$ called Gelfand's transform. The Gelfand transform is one-to-one if and only if $\mathcal{A}$ is semisimple [11, p. 207]. Recall also that a Banach algebra $\mathcal{A}$ is said to be regular whenever for each closed set $F \subseteq \mathfrak{M}$ and each point $\mathcal{M}_0$ in $\mathfrak{M}$ such that $\mathcal{M}_0 \notin F$, there is an element x in $\mathcal{A}$ such that $\widehat{x} \equiv 0$ in F and $\widehat{x}(\mathcal{M}_0) \neq 0$.

The next proposition follows immediately from generals results in the book of Rickart, see [15, p. 91],

Lemma 4.2. *Let $\mathcal{A}$ be a semisimple regular commutative Banach algebra. Then the closed ideals of $\mathcal{A}$ are*

$$\mathcal{I}_F = \left\{ \bigcap_{\varkappa \in F} \ker \varkappa : \ F \text{ is closed in } \Omega(\mathcal{A}) \right\}$$

if and only if for each $x \in \mathcal{A}$ there exists a sequence $\{x_n\}$ tending to x in $\mathcal{A}$ and $\widehat{x}_n$ vanishes on a neighborhood U_n of $h(x)$ with compact complement.

Thus Theorem 3.1 will be proved once we have shown that $W^{1,2}[0,\infty)$ is under the hypothesis of Lemma 4.2.

Proposition 4.3. *The Banach algebra $W^{1,2}[0,\infty)$ is semisimple and regular and the mapping $F \to \bigcap_{t\in F} \ker \delta_t$ is one-to-one from $\mathbb{F}[0,\infty)$ onto the set of closed ideals of $W^{1,2}[0,\infty)$.*

Proof. Since the characters δ_t's separate points, the Banach algebra $W^{1,2}[0,\infty)$ is semisimple. To prove that $W^{1,2}[0,\infty)$ is also regular, we have to show that for each closed F in Ω and each maximal regular ideal $\mathcal{M} \notin F$ there exists f in $W^{1,2}[0,\infty)$ such that $\widehat{f} = 0$ on F and $\widehat{f}(\mathcal{M}) \neq 0$. By Proposition 4.1 this is equivalent to show that for every closed set $F \subseteq [0,\infty)$ and each point $t_0 \in [0,\infty)\backslash F$ there exists f in $W^{1,2}[0,\infty)$ such that f vanishes on F and $f(t_0) \neq 0$, which is obvious.

It remains to show that the last hypothesis of Lemma 4.2 is also fulfilled. Indeed, the Gelfand transform of a function in $W^{1,2}[0,\infty)$ vanishes on a set in Ω if and only if the function vanishes on its preimage under the homeomorphism furnished by Proposition 4.1. Clearly, for each f in $W^{1,2}[0,\infty)$ there is a sequence $\{f_n\}$ in $\mathcal{C}_c^\infty[0,\infty)$ converging to f and such that the zero set of each f_n contains an open neighborhood U_n of the zero set of f. Then, by Lemma 4.2, each closed ideal of $W^{1,2}[0,\infty)$ is of the form $\bigcap_{t\in F}\ker\delta_t$ for some F in $\mathbb{F}[0,\infty)$, so the mapping $F\to\bigcap_{t\in F}\ker\delta_t$ is onto and since $\bigcap_{t\in F}\ker\delta_t\neq\bigcap_{t\in G}\ker\delta_t$ whenever $F\neq G$, it is also one-to-one. The result is proved. □

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