

DOUBLE AFFINE HECKE ALGEBRAS FOR THE SPIN SYMMETRIC GROUP

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ABSTRACT. We introduce a new class (in two versions, $\check{\mathfrak{H}}^\epsilon$ and $\check{\mathfrak{H}}^-$) of rational double affine Hecke algebras (DaHa) associated to the spin symmetric group. We establish the basic properties of the algebras, such as PBW and Dunkl representation, and connections to Nazarov's degenerate affine Hecke-Clifford algebra and to a new degenerate affine Hecke algebra introduced here. We formulate a Morita equivalence between the two versions of rational DaHa's. The trigonometric generalization of the above constructions is also formulated and its relation to the rational counterpart is established.

1. Introduction

1.1. The double affine Hecke algebras (DaHa) introduced by Cherednik [2] are intimately related to K. Saito's elliptic root systems [12] and the Dunkl operators [4], and they have numerous connections and applications to Macdonald polynomials, integrable systems, and (quantum) affine algebras, etc. The DaHa affords three variants of degeneration: rational, trigonometric, and elliptic. The rational DaHa, which goes back as a special case to Drinfeld [3], has been actively studied in recent years by many authors (see Etingof-Ginzburg [5] and the review of Rouquier [11] for extensive references).

The question addressed here is whether or not a reasonable notion of DaHa's associated to the spin symmetric group of Schur [13] exists. For Coxeter groups, there is a standard procedure to construct the associated Hecke algebras. The spin symmetric group is not a reflection group, and so a priori it is not clear whether such a DaHa should exist and, if it exists, how its characteristic feature looks like.

In this paper we provide a natural construction of rational and trigonometric DaHa's, denoted by $\check{\mathfrak{H}}^\epsilon$ and $\check{\mathfrak{H}}^-$, associated to the spin symmetric group, which not only exhibits new phenomenon but also suggests for further natural generalization. These new algebras afford favorable properties similar to the usual DaHa (see [5] and Suzuki [15]). Instead of constructing directly the DaHa for the spin symmetric group, our idea is to start with a construction of DaHa for a certain double cover of the hyperoctahedral group, which then by some Morita equivalence leads to a construction of the DaHa for the spin symmetric group. The twisted group algebra of this double covering group is a semidirect product $\mathcal{C}_n \rtimes \mathbb{C}S_n$ between the group algebra of the symmetric group S_n and a Clifford algebra \mathcal{C}_n in n generators.

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1.2. We introduce in Section 2 the rational double affine Hecke-Clifford algebra (DaHCa) $\check{\mathfrak{H}}^\epsilon$ with parameter u and show that $\check{\mathfrak{H}}^\epsilon$ has a standard PBW basis. In particular, $\check{\mathfrak{H}}^\epsilon$ admits a triangular decomposition with the middle part being the algebra $\mathcal{C}_n \rtimes \mathbb{C}S_n$. In contrast to the usual rational DaHa, the symmetry between the two halves of polynomial generators for $\check{\mathfrak{H}}^\epsilon$ has to be broken. We further provide a realization of $\check{\mathfrak{H}}^\epsilon$ in terms of Dunkl operators.

A degenerate affine Hecke-Clifford algebra \mathfrak{H}^ϵ was introduced by Nazarov [9] for studying the Young symmetrizer for the spin symmetric group (cf. Brundan-Kleshchev [1, 7] for recent development on representations of \mathfrak{H}^ϵ and see [3] and Lusztig [8] for the usual degenerate/graded affine Hecke algebras). We show that the DaHa $\check{\mathfrak{H}}^\epsilon$ contains a family of subalgebras isomorphic to \mathfrak{H}^ϵ , analogous to the usual DaHa setup (cf. [2, 5]).

We then introduce the rational spin double affine Hecke algebra (sDaHa) $\check{\mathfrak{H}}^-$ associated to the spin symmetric group and parameter u . The PBW theorem for $\check{\mathfrak{H}}^-$ provides the following isomorphism of vector spaces

$$\check{\mathfrak{H}}^- \cong \mathcal{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}S_n^- \otimes \mathbb{C}[y_1, \dots, y_n],$$

where $\mathbb{C}S_n^-$ is the spin symmetric group algebra. A key new feature is that the generators y_i 's commute with each other but the ξ_i 's *anti-commute* with each other (cf. Definition 3.1). We also introduce a (new) degenerate spin affine Hecke algebra \mathfrak{H}^- . The results for $\check{\mathfrak{H}}^-$ and \mathfrak{H}^- in Section 3 are parallel to those for $\check{\mathfrak{H}}^\epsilon$ and \mathfrak{H}^ϵ in Section 2.

In Section 4 we establish an explicit isomorphism of super algebras

$$\check{\mathfrak{H}}^\epsilon \xrightarrow{\cong} \mathcal{C}_n \otimes \check{\mathfrak{H}}^-.$$

In this case, the superalgebras $\check{\mathfrak{H}}^\epsilon$ and $\check{\mathfrak{H}}^-$ are said to be Morita super-equivalent (cf. [7, 13.2] for a justification of the terminology). We also establish another Morita super-equivalence between \mathfrak{H}^ϵ and \mathfrak{H}^- by exhibiting an superalgebra isomorphism $\mathfrak{H}^\epsilon \cong \mathcal{C}_n \otimes \mathfrak{H}^-$. Such isomorphisms provide a conceptual explanation and easy proofs for the results in Section 3 parallel to Section 2. Without such a connection to $\check{\mathfrak{H}}^\epsilon$, even the definition of $\check{\mathfrak{H}}^-$ is by no means obvious. A finite-dimensional version of such isomorphisms, discovered independently in Sergeev [14] and Yamaguchi [16], explained for the well-known fact that the representation theory of the spin symmetric group is (essentially) equivalent to that of the algebra $\mathcal{C}_n \rtimes S_n$. The three isomorphisms in the finite, affine, and double affine setups are compatible with each other.

We finally introduce the trigonometric DaHCa $\check{\mathfrak{H}}_{tr}^\epsilon$ and the trigonometric sDaHa $\check{\mathfrak{H}}_{tr}^-$. We construct explicitly a superalgebra isomorphism

$$\check{\mathfrak{H}}_{tr}^\epsilon \cong \mathcal{C}_n \otimes \check{\mathfrak{H}}_{tr}^-.$$

We further establish the precise connections between our rational and trigonometric DaHa's, following Suzuki [15] in the usual DaHa setup. This is developed in Section 5.

1.3. This paper is a first step of a program in which we attempt to develop a new theory of spin Hecke algebras. The new features of the constructions in this paper will be instrumental in telling us how to relax the usual setup (cf. [3, 5] and Ram-Shepler [10] for the degenerate case) in order to define appropriate spin (degenerate/double) affine Hecke algebras more generally. The algebras $\check{\mathfrak{H}}^\epsilon$ and $\check{\mathfrak{H}}^-$ have large centers, and

are expected to have rich finite-dimensional representation theory (compare [5] and Gordon [6]).

2. The rational double affine Hecke-Clifford algebras

2.1. The symmetric group S_n . Recall that the symmetric group S_n is generated by s_i ($1 \leq i \leq n - 1$) subject to the relations:

$$(2.1) \quad \begin{aligned} s_i s_j &= s_j s_i, & |i - j| > 1 \\ s_i^2 &= 1, & s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

Let s_{ij} denote the transposition of i and j in the symmetric group S_n .

2.2. The rational DaHCa.

Definition 2.1. Let $u \in \mathbb{C}$. The rational *double affine Hecke-Clifford algebra* (or simply, DaHCa) $\check{\mathfrak{H}}^c$ is the \mathbb{C} -algebra generated by x_i, y_i, c_i ($1 \leq i \leq n$) and S_n , subject to the following relations:

$$(2.2) \quad \begin{aligned} x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i, & (\forall i, j) \\ \sigma x_i &= x_{\sigma i}, & \sigma y_i &= y_{\sigma i} & (\sigma \in S_n) \\ c_i x_i &= -x_i c_i, & c_i y_i &= y_i c_i, \\ c_j x_i &= x_i c_j, & c_j y_i &= y_i c_j, & (i \neq j) \end{aligned}$$

$$(2.3) \quad \sigma c_i = c_{\sigma i} \sigma \ (\sigma \in S_n), \quad c_i^2 = 1, \quad c_i c_j = -c_j c_i \ (i \neq j)$$

$$(2.4) \quad [y_j, x_i] = u(1 + c_j c_i) s_{ij}, \quad (i \neq j)$$

$$(2.5) \quad [y_i, x_i] = -u \sum_{k \neq i} (1 + c_k c_i) s_{ki}.$$

Denote by \mathcal{C}_n , or $\mathcal{C}(c_1, \dots, c_n)$, the Clifford algebra generated by c_1, \dots, c_n . The rational DaHCa $\check{\mathfrak{H}}^c$ is a super (i.e. \mathbb{Z}_2 -graded) algebra with $|c_i| = \bar{1}$ and $|x_i| = |y_i| = |s_{ij}| = \bar{0}$. Unlike the usual rational DaHa, the symmetry between x 's and y 's is broken in the definition of $\check{\mathfrak{H}}^c$, cf. (2.2).

For $u = 0$, we have $\check{\mathfrak{H}}^c|_{u=0} = (\mathbb{C}[\underline{x}, \underline{y}] \tilde{\otimes} \mathcal{C}_n) \# S_n$, where the tilde refers to the unusual sign in (2.2). The $\check{\mathfrak{H}}^c|_{u=0}$ clearly has the PBW property, i.e. the $\check{\mathfrak{H}}^c|_{u=0}$ has a linear basis $\{\underline{x}^a \underline{y}^b \sigma \underline{c}^\epsilon\}$, where $\sigma \in S_n$, $\underline{a}, \underline{b} \in \mathbb{Z}_+^n$, $\epsilon \in \{0, 1\}^n$ and \underline{x}^a denotes the monomials $x_1^{a_1} \dots x_n^{a_n}$ etc. The algebras $\check{\mathfrak{H}}^c$ are isomorphic for all $u \neq 0$.

Theorem 2.2. *The PBW property holds for $\check{\mathfrak{H}}^c$. That is, the multiplication of the subalgebra gives rise to an isomorphism of vector spaces:*

$$\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}S_n \otimes \mathcal{C}_n \otimes \mathbb{C}[y_1, \dots, y_n] \xrightarrow{\cong} \check{\mathfrak{H}}^c.$$

Proof. This can be proved in a similar way as in [5, Proof of Th. 1.3, pp.256-7] (and see *loc. cit.* for earlier works on PBW algebras), with one crucial modification. Set $V = \mathbb{C}^{2n}$ with the standard coordinates $\{x_i, y_i\}$, and note that $K = \mathcal{C}_n \rtimes \mathbb{C}S_n$ is a semisimple algebra. *The new key observation is that $E := V \otimes_{\mathbb{C}} K$ is naturally a K -bimodule even though K does not act on V .* The right action of K is via the right multiplication $g : v \otimes a \mapsto v \otimes (ag)$. The S_n acts on the left by $\sigma : v \otimes a \mapsto v^\sigma \otimes (\sigma a)$,

while \mathcal{C}_n acts by $c_i : x_j \otimes a \mapsto (-1)^{\delta_{ij}} x_j \otimes (c_i a)$ and $c_i : y_j \otimes a \mapsto y_j \otimes (c_i a)$. The rest of the proof is the same as in *loc. cit.* \square

Remark 2.3. The observation in the above proof, when applied to more general pairs (V, K) , leads to a generalization of Drinfeld’s setup [3]. The proof above also shows that \mathfrak{H}^ϵ is the only ‘quadratic’ deformation of $\check{\mathfrak{H}}_{|u=0}^\epsilon$ with PBW property. Note the special feature of $\check{\mathfrak{H}}^\epsilon$ that an additional term $t \cdot 1$ which appeared in the symplectic reflection algebra [5] is not allowed in (2.5) by the conjugation invariance with respect to c_i .

Remark 2.4. Introducing an additional central element z such that $z^2 = 1$ to $\check{\mathfrak{H}}^\epsilon$, we can define a modified algebra $\check{\mathfrak{H}}^{\text{mod}}$ with relations

$$c_i c_j = z c_j c_i, \quad c_i x_i = z x_i c_i$$

replacing the corresponding relations in (2.3) and (2.2). Then $\check{\mathfrak{H}}^{\text{mod}}/\langle z + 1 \rangle \cong \check{\mathfrak{H}}^\epsilon$. On the other hand, $\check{\mathfrak{H}}^{\text{mod}}/\langle z - 1 \rangle$ is isomorphic to the ordinary rational DaHa (cf. [5]) associated to the reflection group $\mathbb{Z}_2^n \rtimes S_n$ with specialized parameters. A similar remark applies to the algebra $\check{\mathfrak{H}}^-$ introduced in Section 3.

Below we always assume that $u \neq 0$ unless otherwise specified.

2.3. Degenerate affine Hecke-Clifford algebra. The degenerate affine Hecke-Clifford algebra was introduced by Nazarov (also called affine Sergeev algebra in [9]). It is the algebra \mathfrak{H}^ϵ generated by $a_i, c_i (i = 1, \dots, n)$ and S_n , subject to the relations (2.1), (2.3) and the following relations:

$$\begin{aligned} a_j s_i &= s_i a_j \quad (j \neq i, i + 1), \\ a_{i+1} s_i - s_i a_i &= 1 - c_{i+1} c_i, \\ c_i a_i &= -a_i c_i, \\ a_i a_j &= a_j a_i, \quad c_j a_i = a_i c_j, \quad (i \neq j). \end{aligned} \tag{2.6}$$

Our convention $c_i^2 = 1$ is consistent with [7], not with [9]. The \mathfrak{H}^ϵ is a superalgebra with c_i being odd and a_i being even for each i . For $1 \leq i \leq n$, set

$$M_i := \sum_{k < i} (1 - c_i c_k) s_{ki},$$

The M_i ’s are the Jucys-Murphy elements introduced in [9] and they satisfy

$$M_i M_j = M_j M_i, \quad (\forall i, j). \tag{2.7}$$

Proposition 2.5. [9]

- (1) *The algebra \mathfrak{H}^ϵ admits the PBW property. That is, the multiplication of the subalgebras induces a vector space isomorphism*

$$\mathbb{C}[a_1, \dots, a_n] \otimes \mathbb{C}S_n \otimes \mathcal{C}_n \xrightarrow{\cong} \mathfrak{H}^\epsilon.$$

- (2) *There exists a unique algebra homomorphism $\mathfrak{H}^\epsilon \rightarrow \mathcal{C}_n \rtimes \mathbb{C}S_n$, which restricts to the identity map on the subalgebra $\mathcal{C}_n \rtimes \mathbb{C}S_n$ of \mathfrak{H}^ϵ , and sends a_1 to 0. Moreover, this homomorphism sends each a_i to M_i ($1 \leq i \leq n$).*
- (3) *The even center of the algebra \mathfrak{H}^ϵ is $\mathbb{C}[a_1^2, \dots, a_n^2]^{S_n}$.*

The intertwining elements

$$\phi_i := s_i(a_i^2 - a_{i+1}^2) + (a_i + a_{i+1}) + c_i c_{i+1}(a_i - a_{i+1}), \quad 1 \leq i \leq n - 1$$

were introduced in [9] (cf. [7, Chapter 14]).

Proposition 2.6. [9] *The intertwining elements ϕ_i 's satisfy*

$$\begin{aligned} \phi_i^2 &= 2a_i^2 + 2a_{i+1}^2 - (a_i^2 - a_{i+1}^2)^2 \\ \phi_i \phi_{i+1} \phi_i &= \phi_{i+1} \phi_i \phi_{i+1} \\ \phi_i \phi_j &= \phi_j \phi_i \quad (|i - j| > 1). \end{aligned}$$

2.4. The affine Hecke-Clifford subalgebra of $\check{\mathfrak{H}}^c$. Set

$$z_i := u^{-1}y_i x_i + M_i.$$

Lemma 2.7. *For all i, j and each $\alpha \in \mathbb{C}$, the following identities hold:*

$$(2.8) \quad [z_i, z_j] = 0,$$

$$(2.9) \quad [x_i, z_j] - [x_j, z_i] = 0,$$

$$(2.10) \quad [\alpha x_i + z_i, \alpha x_j + z_j] = 0.$$

Proof. We may assume $i < j$. Then, by (2.7), we have

$$\begin{aligned} u[z_i, z_j] &= u^{-1}[y_i x_i, y_j x_j] + [y_i x_i, M_j] \\ &= u^{-1}(y_i[x_i, y_j]x_j + y_j[y_i, x_j]x_i) + [y_i x_i, s_{ij} - c_j c_i s_{ij}] \\ &= -y_i(s_{ij} - c_j c_i s_{ij})x_j + y_j(s_{ij} + c_i c_j s_{ij})x_i \\ &\quad + (y_i x_i - y_j x_j)s_{ij} - (y_i x_i + y_j x_j)c_j c_i s_{ij} = 0. \end{aligned}$$

To prove (2.9) for $i < j$, we calculate that

$$\begin{aligned} &[x_i, z_j] - [x_j, z_i] \\ &= [x_i, u^{-1}y_j x_j + M_j] - [x_j, u^{-1}y_i x_i] \\ &= u^{-1}[x_i, y_j]x_j + [x_i, s_{ij} - c_j c_i s_{ij}] - u^{-1}[x_j, y_i]x_i \\ &= (c_i c_j s_{ij} - s_{ij})x_j + (x_i - x_j)s_{ij} - (x_i + x_j)c_j c_i s_{ij} + (s_{ij} + c_i c_j s_{ij})x_i = 0. \end{aligned}$$

Now (2.10) follows from (2.8) and (2.9). □

We remark that $[y_i, z_j] - [y_j, z_i] \neq 0 \quad (i \neq j)$.

Lemma 2.8. *The following identities hold:*

$$c_i z_i = -z_i c_i, \quad c_i z_j = z_j c_i \quad (i \neq j).$$

Proof. Recall $z_i = u^{-1}y_i x_i + M_i$. It is known [9] that the Jucys-Murphy elements satisfy $c_i M_i = -M_i c_i$, and $c_j M_i = M_i c_j$ for $i \neq j$. Clearly, $c_i(y_i x_i) = -(y_i x_i)c_i$, and $c_j(y_i x_i) = (y_i x_i)c_j$ for $i \neq j$. Now the lemma follows. □

Lemma 2.9. *The following identities hold:*

$$\begin{aligned} z_{i+1} s_i - s_i z_i &= 1 - c_{i+1} c_i, \\ (\alpha x_{i+1} + z_{i+1}) s_i - s_i (\alpha x_i + z_i) &= 1 - c_{i+1} c_i. \end{aligned}$$

Proof. The second identity follows from the first one and the equation $x_{i+1}s_i = s_ix_i$. Note that

$$\begin{aligned} z_{i+1}s_i - s_iz_i &= (u^{-1}y_{i+1}x_{i+1} + M_{i+1})s_i - s_i(u^{-1}y_ix_i + M_i) \\ &= M_{i+1}s_i - s_iM_i = 1 - c_{i+1}c_i. \end{aligned}$$

This proves the first identity. □

The next theorem follows now from Lemmas 2.7, 2.8, and 2.9.

Theorem 2.10. *Fix $\alpha \in \mathbb{C}$. The subalgebra of $\check{\mathfrak{H}}^\alpha$ generated by $c_i, \alpha x_i + z_i$ ($1 \leq i \leq n$) and S_n is isomorphic to the degenerate affine Hecke-Clifford algebra \mathfrak{H}^α .*

2.5. The Dunkl operator for $\check{\mathfrak{H}}^\alpha$. The PBW Theorem 2.2 provides the triangular decomposition of $\check{\mathfrak{H}}^\alpha$:

$$\check{\mathfrak{H}}^\alpha \cong \mathbb{C}[\underline{x}] \otimes (\mathbb{C}_n \rtimes \mathbb{C}S_n) \otimes \mathbb{C}[\underline{y}]$$

where $\mathbb{C}[\underline{x}]$ is the short-hand for $\mathbb{C}[x_1, \dots, x_n]$, etc. Denote by \mathfrak{H}_x^α the subalgebra of $\check{\mathfrak{H}}^\alpha$ generated by $\mathbb{C}_n \rtimes \mathbb{C}S_n$ and x_1, \dots, x_n . The subalgebra \mathfrak{H}_y^α is similarly defined. Take any $\mathbb{C}_n \rtimes \mathbb{C}S_n$ -module W , and extend it to an \mathfrak{H}_x^α -module with the trivial action of x_i 's. Consider the induced $\check{\mathfrak{H}}^\alpha$ -module $\text{Ind}_{\mathfrak{H}_x^\alpha}^{\check{\mathfrak{H}}^\alpha} W$, which as a vector space is isomorphic to $\mathbb{C}[y_1, \dots, y_n] \otimes W$. Denote the action by \circ . The most interesting $S_n \times \mathbb{C}_n$ -module is the so-called *basic spin module* $L_n = \mathbb{C}(c_1, \dots, c_n)$ of $S_n \times \mathbb{C}_n$, (where \mathbb{C}_n acts by left multiplication and S_n acts by permuting the c_i 's), and the induced $\check{\mathfrak{H}}^\alpha$ -module $\text{Ind}_{\mathfrak{H}_x^\alpha}^{\check{\mathfrak{H}}^\alpha} L_n$ is then identified as $\mathbb{C}[y_1, \dots, y_n] \otimes \mathbb{C}(c_1, \dots, c_n)$. Given operators h and g , $\frac{h}{g}$ stands for $g^{-1}h$ throughout the paper.

Theorem 2.11. *Given a $\mathbb{C}_n \rtimes \mathbb{C}S_n$ -module W , the action of x_i on $\mathbb{C}[y_1, \dots, y_n] \otimes W$ is realized as “Dunkl operator” as follows. For any polynomial $f = f(\underline{y})$ and $w \in W$, we have*

$$x_i \circ (f \otimes w) = u \sum_{k \neq i} \frac{(1 - s_{ki})(f)}{y_i - y_k} \otimes (1 - c_i c_k) s_{ki}(w).$$

Proof. We calculate that

$$x_i \circ (f \otimes w) = [x_i, f] \circ w + f x_i \circ w = [x_i, f] \circ w.$$

Now the theorem follows from Lemma 2.12 (2) below. □

Lemma 2.12. (1) *For $a \in \mathbb{N}$ and $i \neq j$, we have*

$$\begin{aligned} [x_i, y_j^a] &= -u \frac{y_i^a - y_j^a}{y_i - y_j} (1 - c_i c_j) s_{ij} \\ [x_i, y_i^a] &= u \sum_{k \neq i} \frac{y_i^a - y_k^a}{y_i - y_k} (1 - c_i c_k) s_{ki}. \end{aligned}$$

(2) *Let $f(\underline{y})$ be any polynomial in y_1, \dots, y_n . Then,*

$$[x_i, f(\underline{y})] = u \sum_{k \neq i} \frac{(1 - s_{ki})f(\underline{y})}{y_i - y_k} (1 - c_i c_k) s_{ki}.$$

Proof. (1) is proved by induction on a using $[x_i, y_j^a] = [x_i, y_j^{a-1}]y_j + y_j^{a-1}[x_i, y_j]$.

(2) It suffices to check for every monomial f . We first prove the case when $f = \prod_{j \neq i} y_j^{a_j}$ using the first identity in (1). Then we combine with the second identity in (1) to prove for a general monomial f involving powers of y_i . \square

Note that the symmetry between x 's and y 's in our algebra $\check{\mathfrak{H}}^\epsilon$ is broken. We compute below the Dunkl operator for y_i .

Lemma 2.13. *For $a \in \mathbb{N}$ and $1 \leq i \neq j \leq n$, we have*

$$\begin{aligned} [y_i, x_i^a] &= u \sum_{k \neq i} \left(-\frac{x_k^a - x_i^a}{x_k - x_i} s_{ki} + \frac{x_i^a - (-x_k)^a}{x_k + x_i} c_i s_{ki} c_i \right) \\ [y_i, x_j^a] &= u \frac{x_j^a - x_i^a}{x_j - x_i} s_{ij} + u \frac{x_j^a - (-x_i)^a}{x_j + x_i} c_i s_{ij} c_i. \end{aligned}$$

Proof. Follows by induction on a . \square

We can identify $\mathbb{C}[x_1, \dots, x_n] \otimes W$ with the induced $\check{\mathfrak{H}}^\epsilon$ -module $\text{Ind}_{\check{\mathfrak{H}}_y^\epsilon}^{\check{\mathfrak{H}}^\epsilon} W$, where W is a $\mathbb{C}S_n \times \mathbb{C}_n$ -module and is then extended to an $\check{\mathfrak{H}}_y^\epsilon$ -module trivially.

Proposition 2.14. *The action of y_i on $\mathbb{C}[x_1, \dots, x_n] \otimes W$ is given as follows. For any polynomial $f(x)$ and $w \in W$, we have*

$$y_i \circ (f \otimes w) = \sum_{k \neq i} \left(\frac{f - s_{ki}(f)}{x_k - x_i} \otimes s_{ki}(w) + \frac{f c_i c_k - c_i c_k s_{ki}(f)}{x_k + x_i} \otimes s_{ik}(w) \right).$$

Proof. First note that y_i commutes with all c_j 's. So we are reduced to the case $f = 1$ and w being a monomial. This follows by induction from Lemma 2.13. \square

2.6. The center of $\check{\mathfrak{H}}^\epsilon$. Denote by $\mathcal{Z}(A)$ the even center of a superalgebra A , which consists of all the even central elements in A .

Proposition 2.15. (1) *For $u = 0$, the even center $\mathcal{Z}(\check{\mathfrak{H}}_{[u=0]}^\epsilon)$ of the superalgebra $\check{\mathfrak{H}}_{[u=0]}^\epsilon$ is $\mathbb{C}[x_1^2, \dots, x_n^2, y_1, \dots, y_n]^{\Delta S_n}$, where Δ denotes the diagonal action on x_j^2 's and y_j 's.*

(2) *We have $\mathbb{C}[x_1^2, \dots, x_n^2]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$ and $\mathbb{C}[y_1, \dots, y_n]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$, $\forall u \in \mathbb{C}$.*

Proof. (1) is clear.

(2) By Lemma 2.12 (1), for each i we have that

$$[x_i, y_1^k + \dots + y_n^k] = 0, \quad \forall k \geq 1.$$

By definition of $\check{\mathfrak{H}}^\epsilon$, $y_1^k + \dots + y_n^k$ commutes with $\mathbb{C}S_n \times \mathbb{C}_n$. So $y_1^k + \dots + y_n^k \in \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$ for all k , and thus $\mathbb{C}[y_1, \dots, y_n]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$. We have $\mathbb{C}[x_1^2, \dots, x_n^2]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$, by arguing similarly using Lemma 2.13 and the definition of $\check{\mathfrak{H}}^\epsilon$. \square

Example 2.16. For $n = 2$, $x_1^2 y_1 + x_2^2 y_2 - (x_1 + x_2) s_{12} - c_1 (x_1 + x_2) s_{12} c_1 \in \mathcal{Z}(\check{\mathfrak{H}}^\epsilon)$.

3. The rational spin double affine Hecke algebras

The results in this section are presented in a way parallel to Section 2, but with all proofs omitted except for the PBW theorem. They can be either proved in the same way as for their counterparts in Section 2, or follow directly from the counterparts in Section 2 via the isomorphisms to be established in Section 4.

3.1. The spin symmetric group algebra. The symmetric group S_n affords a double cover \tilde{S}_n , nontrivial for $n \geq 4$, according to Schur [13] (cf. [7]):

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{S}_n \longrightarrow S_n \longrightarrow 1.$$

Denote $\mathbb{Z}_2 = \{1, z\}$. The *spin symmetric group algebra* $\mathbb{C}S_n^- := \mathbb{C}[\tilde{S}_n]/\langle z + 1 \rangle$ is generated by $t_i, i = 1, \dots, n - 1$ subject to the following relations:

$$(3.1) \quad \begin{aligned} t_i^2 &= 1, & t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \\ t_i t_j &= -t_j t_i, & |i - j| &> 1. \end{aligned}$$

The algebra $\mathbb{C}S_n^-$ is naturally a superalgebra by declaring t_i for every i to be odd.

Define the “transpositions” of odd parity, for $1 \leq i < j \leq n$,

$$[i, j] = -[j, i] = (-1)^{j-i-1} t_{j-1} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-1}$$

which satisfies the following relations:

$$\begin{aligned} [i, i + 1] &= t_i, & [i, j]^2 &= 1, \\ t_i [i, j] t_i &= -[i + 1, j], & \text{for } j &\neq i, i + 1. \end{aligned}$$

Define the *odd Jucys-Murphy elements*

$$M_j = \sum_{k < j} [k, j], \quad (1 \leq j \leq n).$$

We have [14] (also see [7, Chapter 13])

$$[M_i, M_j]_+ := M_i M_j + M_j M_i = 0, \quad (i \neq j).$$

As usual, we have denoted that $[a, b]_+ = ab + ba$.

3.2. The rational sDaHa.

Definition 3.1. Let $u \in \mathbb{C}$. The rational *spin double affine Hecke algebra* (sDaHa) is the algebra $\tilde{\mathfrak{H}}^-$ generated by $\xi_i, y_i (1 \leq i \leq n)$ and $t_i (1 \leq i \leq n - 1)$, subject to the relations (3.1) for t_i and the following relations:

$$\begin{aligned} \xi_i \xi_j &= -\xi_j \xi_i, & y_i y_j &= y_j y_i, & (i \neq j), \\ t_i \xi_i &= -\xi_{i+1} t_i, & t_i y_i &= y_{i+1} t_i, \\ t_j \xi_i &= -\xi_i t_j, & t_j y_i &= y_i t_j, & (i \neq j, j + 1), \\ [y_i, \xi_j] &= u [i, j], & (i \neq j), \\ [y_i, \xi_i] &= u \sum_{k \neq i} [i, k]. \end{aligned}$$

Denote by $\mathbb{C}[\xi_1, \dots, \xi_n]$ the algebra generated by ξ_1, \dots, ξ_n subject to the relations $\xi_i \xi_j = -\xi_j \xi_i$ for $i \neq j$. Clearly, it has a linear basis $\underline{\xi}^{\mathbf{a}}$, where $\mathbf{a} \in \mathbb{Z}_+^n$.

Theorem 3.2. *The PBW property holds for $\check{\mathfrak{H}}^-$. That is, the multiplication of the subalgebras gives rise to an isomorphism of vector spaces:*

$$\mathbb{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}S_n^- \otimes \mathbb{C}[y_1, \dots, y_n] \xrightarrow{\cong} \check{\mathfrak{H}}^-.$$

Proof. It follows readily from the definition that $\check{\mathfrak{H}}^-$ is spanned by the elements $\xi^{\underline{a}}\sigma y^{\underline{b}}$, where σ runs over a basis for $\mathbb{C}S_n^-$ and $\underline{a}, \underline{b} \in \mathbb{Z}_+^n$. Recall the homomorphism Ψ in Theorem 4.2. The images $\Psi(\xi^{\underline{a}}\sigma y^{\underline{b}})$ in explicit formulas are clearly all linearly independent in $\check{\mathfrak{H}}^c$ by the PBW Theorem 2.2 for $\check{\mathfrak{H}}^c$. Thus, all the elements $\xi^{\underline{a}}\sigma y^{\underline{b}}$ are linearly independent in $\check{\mathfrak{H}}^-$. \square

Remark 3.3. Note the subtle signs in the definition of $\check{\mathfrak{H}}^-$. The algebra $\check{\mathfrak{H}}^-$ is naturally a superalgebra with y_i being even and ξ_i, t_i being odd for each i . The algebras $\check{\mathfrak{H}}^-$ are all isomorphic for different $u \neq 0$.

Below we assume that $u \neq 0$ unless otherwise specified.

3.3. The degenerate spin affine Hecke algebra. The *degenerate spin affine Hecke algebra* is the algebra \mathfrak{H}^- generated by $b_i (1 \leq i \leq n)$ and $t_i (1 \leq i \leq n - 1)$, subject to the relations (3.1) for t_i and the following relations:

$$\begin{aligned} b_i b_j &= -b_j b_i, & (i \neq j) \\ b_{i+1} t_i &= -t_i b_i + 1, \\ t_j b_i &= -b_i t_j, & (i \neq j, j + 1). \end{aligned}$$

The algebra \mathfrak{H}^- has a superalgebra structure with both b_i and t_i being odd elements for every i . Note that the algebra \mathfrak{H}^- contains $\mathbb{C}[b_1, \dots, b_n]$ and $\mathbb{C}S_n^-$ as subalgebras.

Proposition 3.4. (1) *The algebra \mathfrak{H}^- admits the PBW property. That is, the multiplication of the subalgebras induces a vector space isomorphism*

$$\mathbb{C}[b_1, \dots, b_n] \otimes \mathbb{C}S_n^- \xrightarrow{\cong} \mathfrak{H}^-.$$

- (2) *There exists a unique algebra homomorphism $\mathfrak{H}^- \rightarrow \mathbb{C}S_n^-$, which extends the identity map on the subalgebra $\mathbb{C}S_n^-$ of \mathfrak{H}^- and sends b_1 to 0. Moreover, this homomorphism sends each b_i to $M_i (1 \leq i \leq n)$.*
- (3) *The even center of the superalgebra \mathfrak{H}^- is $\mathbb{C}[b_1^2, \dots, b_n^2]^{S_n}$.*

We define the (odd) intertwining elements in \mathfrak{H}^- :

$$\psi_i := t_i(b_i^2 - b_{i+1}^2) - (b_i - b_{i+1}), \quad 1 \leq i \leq n - 1.$$

Proposition 3.5. *The intertwining elements ψ_i 's satisfy that*

$$\begin{aligned} \psi_i^2 &= b_i^2 + b_{i+1}^2 - (b_i^2 - b_{i+1}^2)^2 \\ \psi_i \psi_{i+1} \psi_i &= \psi_{i+1} \psi_i \psi_{i+1} \\ \psi_i \psi_j &= -\psi_j \psi_i \quad (|i - j| > 1) \\ \psi_i b_j &= -b_j \psi_i \quad (j \neq i, i + 1) \\ \psi_i b_i &= -b_{i+1} \psi_i, & \psi_i b_{i+1} &= -b_i \psi_i. \end{aligned}$$

3.4. The spin affine Hecke subalgebra of $\check{\mathfrak{H}}^-$. Let

$$\mathfrak{z}_i := u^{-1}y_i\xi_i + M_i.$$

Lemma 3.6. *For $i \neq j$, the following identities hold:*

$$\begin{aligned} [\mathfrak{z}_i, \mathfrak{z}_j]_+ &= 0, \\ [\xi_i, \mathfrak{z}_j]_+ + [\xi_j, \mathfrak{z}_i]_+ &= 0, \\ [\alpha\xi_i + \mathfrak{z}_i, \alpha\xi_j + \mathfrak{z}_j]_+ &= 0, \quad \forall \alpha \in \mathbb{C}. \end{aligned}$$

Lemma 3.7. *The following identities hold:*

$$(\alpha\xi_{i+1} + \mathfrak{z}_{i+1})t_i = -t_i(\alpha\xi_i + \mathfrak{z}_i) + 1, \quad \forall \alpha \in \mathbb{C}.$$

Theorem 3.8. *Fix $\alpha \in \mathbb{C}$. The subalgebra of $\check{\mathfrak{H}}^-$ generated by $\alpha\xi_i + \mathfrak{z}_i$ ($1 \leq i \leq n$) and $\mathbb{C}S_n^-$ is isomorphic to the degenerate spin affine Hecke algebra \mathfrak{H}^- .*

3.5. The Dunkl operators for $\check{\mathfrak{H}}^-$. Denote by \mathfrak{H}_ξ^- the subalgebra of $\check{\mathfrak{H}}^-$ generated by $\mathbb{C}S_n^-$ and ξ_1, \dots, ξ_n . Take any $\mathbb{C}S_n^-$ -module W , and extend it to an \mathfrak{H}_ξ^- -module with the trivial action of ξ_i 's. Consider the induced $\check{\mathfrak{H}}^-$ -module $\text{Ind}_{\mathfrak{H}_\xi^-}^{\check{\mathfrak{H}}^-} W \cong \mathbb{C}[y_1, \dots, y_n] \otimes W$.

Theorem 3.9. *The action of $\xi_i \in \check{\mathfrak{H}}^-$ on $\mathbb{C}[y_1, \dots, y_n] \otimes W$ is realized as ‘‘Dunkl operator’’ as follows. For any polynomial $f = f(y)$ and $w \in W$, we have*

$$\xi_i \circ (f \otimes w) = u \sum_{k \neq i} \frac{(1 - s_{ki})(f)}{y_i - y_k} \otimes [k, i](w).$$

3.6. The center of $\check{\mathfrak{H}}^-$.

Proposition 3.10. (1) *For $u = 0$, the even center $\mathcal{Z}(\check{\mathfrak{H}}_{|u=0}^-)$ of the superalgebra $\check{\mathfrak{H}}_{|u=0}^-$ is $\mathbb{C}[\xi_1^2, \dots, \xi_n^2; y_1, \dots, y_n]^{\Delta S_n}$, where Δ denotes the diagonal action on the ξ_j^2 's and y_j 's.*

(2) *For any $u \in \mathbb{C}$, $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^-)$ and $\mathbb{C}[y_1, \dots, y_n]^{S_n} \subset \mathcal{Z}(\check{\mathfrak{H}}^-)$.*

Example 3.11. For $n = 2$, $\xi_1^2 y_1 + \xi_2^2 y_2 + 2(\xi_1 - \xi_2)t_1 \in \mathcal{Z}(\check{\mathfrak{H}}^-)$.

4. The algebra isomorphisms

The results in this section use earlier notations but their proofs will be independent of Section 3. Note that the multiplication in a tensor product $\mathcal{C} \otimes \check{\mathfrak{H}}^-$ of two super algebras \mathcal{C} and $\check{\mathfrak{H}}^-$ has a suitable sign convention:

$$(c' \otimes b')(c \otimes b) = (-1)^{|b'| |c|} (c'c \otimes b'b).$$

We shall write a typical element in $\mathcal{C} \otimes \check{\mathfrak{H}}^-$ as cb rather than $c \otimes b$, and use short-hand notations $c = c \otimes 1, b = 1 \otimes b$.

Theorem 4.1. *There exists a superalgebra isomorphism:*

$$\begin{aligned} \widehat{\Phi} : \mathfrak{H}^c &\xrightarrow{\cong} \mathcal{C}_n \otimes \mathfrak{H}^-, \\ c_i &\mapsto c_i, \quad a_i \mapsto \sqrt{-2}c_i b_i, \quad s_i \mapsto \frac{1}{\sqrt{-2}}(c_i - c_{i+1})t_i. \end{aligned}$$

The inverse map is given by

$$\widehat{\Psi} : c_i \mapsto c_i, b_i \mapsto \frac{1}{\sqrt{-2}}c_i a_i, t_i \mapsto \frac{1}{\sqrt{-2}}(c_{i+1} - c_i)s_i.$$

Proof. Clearly $\widehat{\Phi}, \widehat{\Psi}$ are inverses to each other, once we show that they are algebra homomorphisms. Note that $\mathcal{C}_n \rtimes \mathbb{C}S_n$ is a subalgebra of \mathfrak{H}^c and $\widehat{\Phi}$ extends the isomorphism established in [14, 16]

$$\Phi^{fin} : \mathcal{C}_n \rtimes \mathbb{C}S_n \longrightarrow \mathcal{C}_n \otimes \mathbb{C}S_n^-.$$

This takes care of all the defining relations for the images of $\widehat{\Phi}$ or $\widehat{\Psi}$ of the generators which do not involve a_i or b_i (the verification of these relations can be done directly without difficulty). Among the defining relations involving b_i , the most complicated one is $\widehat{\Psi}(b_{i+1}t_i + t_i b_i) = 1$, which is equivalent to the following identity in \mathfrak{H}^c :

$$\frac{1}{\sqrt{-2}}c_{i+1}a_{i+1} \cdot \frac{1}{\sqrt{-2}}(c_{i+1} - c_i)s_i + \frac{1}{\sqrt{-2}}(c_{i+1} - c_i)s_i \cdot \frac{1}{\sqrt{-2}}c_i a_i = 1.$$

This can be checked directly using the defining relations (2.6) of \mathfrak{H}^c . The verification of the remaining relations is similar, and will be skipped. \square

An argument similar to the above implies the following isomorphism theorem.

Theorem 4.2. *There exists a superalgebra isomorphism:*

$$\begin{aligned} \Phi : \check{\mathfrak{H}}^c &\xrightarrow{\cong} \mathcal{C}_n \otimes \check{\mathfrak{H}}^-, \\ y_i \mapsto y_i, c_i \mapsto c_i, x_i \mapsto \sqrt{-2}c_i \xi_i, s_i \mapsto \frac{1}{\sqrt{-2}}(c_i - c_{i+1})t_i. \end{aligned}$$

The inverse map is given by

$$\Psi : y_i \mapsto y_i, c_i \mapsto c_i, \xi_i \mapsto \frac{1}{\sqrt{-2}}c_i x_i, t_i \mapsto \frac{1}{\sqrt{-2}}(c_{i+1} - c_i)s_i.$$

- Remark 4.3.*
- (1) Recall the intertwining elements ϕ_i and ψ_i from Sections 2 and 3. The isomorphism $\widehat{\Phi}$ sends ϕ_i to $-\sqrt{-2}(c_i - c_{i+1})\psi_i$ for every i .
 - (2) Recall from Propositions 2.10 and 3.8 that $\alpha x_i + z_i$ (resp. $\alpha \xi_i + \mathfrak{z}_i$) are the ‘polynomial’ generators of a degenerate affine Hecke algebra isomorphic to \mathfrak{H}^c (resp. \mathfrak{H}^-). The Φ sends $\alpha x_i + z_i$ to $\sqrt{-2}c_i(\alpha \xi_i + \mathfrak{z}_i)$ for each i . This is compatible with the fact that $\widehat{\Phi}$ sends a_i to $\sqrt{-2}c_i b_i$.
 - (3) Note also that $\widehat{\Phi}$ or Φ sends the JM element M_i to $\sqrt{-2}c_i M_i$, which amounts to checking the identity $\Phi(\frac{1}{\sqrt{-2}}(c_i - c_k)s_{ik}) = [k, i]$.

5. The trigonometric version of DaHCa and sDaHa

5.1. The trigonometric DaHCa. We start with the following definition.

Definition 5.1. *The trigonometric double affine Hecke-Clifford algebra is the algebra $\ddot{\mathfrak{H}}_{tr}^c$ generated by $e^{\pm\epsilon_i}, \epsilon_i^\vee, c_i$ ($1 \leq i \leq n$) and S_n , subject to the following relations:*

$$\begin{aligned} c_i^2 &= 1, & c_i c_j &= -c_j c_i \quad (i \neq j) \\ e^{\epsilon_i} e^{-\epsilon_i} &= 1, & e^{\epsilon_i} e^{\epsilon_j} &= e^{\epsilon_j} e^{\epsilon_i}, \\ \epsilon_i^\vee \epsilon_j^\vee &= \epsilon_j^\vee \epsilon_i^\vee, & c_j e^{\epsilon_i} &= e^{\epsilon_i} c_j \quad (\forall i, j), \\ \sigma e^{\epsilon_i} &= e^{\epsilon_{\sigma i}} \sigma, & \sigma c_i &= c_{\sigma i} \sigma \quad (\sigma \in S_n), \\ c_i \epsilon_i^\vee &= -\epsilon_i^\vee c_i, & c_j \epsilon_i^\vee &= \epsilon_i^\vee c_j \quad (i \neq j), \\ \epsilon_{i+1}^\vee s_i - s_i \epsilon_i^\vee &= u(1 - c_{i+1} c_i), & \epsilon_j^\vee s_i &= s_i \epsilon_j^\vee \quad (j \neq i, i + 1), \\ [\epsilon_i^\vee, e^\eta] &= u \sum_{k \neq i} \operatorname{sgn}(k - i) \frac{e^\eta - e^{s_{ki}(\eta)}}{1 - e^{\operatorname{sgn}(k-i) \cdot (\epsilon_k - \epsilon_i)}} (1 - c_i c_k) s_{ki}. \end{aligned}$$

The algebra $\ddot{\mathfrak{H}}_{tr}^c$ admits a natural superalgebra structure with c_i being odd and all other generators being even. Note that the subalgebra generated by e^{ϵ_i} ($1 \leq i \leq n$), denoted by $\mathbb{C}[P]$, is identified with the group algebra of the weight lattice of type GL_n ; the subalgebra generated by e^{ϵ_i} ($1 \leq i \leq n$) and S_n is identified with the group algebra of the extended affine Weyl group of type GL_n ; for $u \neq 0$, the subalgebra generated by $u^{-1}\epsilon_i^\vee, c_i$ ($1 \leq i \leq n$) and S_n is identified with the degenerate affine Hecke-Clifford algebra (cf. Sect. 2.3).

Theorem 5.2. (1) *There exists a unique superalgebras homomorphism $\iota : \ddot{\mathfrak{H}}^c \rightarrow \ddot{\mathfrak{H}}_{tr}^c$, which extends the identity map on the subalgebra $\mathcal{C}_n \rtimes \mathbb{C}S_n$ such that*

$$\begin{aligned} \iota(y_i) &= e^{\epsilon_i} \\ \iota(x_i) &= e^{-\epsilon_i} \left(\epsilon_i^\vee - u \sum_{k < i} (1 - c_i c_k) s_{ki} \right). \end{aligned}$$

Moreover, ι is injective.

(2) *The homomorphism ι extends to an algebra isomorphism*

$$\mathbb{C}[\underline{y}^\pm] \otimes_{\mathbb{C}[\underline{y}]} \ddot{\mathfrak{H}}^c \xrightarrow{\cong} \ddot{\mathfrak{H}}_{tr}^c.$$

The inverse j is an extension of the identity map on $\mathcal{C}_n \rtimes \mathbb{C}S_n$ such that

$$\begin{aligned} j(e^{\epsilon_i}) &= y_i, \\ j(\epsilon_i^\vee) &= y_i x_i + u \sum_{k < i} (1 - c_i c_k) s_{ki}. \end{aligned}$$

Proof. The j and (the extended) ι are clearly inverses to each other, once we show that they are algebra homomorphisms.

Recall from (2.4) that $uz_i = y_i x_i + u \sum_{k < i} (1 - c_i c_k) s_{ki}$. Let us verify

$$j([\epsilon_i^\vee, e^{\epsilon_i}]) = [j(\epsilon_i^\vee), j(e^{\epsilon_i})] = [uz_i, y_i].$$

By a long but straightforward computation we have that

$$\begin{aligned} [uz_i, y_j] &= -uy_i(1 - c_i c_j) s_{ij}, \quad (i < j), \\ [uz_i, y_j] &= -uy_j(1 - c_i c_j) s_{ij}, \quad (i > j), \\ [uz_i, y_i] &= u \sum_{k > i} y_i(1 - c_i c_k) s_{ki} + u \sum_{k < i} (1 - c_i c_k) s_{ki} y_i. \end{aligned}$$

By another direct computation, we see that the commutator $[\epsilon_i^\vee, e^{\epsilon_j}]$ are given by exactly the right-hand side of $[uz_i, y_j]$ above with y_i therein replaced by e^{ϵ_i} , for $i < j$, $i = j$, or $i > j$.

The verification of other relations is easier and will be skipped. □

Remark 5.3. Theorem 5.2 and Theorem 5.6 below are analogous to the relation between the usual rational DaHa and trigonometric DaHa due to Suzuki [15].

Proposition 5.4. *The PBW property holds for $\check{\mathfrak{H}}_{tr}^\epsilon$. That is, the multiplication of the subalgebras gives rise to an isomorphism of vector spaces:*

$$\mathbb{C}[P] \otimes \mathbb{C}S_n \otimes \mathbb{C}_n \otimes \mathbb{C}[\epsilon_1^\vee, \dots, \epsilon_n^\vee] \xrightarrow{\cong} \check{\mathfrak{H}}_{tr}^\epsilon.$$

Proof. Clearly the multiplication homomorphism is surjective.

The injectivity of this homomorphism follows from Theorem 5.2 (2) and the PBW property for $\check{\mathfrak{H}}^\epsilon$ (and thus for $\mathbb{C}[\underline{y}^\pm] \otimes_{\mathbb{C}[\underline{y}]} \check{\mathfrak{H}}^\epsilon$). More explicitly, j sends ϵ_i^\vee to $y_i x_i$ plus lower terms. Via j and with such lower terms ignored, any linear relation among $\underline{\epsilon}^{\vee a} \sigma \underline{\epsilon}^{\vee b}$ is translated to the same relation among $(\underline{y}x)^a \sigma \underline{\epsilon}^{\vee b}$. But we know the latter are all linearly independent. □

5.2. The trigonometric sDaHa.

Definition 5.5. The *trigonometric spin double affine Hecke algebra* is the algebra $\check{\mathfrak{H}}_{tr}^-$ generated by $e^{\pm\epsilon_i}, \zeta_i$ ($1 \leq i \leq n$) and t_i ($1 \leq i \leq n - 1$), subject to the relations (3.1) for t_i and the following relations:

$$\begin{aligned} e^{\epsilon_i} e^{-\epsilon_i} &= 1, & e^{\epsilon_i} e^{\epsilon_j} &= e^{\epsilon_j} e^{\epsilon_i}, \\ t_i e^{\epsilon_i} &= e^{\epsilon_{i+1}} t_i, & t_i e^{\epsilon_j} &= e^{\epsilon_j} t_i \quad (j \neq i, i + 1), \\ \zeta_{i+1} t_i + t_i \zeta_i &= u, & \zeta_j t_i &= -t_i \zeta_j \quad (j \neq i, i + 1), \\ \zeta_i \zeta_j &= -\zeta_j \zeta_i \quad (i \neq j), \\ [\zeta_i, e^\eta] &= u \sum_{k \neq i} \operatorname{sgn}(k - i) \frac{e^\eta - e^{s_{ki}(\eta)}}{1 - e^{\operatorname{sgn}(k-i) \cdot (\epsilon_k - \epsilon_i)}} [k, i]. \end{aligned}$$

Note that the algebra $\check{\mathfrak{H}}_{tr}^-$ is naturally a superalgebra by declaring $e^{\pm\epsilon_i}$ to be even and ζ_i and t_i to be odd for every i . For $u \neq 0$, the subalgebra generated by ζ_i ($1 \leq i \leq n$) and $\mathbb{C}S_n^-$ is identified with the degenerate spin affine Hecke algebra \mathfrak{H}^- (cf. Sect. 3.3).

Theorem 5.6. (1) *There exists a unique homomorphism of superalgebras $\iota^- : \check{\mathfrak{H}}^- \rightarrow \check{\mathfrak{H}}_{tr}^-$, which extends the identity map on the subalgebra $\mathbb{C}S_n^-$ such that*

$$\begin{aligned} \iota^-(y_i) &= e^{\epsilon_i} \\ \iota^-(\xi_i) &= e^{-\epsilon_i} \left(\zeta_i - u \sum_{k < i} [k, i] \right). \end{aligned}$$

Moreover, ι^- is injective.

(2) *The ι^- extends to an algebra isomorphism:*

$$\mathbb{C}[\underline{y}^\pm] \otimes_{\mathbb{C}[\underline{y}]} \check{\mathfrak{H}}^- \xrightarrow{\cong} \check{\mathfrak{H}}_{tr}^-.$$

The inverse map j^- is an extension of the identity map on $\mathbb{C}S_n^-$ such that

$$\begin{aligned} j^-(e^{\epsilon_i}) &= y_i, \\ j^-(\zeta_i) &= y_i \zeta_i + u \sum_{k < i} [k, i]. \end{aligned}$$

Proof. Follows from Theorem 4.2 and Theorem 5.2; or one argues directly as for Theorem 5.2. \square

Theorem 5.6 and the PBW Theorem 3.2 for $\check{\mathfrak{H}}^-$ immediately imply the following.

Corollary 5.7. *The PBW property holds for $\check{\mathfrak{H}}_{tr}^-$. That is, the multiplication of the subalgebras gives rise to an isomorphism of vector spaces:*

$$\mathbb{C}[P] \otimes \mathbb{C}S_n^- \otimes \mathbb{C}[\zeta_1, \dots, \zeta_n] \xrightarrow{\cong} \check{\mathfrak{H}}_{tr}^-.$$

5.3. An algebra isomorphism. The following theorem can be established by a direct computation.

Theorem 5.8. *There exists a superalgebra isomorphism:*

$$\begin{aligned} \Phi^{tr} : \check{\mathfrak{H}}_{tr}^c &\xrightarrow{\cong} \mathbb{C}_n \otimes \check{\mathfrak{H}}_{tr}^-, \\ e^{\epsilon_i} \mapsto e^{\epsilon_i}, c_i \mapsto c_i, \epsilon_i^\vee &\mapsto \sqrt{-2} c_i \zeta_i, s_i \mapsto \frac{1}{\sqrt{-2}} (c_i - c_{i+1}) t_i. \end{aligned}$$

The inverse map is given by

$$e^{\epsilon_i} \mapsto e^{\epsilon_i}, c_i \mapsto c_i, \zeta_i \mapsto \frac{1}{\sqrt{-2}} c_i \epsilon_i^\vee, t_i \mapsto \frac{1}{\sqrt{-2}} (c_{i+1} - c_i) s_i.$$

Notes added. This first paper on spin Hecke algebras was posted in arXiv: math.RT/0608074, 2006. The results of this paper have been generalized and extended in different directions since then in a series of papers [17, 18, 19, 20, 21].

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References

- [1] J. Brundan and A. Kleshchev, *Hecke-Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for \widehat{S}_n* , Represent. Theory **5** (2001), 317–403 (electronic).
- [2] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series **319**, Cambridge University Press, 2005.
- [3] V. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Anal. Appl. **20** (1986), 58–60.
- [4] C. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.
- [5] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

- [6] I. Gordon, *Baby Verma modules for rational Cherednik algebras*, Bull. London Math. Soc. **35** (2003), 321–336.
- [7] A. Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics **163**, Cambridge University Press, 2005.
- [8] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), 599–635.
- [9] M. Nazarov, *Young’s symmetrizers for projective representations of the symmetric group*, Adv. Math. **127** (1997), 190–257.
- [10] A. Ram and A. Shepler, *Classification of graded Hecke algebras for complex reflection groups*, Comment. Math. Helv. **78** (2003), 308–334.
- [11] R. Rouquier, *Representations of rational Cherednik algebras*, In: Infinite-dimensional aspects of representation theory and applications (Charlottesville 2004), Contemp. Math. **392** (2005), 103–131.
- [12] K. Saito, *Extended affine root system I. Coxeter transformations*, Publ. Res. Inst. Math. Sci. **21** (1985), 75–179.
- [13] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.
- [14] A. Sergeev, *The Howe duality and the projective representations of symmetric groups*, Represent. Theory **3** (1999), 416–434.
- [15] T. Suzuki, *Rational and trigonometric degeneration of the double affine Hecke algebra of type A*, Int. Math. Res. Not. **37** (2005), 2249–2262.
- [16] M. Yamaguchi, *A duality of the twisted group algebra of the symmetric group and a Lie superalgebra*, J. Algebra **222** (1999), 301–327.
- [17] T. Khongsap, *Hecke-Clifford algebras and spin Hecke algebras III: The trigonometric type*, J. Algebra **322** (2009), 2731–2750.
- [18] T. Khongsap and W. Wang, *Hecke-Clifford algebras and spin Hecke algebras I: the classical affine type*, Transformation Groups **13** (2008), 389–412.
- [19] T. Khongsap and W. Wang, *Hecke-Clifford algebras and spin Hecke algebras II: the rational double affine type*, Pacific J. Math. **238** (2008), 73–103.
- [20] T. Khongsap and W. Wang, *Hecke-Clifford algebras and spin Hecke algebras IV: odd double affine type*, Special Issue on Dunkl Operators and Related Topics, SIGMA **5** (2009), 012, 27 pages.
- [21] W. Wang, *Spin Hecke algebras of finite and affine types*, Adv. in Math. **212** (2007), 723–748.

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