

## ON THE LINEARIZED LOCAL CALDERÓN PROBLEM

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ABSTRACT. In this article, we investigate a density problem coming from the linearization of Calderón’s problem with partial data. More precisely, we prove that the set of products of harmonic functions on a bounded smooth domain  $\Omega$  vanishing on any fixed closed proper subset of the boundary are dense in  $L^1(\Omega)$  in all dimensions  $n \geq 2$ . This is proved using ideas coming from the proof of Kashiwara’s Watermelon theorem [15].

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### 1. Introduction

**1.1. Main results.** In the seminal article [7], A. P. Calderón asked the question of whether it is possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary. Put in mathematical terms, the question amounts to whether the knowledge of the Dirichlet-to-Neumann map associated to the conductivity equation

$$(1.1) \quad \operatorname{div}(\gamma \nabla u) = 0$$

on a bounded open set  $\Omega$  with smooth boundary uniquely determines a bounded from below conductivity  $\gamma \in L^\infty(\Omega)$ . Using Green’s formula, the problem can be reformulated in the following way: does the cancellation

$$\int_{\Omega} (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = 0$$

for all solutions  $u_1, u_2$  in  $H^1(\Omega)$  of equation (1.1) with respective conductivities  $\gamma_1, \gamma_2$  imply that  $\gamma_1$  and  $\gamma_2$  are equal? Since 1980, the problem has been extensively studied

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Received by the editors May 8, 2009.

and answers have been given in many cases (see for instance [17, 25, 20, 2]). In his article [7], Calderón studied the linearization of this problem at constant conductivities  $\gamma = \gamma_0$ : does the cancellation

$$\int_{\Omega} \gamma \nabla u \cdot \nabla v \, dx = 0$$

for all pairs of harmonic functions  $(u, v)$  imply that  $\gamma \in L^\infty(\Omega)$  vanishes identically? The answer can easily be seen to be true by using harmonic exponentials. A similar and related inverse problem for the Schrödinger equation

$$(1.2) \quad -\Delta u + qu = 0$$

on a bounded open set with smooth boundary  $\Omega$  is whether the Dirichlet-to-Neumann map associated to this equation uniquely determines the bounded potential  $q$  (see for instance [25, 20, 5]). In [25], Calderón’s problem is reduced to this problem for  $\gamma \in C^2$ . The linearization of this inverse problem at  $q = 0$  leads to the question of density of products of harmonic functions in  $L^1(\Omega)$ . Again the use of harmonic exponentials is enough to conclude this.

We are interested in local versions of these inverse problems, in particular to prove that if  $\Lambda_{q_j}$  denotes the Dirichlet-to-Neumann map associated with the Schrödinger equation (1.2) with potential  $q_j$  and if

$$(1.3) \quad \Lambda_{q_1} f|_{\Sigma} = \Lambda_{q_2} f|_{\Sigma}, \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega), \quad \text{supp } f \subset \Sigma,$$

where  $\Sigma$  is an open neighbourhood of some point in the boundary, then  $q_1 = q_2$ . An equivalent formulation is that the cancellation

$$\int_{\Omega} qu_1 u_2 \, dx = 0$$

for all solutions  $u_1, u_2$  in  $H^1(\Omega)$  of the Schrödinger equations (1.2) with respective bounded potentials  $q_1, q_2$ , and with Dirichlet data  $u_1|_{\partial\Omega}, u_2|_{\partial\Omega}$  supported in  $\Sigma$ , implies that  $q$  vanishes identically. This result has recently been proved in dimension  $n = 2$  by Imanuvilov, Uhlmann, and Yamamoto in [13]. The case of partial data where one drops the support constraint on the test functions  $f \in H^{\frac{1}{2}}(\partial\Omega)$  was treated in various situations by Bukhgeim and Uhlmann [5], Kenig, Sjöstrand and Uhlmann [16], Isakov [14] in dimension  $n \geq 3$  and Imanuvilov, Uhlmann, and Yamamoto [12] in dimension 2. However the question of global identifiability from (1.3) is still open in dimension  $n \geq 3$ .

As a first step in this study, we consider here the linearized version of the local problem: we add the constraint that the restriction of the harmonic functions to the boundary vanishes on any fixed closed proper subset of the boundary.

**Theorem 1.1.** *Let  $\Omega$  be a connected bounded open set in  $\mathbf{R}^n$ ,  $n \geq 2$ , with smooth boundary. The set of products of harmonic functions in  $C^\infty(\bar{\Omega})$  which vanish on a closed proper subset  $\Gamma \subsetneq \partial\Omega$  of the boundary is dense in  $L^1(\Omega)$ .*

Another motivation for considering this linearized problem is the following possible application of Theorem 1.1 to travel time tomography in dimension 2. We conjecture that one can use Theorem 1.1 and a method developed by Pestov and Uhlmann in [22] to solve the corresponding global problem to show that in a simple 2-dimensional

Riemannian manifold with boundary, the conformal factor of the metric is uniquely determined from partial knowledge of the boundary distance function. A Riemannian manifold with boundary  $(X, g)$  is said to be *simple* if its boundary is strictly convex and if no geodesic has conjugate points.

**Conjecture 1.2.** *Let  $(X, g_1)$  and  $(X, g_2)$  be two simple compact Riemannian manifolds of dimension 2 with boundary, and  $d_1$  and  $d_2$  denote their respective Riemannian distances. Let  $Y$  be a non-empty open subset of the boundary  $\partial X$  and suppose that  $g_1$  and  $g_2$  are conformal metrics. If*

$$d_1|_{Y \times \partial X} = d_2|_{Y \times \partial X}$$

then  $g_1 = g_2$ .

We hope to come back to this possible application in future work.

**1.2. The Watermelon approach.** The Segal-Bargmann transform of an  $L^\infty$  function  $f$  on  $\mathbf{R}^n$  is given by the following formula

$$Tf(z) = \int_{\mathbf{R}^n} e^{-\frac{1}{2h}(z-y)^2} f(y) dy$$

with  $z = x + i\xi \in \mathbf{C}^n$ . This transform is also known in the literature as the Fourier-Bros-Iagolnitzer (FBI) transform. A variant of this transformation is the wave packet transform (or Gabor transform) which differs by a factor  $e^{-\xi^2/2h}$ . Other denominations or variants include decomposition into coherent states, Husimi functions, etc.

The extension of this definition to tempered distributions is straightforward. The Segal-Bargmann transform is related to the microlocal analysis of analytic singularities of a distribution: the analytic wave front set  $\text{WF}_a(f)$  of  $f$  is the complement of the set of all covectors  $(x_0, \xi_0) \in T^*\mathbf{R}^n \setminus 0$  such that there exists a neighbourhood  $V_{z_0}$  of  $z_0 = x_0 - i\xi_0$  in  $\mathbf{C}^n$ , a cutoff function  $\chi \in C_0^\infty(\mathbf{R}^n)$  with  $\chi(x_0) = 1$ , and two constants  $c > 0$  and  $C > 0$  for which one has the estimate

$$(1.4) \quad |T(\chi f)(z)| \leq C e^{-\frac{c}{h} + \frac{1}{2h}|\text{Im } z|^2}, \quad \forall z \in V_{z_0}, \quad \forall h \in (0, 1].$$

The analytic wave front set  $\text{WF}_a(f)$  is a closed conic set and its image by the first projection  $T^*\mathbf{R}^n \rightarrow \mathbf{R}^n$  is the analytic singular support of  $f$ , i.e. the set of points  $x_0 \in \mathbf{R}^n$  for which there is no neighbourhood on which  $f$  is a real analytic function.

When a distribution  $f$  is supported on a half-space  $H$  and when  $x_0 \in \text{supp } f \cap \partial H$ ,  $f$  cannot be analytic at  $x_0$ , so the analytic wave front set of  $f$  cannot be empty. The following result (see [10] chapter 8, Theorems 8.5.6 and 8.5.6', [23] chapter 8, Theorem 8.2) gives explicitly covectors which are in the wave front set.

**Theorem 1.3.** *Let  $f$  be a distribution supported in a half-space  $H$ , if  $x_0 \in \partial H$  belongs to the support of  $f$ , then  $(x_0, \pm\nu)$  belongs to the analytic wave front set of  $f$  where  $\nu$  denotes a unit conormal to the hyperplane  $\partial H$ .*

One sometimes refers to Theorem 1.3 as the microlocal version of Holmgren's uniqueness theorem. This is due to the fact that the combination of this result together with microlocal ellipticity

$$\text{WF}_a(u) \subset \text{WF}_a(Pu) \cup \text{char } P$$

in the conormal direction (equivalent to the fact that the hypersurface is non-characteristic) yields Holmgren’s uniqueness theorem (see [23] chapter 8, [10] chapter 8 and [11]). Other applications involve the proof of Helgason’s support theorem on the Radon transform and extensions (see [4] and [11]) of this result. Theorem 1.3 has also proved to be a useful tool in the resolution of inverse problems (see [16] and [9]) with partial data. In fact the microlocal version of Holmgren’s uniqueness theorem is a consequence<sup>1</sup> of a more general result on the analytic wave front set due to Kashiwara (see [15], [23] chapter 8, Theorem 8.3, [10] chapter 9, Theorem 9.6.6)

**Watermelon Theorem.** *Let  $f$  be a distribution supported in a half-space  $H$ , if  $x_0 \in \partial H$  and if  $(x_0, \xi_0)$  belongs to the analytic wave front set of  $f$ , then so does  $(x_0, \xi_0 + t\nu)$  where  $\nu$  denotes a unit conormal to the hyperplane  $\partial H$  provided  $\xi_0 + t\nu \neq 0$ .*

From Kashiwara’s Watermelon theorem, it is easy to deduce the microlocal version of Holmgren’s uniqueness theorem: if  $f$  is supported in the half-space  $H$  and  $x_0 \in \partial H \cap \text{supp } f$  then there exists  $(x_0, \xi_0)$  in the analytic wave front set of  $f$  since  $f$  cannot be analytic at  $x_0$ , then  $(x_0, \xi_0 + t\nu) \in \text{WF}_a(f)$  by the Watermelon theorem, which implies  $(x_0, \nu + \xi_0/t) \in \text{WF}_a(f)$  since the wave front set is conic and finally  $(x_0, \nu) \in \text{WF}_a(f)$  by passing to the limit since the wave front set is closed.

One possible proof of Kashiwara’s Watermelon theorem involves the Segal-Bargmann transform. Note that there is an *a priori* exponential bound on the Segal-Bargmann transform of an  $L^\infty$  function

$$|Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h} |\text{Im } z|^2} \|f\|_{L^\infty}.$$

If  $f$  is supported in the half-space  $x_1 \leq 0$  then the former estimate can be improved into

$$|Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h} (|\text{Im } z|^2 - |\text{Re } z_1|^2)} \|f\|_{L^\infty}$$

when  $\text{Re } z_1 \geq 0$ . The exponent in the right-hand side is harmonic with respect to  $z_1$ . The idea of the proof of the Watermelon theorem is to propagate the exponential decay by use of the maximum principle. If  $f$  is supported in the half-space  $x_1 \leq 0$ , one works with the subharmonic function

$$\varphi(z_1) + \frac{1}{2}(\text{Re } z_1)^2 - \frac{1}{2}(\text{Im } z_1)^2 + h \log |Tf(z_0 + z_1 e_1)|$$

on a rectangle  $R$ . One of the edges of  $R$  is contained in the neighbourhood  $V_{z_0}$  where there is the additional exponential decay (1.4) of the Segal-Bargmann transform and one chooses  $\varphi$  to be a non-negative harmonic function vanishing on the boundary of  $R$  except for the segment where there is the exponential decay. The fact that  $\varphi$  is positive on the interior of the rectangle  $R$  allows to propagate the exponential decay of the Segal-Bargmann transform and this translates into the propagation of singularities described in the Watermelon theorem. For more details we refer the reader to [23, 24]. In this note, we will use a variant of this argument adapted to our problem.

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<sup>1</sup>There are of course other ways to prove Theorem 1.3.

## 2. From local to global results

Let  $\Omega$  be a connected bounded open set in  $\mathbf{R}^n$  with smooth boundary. Consider a proper closed subset  $\Gamma \subsetneq \partial\Omega$  of the boundary and a function  $f \in L^\infty(\Omega)$ . Our aim is to prove that the cancellation

$$(2.1) \quad \int_{\Omega} fuv \, dx = 0$$

for any pair of harmonic functions  $u$  and  $v$  in  $C^\infty(\bar{\Omega})$  satisfying

$$u|_{\Gamma} = v|_{\Gamma} = 0$$

implies that  $f$  vanishes identically. Note that the bigger the subset  $\Gamma$  is, the smaller the set of harmonic functions vanishing on  $\Gamma$  is. Therefore we can assume that the complement of  $\Gamma$  in the boundary, is a small open neighbourhood of some point of the boundary. We will obtain Theorem 1.1 as a corollary of a local result.

**Theorem 2.1.** *Let  $\Omega$  be a bounded open set in  $\mathbf{R}^n$ ,  $n \geq 2$ , with smooth boundary, let  $x_0 \in \partial\Omega$  and  $\Gamma$  be the complement of an open boundary neighbourhood of  $x_0$ . There exists  $\delta > 0$  such that if we have the cancellation (2.1) for any pair of harmonic functions  $u$  and  $v$  in  $C^\infty(\bar{\Omega})$  vanishing on  $\Gamma$ , then  $f$  vanishes on  $B(x_0, \delta) \cap \Omega$ .*

Let us see how this local result implies the global one. We have learned of this technique from unpublished work of Alessandrini, Isozaki and Uhlmann [1]. We will need the following approximation lemma in the spirit of the Runge approximation theorem.

**Lemma 2.2.** *Let  $\Omega_1 \subset \Omega_2$  be two bounded open sets with smooth boundaries. Let  $G_{\Omega_2}$  be the Green kernel associated to the open set  $\Omega_2$*

$$-\Delta_y G_{\Omega_2}(x, y) = \delta(x - y), \quad G_{\Omega_2}(x, \cdot)|_{\partial\Omega_2} = 0.$$

Then the set

$$(2.2) \quad \left\{ \int_{\Omega_2} G_{\Omega_2}(\cdot, y) a(y) \, dy : a \in C^\infty(\bar{\Omega}_2), \text{supp } a \subset \bar{\Omega}_2 \setminus \Omega_1 \right\}$$

is dense for the  $L^2(\Omega_1)$  topology in the subspace of harmonic functions  $u \in C^\infty(\bar{\Omega}_1)$  such that  $u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0$ .

*Proof.* Let  $v \in L^2(\Omega_1)$  be a function which is orthogonal to the subspace (2.2), then by Fubini we have

$$\int_{\Omega_2} a(y) \left( \int_{\Omega_1} G_{\Omega_2}(x, y) v(x) \, dx \right) dy = 0$$

for all  $a \in C^\infty(\bar{\Omega}_2)$  supported in  $\bar{\Omega}_2 \setminus \Omega_1$ , therefore

$$\int_{\Omega_1} G_{\Omega_2}(x, y) v(x) \, dx = 0, \quad \forall y \in \bar{\Omega}_2 \setminus \Omega_1.$$

We want to show that  $v$  is orthogonal to any harmonic function  $u \in C^\infty(\bar{\Omega}_1)$  such that  $u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0$ .

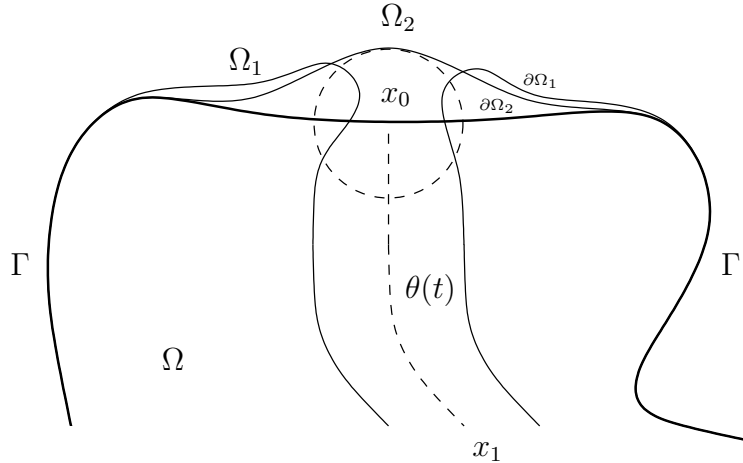


FIGURE 1. The open sets  $\Omega_1$  and  $\Omega_2$

Let  $u \in C^\infty(\overline{\Omega}_1)$  be a such a harmonic function. If we consider

$$w(y) = \int_{\Omega_1} G_{\Omega_2}(x, y)v(x) dx \in H^2(\Omega_2) \cap H_0^1(\Omega_2)$$

then we have by Green’s formula

$$\begin{aligned} \int_{\Omega_1} uv dx &= \int_{\Omega_1} u\Delta w dx - \int_{\Omega_1} w\Delta u dx \\ &= \int_{\partial\Omega_1} u\partial_\nu w dx - \int_{\partial\Omega_1} w\partial_\nu u dx. \end{aligned}$$

Note that the trace of  $w$  vanishes on  $\partial\Omega_1 \cap \partial\Omega_2$  since  $w \in H_0^1(\Omega_2)$ , therefore we have

$$(2.3) \quad \int_{\Omega_1} uv dx = \int_{\partial\Omega_1 \setminus \partial\Omega_2} u\partial_\nu w dx - \int_{\partial\Omega_1 \setminus \partial\Omega_2} w\partial_\nu u dx.$$

At the beginning of this proof, we have shown that

$$w|_{\overline{\Omega}_2 \setminus \Omega_1} = 0 \quad \text{hence also} \quad \nabla w|_{\overline{\Omega}_2 \setminus \Omega_1} = 0$$

and this implies that  $w|_{\partial\Omega_1 \setminus \partial\Omega_2} = 0$  and  $\partial_\nu w|_{\partial\Omega_1 \setminus \partial\Omega_2} = 0$ . Therefore the integral (2.3) vanishes and this proves that  $v$  is orthogonal to any harmonic function in  $C^\infty(\overline{\Omega}_1)$  vanishing on  $\partial\Omega_1 \cap \partial\Omega_2$ .  $\square$

*Proof of Theorem 1.1.* We want to prove that  $f$  vanishes inside  $\Omega$ . We fix a point  $x_1 \in \Omega$  and let  $\theta : [0, 1] \rightarrow \overline{\Omega}$  be a  $C^1$  curve joining  $x_0 \in \partial\Omega \setminus \Gamma$  to  $x_1$  such that  $\theta(0) = x_0$ ,  $\theta'(0)$  is the interior normal to  $\partial\Omega$  at  $x_0$  and  $\theta(t) \in \Omega$  for all  $t \in (0, 1]$ . We consider the closed neighbourhood

$$\Theta_\varepsilon(t) = \{x \in \overline{\Omega} : d(x, \theta([0, t])) \leq \varepsilon\}$$

of the curve ending at  $\theta(t)$ ,  $t \in [0, 1]$  and the set

$$I = \{t \in [0, 1] : f \text{ vanishes a.e. on } \Theta_\varepsilon(t) \cap \Omega\}$$

which is obviously a closed subset of  $[0, 1]$ . By Theorem 2.1 it is non-empty if  $\varepsilon$  is small enough. Let us prove that  $I$  is open. If  $t \in I$  and  $\varepsilon$  is small enough, then we may suppose  $\partial\Theta_\varepsilon(t) \cap \partial\Omega \subset \partial\Omega \setminus \Gamma$  and  $\Omega \setminus \Theta_\varepsilon(t)$  can be smoothed out into an open subset  $\Omega_1$  of  $\Omega$  with smooth boundary such that

$$\Omega_1 \supset \Omega \setminus \Theta_\varepsilon(t) \quad \partial\Omega \cap \partial\Omega_1 \supset \Gamma.$$

We also augment the set  $\Omega$  by smoothing out the set  $\Omega \cup B(x_0, \varepsilon')$  into an open set  $\Omega_2$  with smooth boundary; if  $\varepsilon'$  is small enough then one can construct  $\Omega_2$  in such a way that

$$\partial\Omega_2 \cap \partial\Omega \supset \partial\Omega_1 \cap \partial\Omega \supset \Gamma.$$

Let  $G_{\Omega_2}$  be the Green kernel associated to the open set  $\Omega_2$

$$-\Delta_y G_{\Omega_2}(x, y) = \delta(x - y), \quad G_{\Omega_2}(x, \cdot)|_{\partial\Omega_2} = 0.$$

The function

$$\int_{\Omega_1} f G_{\Omega_2}(x, y) G_{\Omega_2}(t, y) dy, \quad t, x \in \Omega_2 \setminus \overline{\Omega_1}$$

is harmonic (both as a function of the  $t$  and  $x$  variables) and satisfies

$$\int_{\Omega_1} f G_{\Omega_2}(x, y) G_{\Omega_2}(t, y) dy = \int_{\Omega} f G_{\Omega_2}(x, y) G_{\Omega_2}(t, y) dy$$

since  $f$  vanishes on  $\Theta_\varepsilon(t) \cap \Omega$ . When  $t, x$  belong to  $\Omega_2 \setminus \overline{\Omega}$ , this integral is 0 since the Green functions are  $C^\infty(\overline{\Omega})$ , harmonic on  $\Omega$  and vanish on  $\Gamma \subset \partial\Omega_2$ . By unique continuation and continuity, we have

$$(2.4) \quad \int_{\Omega_1} f G_{\Omega_2}(x, y) G_{\Omega_2}(t, y) dy = 0, \quad t, x \in \overline{\Omega_2} \setminus \Omega_1.$$

By Fubini, this means that we will have  $\int_{\Omega_1} fuv dx = 0$  for all functions  $u, v$  on  $\Omega_1$  belonging to the subspace (2.2). By continuity of the bilinear form

$$L^2(\Omega_1) \times L^2(\Omega_1) \rightarrow \mathbf{C} \\ (u, v) \mapsto \int_{\Omega_1} fuv dx$$

and by Lemma 2.2, we have

$$(2.5) \quad \int_{\Omega_1} fuv dx = 0$$

for all functions  $u, v$  in  $C^\infty(\overline{\Omega_1})$  harmonic on  $\Omega_1$  which vanish on  $\partial\Omega_1 \cap \partial\Omega_2$ .

Thanks to Theorem 2.1, the cancellation (2.5) implies that  $f$  vanishes on a neighbourhood of  $\partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega_2)$ . This shows that  $f$  vanishes on a slightly bigger neighbourhood  $\Theta_\varepsilon(\tau)$ ,  $\tau > t$  of the curve, hence that  $I$  is an open set. By connectivity, we conclude that  $I = [0, 1]$  and therefore that  $x_1 \notin \text{supp } f$ . Since the choice of  $x_1$  is arbitrary, this completes the proof of Theorem 1.1.  $\square$

### 3. Harmonic exponentials

This section and the next are devoted to the proof of Theorem 2.1. One can suppose that  $\Omega \setminus \{x_0\}$  is on one side of the tangent hyperplane  $T_{x_0}(\Omega)$  at  $x_0$  by making a conformal transformation. Pick  $a \in \mathbf{R}^n \setminus \bar{\Omega}$  on the line segment in the direction of the outward normal to  $\partial\Omega$  at  $x_0$ , then there is a ball  $B(a, r)$  such that  $\partial B(a, r) \cap \bar{\Omega} = \{x_0\}$ , and there is a conformal transformation

$$\begin{aligned} \psi : \mathbf{R}^n \setminus B(a, r) &\rightarrow \overline{B(a, r)} \\ x &\mapsto \frac{x - a}{|x - a|^2} r^2 + a \end{aligned}$$

which fixes  $x_0$  and exchanges the interior and the exterior of the ball  $B(a, r)$ . The hyperplane  $H : (x - x_0) \cdot (a - x_0) = 0$  is tangent to  $\psi(\Omega)$ , and the image  $\psi(\Omega) \setminus \{x_0\}$  by the conformal transformation lies inside the ball  $B(a, r)$ , therefore on one side of  $H$ . The fact that functions are supported on the boundary close to  $x_0$  is left unchanged. Since a function is harmonic on  $\Omega$  if and only if its Kelvin transform

$$u^* = r^{n-2} |x - a|^{-n+2} u \circ \psi$$

is harmonic on  $\psi(\Omega)$ , (2.1) becomes

$$0 = \int_{\Omega} f u v \, dx = \int_{\psi(\Omega)} r^4 |x - a|^{-4} f \circ \psi \, u^* v^* \, dx$$

for all harmonic functions  $u^*, v^*$  on  $\psi(\Omega)$ . If  $|x - a|^{-4} f \circ \psi$  vanishes close to  $x_0$  then so does  $f$ . Moreover, by scaling one can assume that  $\Omega$  is contained in a ball of radius 1.

Our setting will therefore be as follows:  $x_0 = 0$ , the tangent hyperplane at  $x_0$  is given by  $x_1 = 0$  and

$$(3.1) \quad \Omega \subset \{x \in \mathbf{R}^n : |x + e_1| < 1\}, \quad \Gamma = \{x \in \partial\Omega : x_1 \leq -2c\}.$$

The prime will be used to denote the last  $n - 1$  variables so that  $x = (x_1, x')$  for instance. The Laplacian on  $\mathbf{R}^n$  has  $p(\xi) = \xi^2$  as a principal symbol, we denote by  $p(\zeta) = \zeta^2$  the continuation of this principal symbol on  $\mathbf{C}^n$ , we consider

$$p^{-1}(0) = \{\zeta \in \mathbf{C}^n : \zeta^2 = 0\}.$$

In dimension  $n = 2$ , this set is the union of two complex lines

$$p^{-1}(0) = \mathbf{C}\gamma \cup \mathbf{C}\bar{\gamma}$$

where  $\gamma = ie_1 + e_2 = (i, 1) \in \mathbf{C}^2$ . Note that  $(\gamma, \bar{\gamma})$  is a basis of  $\mathbf{C}^2$ : the decomposition of a complex vector in this basis reads

$$(3.2) \quad \zeta = \zeta_1 e_1 + \zeta_2 e_2 = \frac{\zeta_2 - i\zeta_1}{2} \gamma + \frac{\zeta_2 + i\zeta_1}{2} \bar{\gamma}.$$

Similarly for  $n \geq 2$ , the differential of the map

$$\begin{aligned} s : p^{-1}(0) \times p^{-1}(0) &\rightarrow \mathbf{C}^n \\ (\zeta, \eta) &\mapsto \zeta + \eta \end{aligned}$$



at  $(\zeta_0, \eta_0)$  is surjective

$$Ds(\zeta_0, \eta_0) : T_{\zeta_0}p^{-1}(0) \times T_{\eta_0}p^{-1}(0) \rightarrow \mathbf{C}^n$$

$$(\zeta, \eta) \mapsto \zeta + \eta$$

provided  $\mathbf{C}^n = T_{\zeta_0}p^{-1}(0) + T_{\eta_0}p^{-1}(0)$ , i.e. provided  $\zeta_0$  and  $\eta_0$  are linearly independent. In particular, this is the case if  $\zeta_0 = \gamma$  and  $\eta_0 = -\bar{\gamma}$ ; as a consequence all  $z \in \mathbf{C}^n$ ,  $|z - 2ie_1| < 2\varepsilon$  may be decomposed as a sum of the form

$$(3.3) \quad z = \zeta + \eta, \quad \text{with } \zeta, \eta \in p^{-1}(0), \quad |\zeta - \gamma| < C\varepsilon, \quad |\eta + \bar{\gamma}| < C\varepsilon.$$

provided  $\varepsilon > 0$  is small enough.

The exponentials with linear weights

$$e^{-\frac{i}{h}x \cdot \zeta}, \quad \zeta \in p^{-1}(0)$$

are harmonic functions. We need to add a correction term in order to obtain harmonic functions  $u$  satisfying the boundary requirement  $u|_\Gamma = 0$ . Let  $\chi \in C_0^\infty(\mathbf{R}^n)$  be a cutoff function which equals 1 on  $\Gamma$ , we consider the solution  $w$  to the Dirichlet problem

$$(3.4) \quad \begin{cases} \Delta w = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = -(e^{-\frac{i}{h}x \cdot \zeta} \chi)|_{\partial\Omega}. \end{cases}$$

The function

$$u(x, \zeta) = e^{-\frac{i}{h}x \cdot \zeta} + w(x, \zeta)$$

is in  $C^\infty(\bar{\Omega})$ , harmonic and satisfies  $u|_\Gamma = 0$ . We have the following bound on  $w$ :

$$(3.5) \quad \begin{aligned} \|w\|_{H^1(\Omega)} &\leq C_1 \|e^{-\frac{i}{h}x \cdot \zeta} \chi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C_2 (1 + h^{-1}|\zeta|)^{\frac{1}{2}} e^{\frac{1}{h}H_K(\text{Im } \zeta)} \end{aligned}$$

where  $H_K$  is the supporting function of the compact subset  $K = \text{supp } \chi \cap \partial\Omega$  of the boundary

$$H_K(\xi) = \sup_{x \in K} x \cdot \xi, \quad \xi \in \mathbf{R}^n.$$

In particular, if we take  $\chi$  to be supported in  $x_1 \leq -c$  and equal to 1 on  $x_1 \leq -2c$  then the bound (3.5) becomes

$$(3.6) \quad \|w\|_{H^1(\Omega)} \leq C_2 (1 + h^{-1}|\zeta|)^{\frac{1}{2}} e^{-\frac{c}{h} \text{Im } \zeta_1} e^{\frac{1}{h}|\text{Im } \zeta'|} \quad \text{when } \text{Im } \zeta_1 \geq 0.$$

Our starting point is the cancellation of the integral

$$(3.7) \quad \int_{\Omega} f(x) u(x, \zeta) u(x, \eta) dx = 0, \quad \zeta, \eta \in p^{-1}(0)$$

which may be rewritten under the form

$$\begin{aligned} \int_{\Omega} f(x) e^{-\frac{i}{h}x \cdot (\zeta + \eta)} dx &= - \int_{\Omega} f(x) e^{-\frac{i}{h}x \cdot \zeta} w(x, \eta) dx \\ &\quad - \int_{\Omega} f(x) e^{-\frac{i}{h}x \cdot \eta} w(x, \zeta) dx - \int_{\Omega} f(x) w(x, \zeta) w(x, \eta) dx. \end{aligned}$$

This allows us to give a bound on the left-hand side term

$$\begin{aligned} \left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot (\zeta + \eta)} dx \right| &\leq \|f\|_{L^\infty(\Omega)} (\|e^{-\frac{i}{h} x \cdot \zeta}\|_{L^2(\Omega)} \|w(x, \eta)\|_{L^2(\Omega)} \\ &\quad + \|e^{-\frac{i}{h} x \cdot \eta}\|_{L^2(\Omega)} \|w(x, \zeta)\|_{L^2(\Omega)} + \|w(x, \eta)\|_{L^2(\Omega)} \|w(x, \zeta)\|_{L^2(\Omega)}). \end{aligned}$$

Thus using (3.6)

$$\begin{aligned} \left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot (\zeta + \eta)} dx \right| &\leq C_3 \|f\|_{L^\infty(\Omega)} (1 + h^{-1} |\eta|)^{\frac{1}{2}} (1 + h^{-1} |\zeta|)^{\frac{1}{2}} \\ &\quad \times e^{-\frac{c}{h} \min(\operatorname{Im} \zeta_1, \operatorname{Im} \eta_1)} e^{\frac{1}{h} (|\operatorname{Im} \zeta'| + |\operatorname{Im} \eta'|)} \end{aligned}$$

when  $\operatorname{Im} \zeta_1 \geq 0, \operatorname{Im} \eta_1 \geq 0$  and  $\zeta, \eta \in p^{-1}(0)$ . In particular if  $|\zeta - a\gamma| < C\varepsilon a$  and  $|\eta + a\bar{\gamma}| < C\varepsilon a$  with  $\varepsilon \leq 1/2C$  then

$$\left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot (\zeta + \eta)} dx \right| \leq C_4 h^{-1} \|f\|_{L^\infty(\Omega)} e^{-\frac{ca}{2h}} e^{\frac{2C\varepsilon a}{h}}.$$

Take  $z \in \mathbf{C}^n$  with  $|z - 2ia e_1| < 2\varepsilon a$  with  $\varepsilon$  small enough, once rescaled the decomposition (3.3) gives

$$z = \zeta + \eta, \quad \zeta, \eta \in p^{-1}(0), \quad |\zeta - a\gamma| < C\varepsilon a, \quad |\eta + a\bar{\gamma}| < C\varepsilon a$$

we therefore get the estimate

$$(3.8) \quad \left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot z} dx \right| \leq C_4 h^{-1} \|f\|_{L^\infty(\Omega)} e^{-\frac{ca}{2h}} e^{\frac{2C\varepsilon a}{h}}$$

for all  $z \in \mathbf{C}^n$  such that  $|z - 2ia e_1| < 2\varepsilon a$ .

In order to conclude, one needs to extrapolate the exponential decay to more values of the frequency variable  $z$ . This will be achieved using a variant of the proof of the Watermelon theorem. We extend the function  $f$  to  $\mathbf{R}^n$  by assigning to it the value 0 outside  $\Omega$ .

#### 4. A watermelon approach

Let us recall the definition of the Segal-Bargmann transform of an  $L^\infty$  function  $f$  on  $\mathbf{R}^n$

$$Tf(z) = \int_{\mathbf{R}^n} e^{-\frac{1}{2h}(z-y)^2} f(y) dy, \quad z \in \mathbf{C}^n$$

and the *a priori* exponential bound

$$(4.1) \quad |Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h} |\operatorname{Im} z|^2} \|f\|_{L^\infty}.$$

If  $f$  is supported in the half-space  $x_1 \leq 0$  then the former estimate can be improved into

$$(4.2) \quad |Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h} (|\operatorname{Im} z|^2 - |\operatorname{Re} z_1|^2)} \|f\|_{L^\infty}$$

when  $\operatorname{Re} z_1 \geq 0$ . Note that when we restrict the Segal-Bargmann transform to real values  $x \in \mathbf{R}^n$ , and when we multiply by the factor  $(2\pi h)^{-n/2}$  we obtain the heat

operator whose kernel  $(2\pi h)^{-n/2} e^{-(x-y)^2/2h}$  is a Gaussian mollifier, therefore if  $f \in L^\infty_{\text{comp}}(\mathbf{R}^n)$

$$(4.3) \quad \lim_{h \rightarrow 0} (2\pi h)^{-\frac{n}{2}} T f(x) = f(x) \quad \text{in } L^p(\mathbf{R}^n)$$

for all  $1 \leq p < \infty$ . If one gets exponential decay of the Segal-Bargmann transform when  $x \in \mathbf{R}^n$  is close to 0, then by the former limit,  $f$  vanishes close to 0 and this proves Theorem 2.1. Getting such an exponential decay will be the aim of this section.

The kernel of the Segal-Bargmann transform of a function  $f \in L^\infty$  can be written as a linear superposition of exponentials with linear weights

$$e^{-\frac{1}{2h}(z-y)^2} = e^{-\frac{z^2}{2h}} (2\pi h)^{-\frac{n}{2}} \int e^{-\frac{t^2}{2h}} e^{-\frac{i}{h}y \cdot (t+iz)} dt$$

therefore we get

$$(4.4) \quad T f(z) = (2\pi h)^{-\frac{n}{2}} \iint e^{-\frac{1}{2h}(z^2+t^2)} e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dt dy.$$

Suppose now that the function  $f$  is supported in  $\Omega$  and satisfies (3.7), formula (4.4) allows us to improve the estimate (4.2):

$$|T f(z)| \leq (2\pi h)^{-\frac{n}{2}} \int e^{\frac{1}{2h}(|\text{Im } z|^2 - |\text{Re } z|^2 - t^2)} \left| \int e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dy \right| dt.$$

Suppose now that  $\text{Re } z_1 \geq 0$ : if we split the integral with respect to the variable  $t$  in two integrals

$$|T f(z)| \leq \frac{e^{\frac{1}{2h}(|\text{Im } z|^2 - |\text{Re } z|^2)}}{(2\pi h)^{\frac{n}{2}}} \left( \int_{|t| \leq \varepsilon a} e^{-\frac{t^2}{2h}} \left| \int e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dy \right| dt + \int_{|t| \geq \varepsilon a} e^{-\frac{t^2}{2h}} \left| \int e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dy \right| dt \right)$$

this implies

$$(4.5) \quad |T f(z)| \leq e^{\frac{1}{2h}(|\text{Im } z|^2 - |\text{Re } z|^2)} \left( \sup_{|t| \leq \varepsilon a} \left| \int e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dy \right| + \sqrt{2} e^{\frac{1}{h}|\text{Re } z'|} e^{-\frac{\varepsilon^2 a^2}{4h}} \int_{\Omega} |f(y)| dy \right)$$

since  $f$  is supported in  $\Omega \subset \{y_1 \leq 0\}$ . If we assume  $|z - 2ae_1| < \varepsilon a$  with  $\varepsilon$  small enough<sup>2</sup>, the estimate (3.8) reads in our context<sup>3</sup>

<sup>2</sup>Note that in dimension  $n = 2$  the decomposition (3.3) for  $t + iz$  is explicit

$$t + iz = \underbrace{\frac{1}{2}(t_2 - it_1 + iz_2 + z_1)}_{=\zeta} \gamma + \underbrace{\frac{1}{2}(t_2 + it_1 + iz_2 - z_1)}_{=-\eta} \bar{\gamma}.$$

<sup>3</sup>Beware that  $z$  has been changed into  $t + iz$  and we have used that  $|(t + iz) - 2iae_1| \leq |t| + |z - 2ae_1| < 2\varepsilon a$ .

$$(4.6) \quad \left| \int_{\Omega} f(y) e^{-\frac{i}{h} y \cdot (t+iz)} dy \right| \leq C_4 h^{-1} \|f\|_{L^\infty(\Omega)} e^{-\frac{ca}{2h}} e^{\frac{2C\varepsilon a}{h}}$$

when  $|t| \leq \varepsilon a$  and  $|z - 2ae_1| < \varepsilon a$ . Thus combining the two estimates (4.6) and (4.5) we get

$$|Tf(z)| \leq C_5 h^{-1} \|f\|_{L^\infty(\Omega)} e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2)} \left( e^{-\frac{ca}{2h}} e^{\frac{2C\varepsilon a}{h}} + e^{-\frac{\varepsilon^2 a^2}{4h}} e^{\frac{\varepsilon a}{h}} \right)$$

provided  $|z - 2ae_1| < \varepsilon a$ . Now choosing  $\varepsilon < c/8C$  and  $a > (c + 4\varepsilon)/\varepsilon^2$  we finally obtain the bound

$$(4.7) \quad |Tf(z)| \leq 2C_5 h^{-1} \|f\|_{L^\infty(\Omega)} e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2 - \frac{ca}{2})}.$$

To sum-up we have obtained the following bounds on the Segal-Bargmann transform of  $f$

$$(4.8) \quad e^{-\frac{\Phi(z_1)}{2h}} |Tf(z_1, x')| \leq Ch^{-1} \|f\|_{L^\infty(\Omega)} \times \begin{cases} 1 & \text{when } z_1 \in \mathbf{C}, \\ e^{-\frac{ca}{4h}} & \text{when } |z_1 - 2a| \leq \frac{\varepsilon a}{2}, |x'| < \frac{\varepsilon a}{2}, \end{cases} \quad x' \in \mathbf{R}^{n-1}$$

where the weight  $\Phi$  is given by the following expression

$$\Phi(z_1) = \begin{cases} (\operatorname{Im} z_1)^2 & \text{when } \operatorname{Re} z_1 \leq 0 \\ (\operatorname{Im} z_1)^2 - (\operatorname{Re} z_1)^2 & \text{when } \operatorname{Re} z_1 \geq 0. \end{cases}$$

These estimates correspond to (4.1), (4.2) and (4.7).

**Lemma 4.1.** *Let  $b$  and  $L$  be two positive numbers. Let  $F$  be an entire function satisfying the following bounds*

$$e^{-\frac{\Phi(s)}{2h}} |F(s)| \leq \begin{cases} 1 & \text{when } s \in \mathbf{C} \\ e^{-\frac{c}{2h}} & \text{when } |s - L| \leq b \end{cases}$$

then for all  $r \geq 0$  there exist  $c', \delta > 0$  (which do not depend on the function  $F$ ) such that  $F$  satisfies

$$|F(s)| \leq e^{-\frac{c'}{2h} + \frac{(\operatorname{Im} s)^2}{2h}}, \quad \text{when } |\operatorname{Re} s| \leq \delta \text{ and } |\operatorname{Im} s| \leq r.$$

*Proof.* We consider the subharmonic function

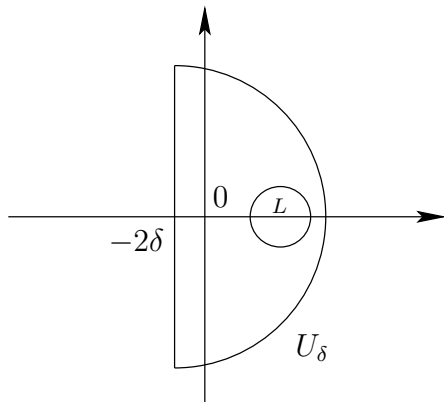
$$f(s) = 2h \log |F(s)| - (\operatorname{Im} s)^2 + (\operatorname{Re} s)^2$$

which satisfies the bounds

$$(4.9) \quad f(s) \leq \begin{cases} (\operatorname{Re} s)^2 & \text{when } \operatorname{Re} s \leq 0 \\ 0 & \text{when } \operatorname{Re} s \geq 0 \\ -c & \text{when } |s - L| \leq b. \end{cases}$$

We will work on the following domain

$$U_\delta = (D(-2\delta, R) \setminus \overline{D(L, b)}) \cap \{\operatorname{Re} s > -2\delta\}$$

FIGURE 2. The domain  $U_\delta$ 

and consider the following harmonic function on  $U_\delta$

$$\varphi(s) = -\frac{c}{d}(\log|s + 2\delta + L| - \log|s + 2\delta - L|) + 4\delta^2.$$

If we choose

$$d = \log(b + 2L + 2\delta) - \log(b - 2\delta) = \log(1 + 2L/b) + \mathcal{O}(\delta)$$

then the harmonic function  $\varphi$  has the following properties

- ◇  $\lim_{|s| \rightarrow \infty} \varphi = 4\delta^2$
- ◇  $\varphi = 4\delta^2$  on the line  $\operatorname{Re} s = -2\delta$
- ◇  $\varphi \geq -c$  on the circle  $|s - L| = b$ .

If we choose  $R$  large enough, the properties of the function  $\varphi$  imply that

$$(f - \varphi)|_{\partial U_\delta} \leq 0$$

therefore by the maximum principle, the subharmonic function  $f - \varphi$  is non-positive on  $U_\delta$ . In particular, this gives

$$f \leq -\frac{c}{2d} \log \frac{(L + \delta)^2 + r^2}{(L - \delta)^2 + r^2} + 4\delta^2 = -c'$$

when  $|\operatorname{Im} s| \leq r$  and  $|\operatorname{Re} s| \leq \delta$ . If  $\delta$  is small enough then

$$c' = \frac{2Lc}{(L^2 + r^2) \log(1 + 2L/b)} \delta + \mathcal{O}(\delta^2)$$

is positive, and we obtain the desired exponential decay

$$|F(s)| = e^{\frac{f(s)}{2h} + \frac{(\operatorname{Im} s)^2 - (\operatorname{Re} s)^2}{2h}} \leq e^{-\frac{c'}{2h} + \frac{(\operatorname{Im} s)^2}{2h}}$$

if  $|\operatorname{Re} s| \leq \delta$  and  $|\operatorname{Im} s| \leq r$ . □

**Remark 4.2.** In fact, one can turn the former proof of Lemma 4.1 into a more conceptual proof where one does not need to have an explicit expression of the harmonic majorant  $\varphi$ . This may turn handy in situations where it might be difficult to find an expression for the function  $\varphi$ . Then one can reason along the following lines.

One considers the harmonic function  $\varphi$  on  $U_\delta$  with the following boundary values:

- ◊  $\varphi = 4\delta^2$  on the boundary of the semi-disc,
- ◊  $\varphi = -c$  on the circle of centre  $L$  and radius  $b$ .

The function  $\tilde{\varphi} = 4\delta^2 - \varphi$  is harmonic and non-negative on  $U_\delta$  and attains its minimum everywhere on the cut-diameter of the semi-disc. By the Hopf boundary lemma, if  $\nu$  stands for the interior normal, one has<sup>4</sup>

$$\frac{\partial \tilde{\varphi}}{\partial \nu}(-2\delta + iy) \geq \frac{C}{\delta} \tilde{\varphi}(iy) > 0, \quad |y| \leq r < R$$

where  $C$  is a universal constant. By Harnack’s inequality,  $\tilde{\varphi}(iy)$  and

$$\tilde{\varphi}(L - b - \delta^2) = c + \mathcal{O}(\delta^2)$$

are comparable, and the constants are uniform with respect to  $\delta$ . Thus if  $\delta$  is small enough, one gets

$$-\frac{\partial \varphi}{\partial \nu}(-2\delta + iy) \geq \frac{2c'}{\delta}, \quad |y| < r.$$

This inequality and elliptic regularity give

$$(4.10) \quad \varphi(s) \leq -c', \quad |\operatorname{Re} s| \leq \delta, |\operatorname{Im} s| \leq r$$

if  $\delta$  is small enough.

One has

$$(f - \varphi)|_{\partial U_\delta} \leq 0$$

therefore by the maximum principle, the subharmonic function  $f - \varphi$  is non-positive on  $U_\delta$ . But according to (4.10), when  $|\operatorname{Re} s| \leq \delta$  and  $|\operatorname{Im} s| \leq r$  one has

$$(4.11) \quad f \leq \varphi \leq -c'$$

This gives the desired exponential estimate

$$|F(s)| \leq e^{-\frac{c'}{2h} + \frac{(\operatorname{Im} s)^2}{2h}}$$

if  $|\operatorname{Re} s| \leq \delta$  and  $|\operatorname{Im} s| \leq r$ . This shows that the arguments given in the proof of Lemma 4.1 do not rely on the expression of the majorant  $\varphi$  and are thus somewhat flexible.

Applying Lemma 4.1 to the function

$$F(s) = \frac{h|Tf(s, x')|}{C\|f\|_{L^\infty(\Omega)}}$$

we obtain in particular that

$$|Tf(x)| \leq Ch^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{c'}{2h}}$$

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<sup>4</sup>The radius  $R$  is chosen large enough so that the points of the boundary with  $|\operatorname{Im} s| \leq r$  stay far enough from the corners where the Hopf lemma is no longer valid.

for all  $x \in \Omega$ ,  $|x_1| \leq \delta$ , provided  $\delta$  has been chosen small enough. Multiplying the former estimate by  $(2\pi h)^{-n/2}$  and letting  $h$  tend to 0, we deduce from (4.3)

$$f(x) = 0, \quad \forall x \in \Omega, \quad 0 \geq x_1 \geq -\delta.$$

This completes the proof of Theorem 2.1.

### Acknowledgements

D. DSF. is partly supported by ANR grant Equa-disp. C.E.K. is partly supported by NSF grant DMS-0456583. G.U. would like to acknowledge partial support of NSF and a Walker Family Endowed Professorship.

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