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ON THE BRAUER GROUP OF ENRIQUES SURFACES

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Abstract. Let S be a complex Enriques surface (quotient of a K3 surface X by a fixed-point-free involution). The Brauer group $\mathrm{Br}(S)$ has a unique nonzero element. We describe its pull-back in $\mathrm{Br}(X)$, and show that the surfaces S for which it is trivial form a countable union of hypersurfaces in the moduli space of Enriques surfaces.

1. Introduction

Let S be a complex Enriques surface, and $\pi : X \to S$ its 2-to-1 cover by a K3 surface. Poincaré duality provides an isomorphism $\mathrm{H}^3(S,\mathbb{Z}) \cong \mathrm{H}_1(S,\mathbb{Z}) = \mathbb{Z}/2$, so that there is a unique nontrivial element b_S in the Brauer group $\mathrm{Br}(S)$. What is the pull-back of this element in $\mathrm{Br}(X)$? Is it nonzero?[1]

The answer to the first question is easy in terms of the canonical isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ (see §2): π^*b_S corresponds to the linear form $\tau \mapsto (\beta\cdot\pi_*\tau)$, where β is any element of $\mathrm{H}^2(S,\mathbb{Z}/2)$ which does not come from $\mathrm{H}^2(S,\mathbb{Z})$. The second question turns out to be more subtle: the answer depends on the surface. We will characterize the surfaces S for which $\pi^*b_S = 0$ (Corollary 5.7), and show that they form a countable union of hypersurfaces in the moduli space of Enriques surfaces (Corollary 6.5).

Part of our results hold over any algebraically closed field, and also in a more general set-up (see Proposition 4.1 below); for the last part, however, we need in a crucial way Horikawa's description of the moduli space by transcendental methods.

2. The Brauer group of a surface

Let S be a smooth projective variety over a field; we define the Brauer group $\mathrm{Br}(S)$ as the étale cohomology group $\mathrm{H}^2_{\text{ét}}(S,\mathbb{G}_m)$. For surfaces this definition coincides with that of Grothendieck [G] by [G], II, Cor. 2.2; this holds in fact in any dimension by a result of Gabber, which we will not need here (see [dJ]).

In this section we assume that S is a complex surface; we recall the description of $\mathrm{Br}(S)$ in that case – this is classical but not so easy to find in the literature. The Kummer exact sequence

$$0 \to \mathbb{Z}/n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \to 0$$

gives rise to an exact sequence

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Z}/n \longrightarrow \mathrm{H}^2(S,\mathbb{Z}/n) \xrightarrow{p} \mathrm{Br}(S)[n] \to 0 \qquad (2.a)$$

(we denote by $M[n]$ the kernel of the multiplication by n in a $\mathbb{Z}$-module M).

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[1]The question is mentioned in [H-S], where the authors construct an Enriques surface S over $\mathbb{Q}$ for which $\pi^*b_S \neq 0$ (see Cor. 2.8).

On the other hand, the cohomology exact sequence associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ gives:

$$0 \to \mathrm{H}^2(S,\mathbb{Z}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{H}^2(S,\mathbb{Z}/n) \longrightarrow \mathrm{H}^3(S,\mathbb{Z})[n] \to 0 \qquad (2.b)$$

Comparing (2.a) and (2.b) we get an exact sequence

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Z}/n \longrightarrow \mathrm{H}^2(S,\mathbb{Z}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{Br}(S)[n] \longrightarrow \mathrm{H}^3(S,\mathbb{Z})[n] \to 0\ . \qquad (2.c)$$

Let $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}$ be the quotient of $\mathrm{H}^2(S,\mathbb{Z})$ by its torsion subgroup; the cup-product induces a perfect pairing on $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}$. We denote by $T_S \subset \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}$ the *transcendental lattice*, that is, the orthogonal of the image of $\mathrm{Pic}(S)$. We have an exact sequence

$$\mathrm{Pic}(S) \xrightarrow{c_1} \mathrm{H}^2(S,\mathbb{Z}) \xrightarrow{u} T_S^* \to 0$$

where u associates to $\alpha \in \mathrm{H}^2(S,\mathbb{Z})$ the cup-product with α. Taking tensor product with $\mathbb{Z}/n$ and comparing with (2.c), we get an exact sequence

$$0 \to \mathrm{Hom}(T_S,\mathbb{Z}/n) \longrightarrow \mathrm{Br}(S)[n] \longrightarrow \mathrm{H}^3(S,\mathbb{Z})[n] \to 0\ ; \qquad (2.d)$$

or, passing to the direct limit over n,

$$0 \to \mathrm{Hom}(T_S,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Br}(S) \longrightarrow \mathrm{Tors}\,\mathrm{H}^3(S,\mathbb{Z}) \to 0\ . \qquad (2.e)$$

3. Algebraic topology of Enriques surfaces

3.1. Let S be an Enriques surface (over $\mathbb{C}$). We first recall some elementary facts on the topology of S. A general reference is [BHPV], ch. VIII.

The torsion subgroup of $\mathrm{H}^2(S,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$; its nonzero element is the canonical class K_S. Let k_S denote the image of K_S in $\mathrm{H}^2(S,\mathbb{Z}/2)$. The universal coefficient theorem together with Poincaré duality gives an exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{k_S} \mathrm{H}^2(S,\mathbb{Z}/2) \xrightarrow{v_S} \mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z}),\mathbb{Z}/2) \to 0 \qquad (3.a)$$

where v_S is deduced from the cup-product.

3.2. The linear form $\alpha \mapsto (k_S \cdot \alpha)$ on $\mathrm{H}^2(S,\mathbb{Z}/2)$ vanishes on the image of $\mathrm{H}^2(S,\mathbb{Z})$, hence coincides with the map $\mathrm{H}^2(S,\mathbb{Z}/2) \to \mathrm{H}^3(S,\mathbb{Z}) = \mathbb{Z}/2$ from the exact sequence (2.b). Note that k_S is the second Stiefel-Whitney class $w_2(S)$; in particular, we have $(k_S \cdot \alpha) = \alpha^2$ for all $\alpha \in \mathrm{H}^2(S,\mathbb{Z}/2)$ (Wu formula, see [M-S]).

3.3. The map $c_1 : \mathrm{Pic}(S) \to \mathrm{H}^2(S,\mathbb{Z})$ is an isomorphism, hence (2.e) provides an isomorphism $\mathrm{Br}(S) \xrightarrow{\sim} \mathrm{Tors}\,\mathrm{H}^3(S,\mathbb{Z}) \cong \mathbb{Z}/2$. We will denote by b_S the nonzero element of $\mathrm{Br}(S)$.

Let $\pi : X \to S$ be the 2-to-1 cover of S by a K3 surface. The aim of this note is to study the pull-back $\pi^* b_S$ in $\mathrm{Br}(X)$.

Proposition 3.4. *The class $\pi^* b_S$ is represented, through the isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Hom}(T_X,\mathbb{Q}/\mathbb{Z})$, by the linear form $\tau \mapsto (\beta \cdot \pi_*\bar{\tau})$, where $\bar{\tau}$ is the image of τ in $\mathrm{H}^2(X,\mathbb{Z}/2)$ and β any element of $\mathrm{H}^2(S,\mathbb{Z}/2)$ which does not come from $\mathrm{H}^2(S,\mathbb{Z})$.*

Proof. Let β be an element of $\mathrm{H}^2(S,\mathbb{Z}/2)$ which does not come from $\mathrm{H}^2(S,\mathbb{Z})$, so that $p(\beta)=b_S$ (2.a). The pull-back $\pi^*b_S\in\mathrm{Br}(X)$ is represented by $\pi^*\beta\in\mathrm{H}^2(X,\mathbb{Z}/2)$ $\cong\mathrm{H}^2(X,\mathbb{Z})\otimes\mathbb{Z}/2$; its image in $\mathrm{Hom}(T_X,\mathbb{Z}/2)$ is the linear form $\tau\mapsto(\pi^*\beta\cdot\bar{\tau})$. Since $(\pi^*\beta\cdot\bar{\tau})=(\beta\cdot\pi_*\bar{\tau})$, the Proposition follows. □

Part (i) of the following Proposition shows that the class $\pi^*\beta\in\mathrm{H}^2(X,\mathbb{Z}/2)$ which appears above is nonzero. This does *not* say that π^*b_S is nonzero, as $\pi^*\beta$ could come from a class in $\mathrm{Pic}(X)$ – see §6.

Proposition 3.5. (i) *The kernel of* $\pi^*:\mathrm{H}^2(S,\mathbb{Z}/2)\to\mathrm{H}^2(X,\mathbb{Z}/2)$ *is* $\{0,k_S\}$.
(ii) *The Gysin map* $\pi_*:\mathrm{H}^2(X,\mathbb{Z})\to\mathrm{H}^2(S,\mathbb{Z})$ *is surjective.*

Proof. To prove (i) we use the Hochschild-Serre spectral sequence :

$$E_2^{p,q}=\mathrm{H}^p(\mathbb{Z}/2,\mathrm{H}^q(X,\mathbb{Z}/2))\ \Rightarrow\ \mathrm{H}^{p+q}(S,\mathbb{Z}/2)\ .$$

We have $E_2^{1,1}=0$, and $E_\infty^{2,0}=E_2^{2,0}=\mathrm{H}^2(\mathbb{Z}/2,\mathbb{Z}/2)=\mathbb{Z}/2$. Thus the kernel of $\pi^*:\mathrm{H}^2(S,\mathbb{Z}/2)\to\mathrm{H}^2(X,\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$. Since it contains k_S, it is equal to $\{0,k_S\}$.

Let us prove (ii). Because of the formula $\pi_*\pi^*\alpha=2\alpha$, the cokernel of $\pi_*:\mathrm{H}^2(X,\mathbb{Z})\to\mathrm{H}^2(S,\mathbb{Z})$ is a $(\mathbb{Z}/2)$-vector space; therefore it suffices to prove that the transpose map

$${}^t\pi_*:\mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z}),\mathbb{Z}/2)\longrightarrow\mathrm{Hom}(\mathrm{H}^2(X,\mathbb{Z}),\mathbb{Z}/2)$$

is injective. This is implied by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(S,\mathbb{Z}/2) \arrow[r, two heads, "v_S"'] \arrow[d, "\pi^*"] & \mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z}),\mathbb{Z}/2) \arrow[d, "{}^t\pi_*"'] \\
\mathrm{H}^2(X,\mathbb{Z}/2) \arrow[r, "\sim", "v_X"'] & \mathrm{Hom}(\mathrm{H}^2(X,\mathbb{Z}),\mathbb{Z}/2)
\end{tikzcd}$$

plus the fact that $\mathrm{Ker}\,\pi^*=\mathrm{Ker}\,v_S=\{0,k_S\}$ (by (i) and (3.a)). □

4. Brauer groups and cyclic coverings

Proposition 4.1. *Let* $\pi:X\to S$ *be an étale, cyclic covering of smooth projective varieties over an algebraically closed field* k. *Let* σ *be a generator of the Galois group* G *of* π, *and let* $\mathrm{Nm}:\mathrm{Pic}(X)\to\mathrm{Pic}(S)$ *be the norm homomorphism. The kernel of* $\pi^*:\mathrm{Br}(S)\to\mathrm{Br}(X)$ *is canonically isomorphic to* $\mathrm{Ker}\,\mathrm{Nm}\,/(1-\sigma^*)(\mathrm{Pic}(X))$.

Proof. We consider the Hochschild-Serre spectral sequence

$$E_2^{p,q}=\mathrm{H}^p(G,\mathrm{H}^q(X,\mathbb{G}_m))\ \Rightarrow\ \mathrm{H}^{p+q}(S,\mathbb{G}_m)\ .$$

Since $E_2^{2,0}=\mathrm{H}^2(G,k^*)=0$, the kernel of $\pi^*:\mathrm{Br}(S)\to\mathrm{Br}(X)$ is identified with $E_\infty^{1,1}=\mathrm{Ker}\big(d_2:E_2^{1,1}\to E_2^{3,0}\big)$. We have $E_2^{3,0}=\mathrm{H}^3(G,k^*)$; by periodicity of the cohomology of G, this group is canonically isomorphic to $\mathrm{H}^1(G,k^*)=\mathrm{Hom}(G,k^*)$, the character group of G, which we denote by $\widehat{G}$. So we view d_2 as a map from $\mathrm{H}^1(G,\mathrm{Pic}(X))$ to $\widehat{G}$.

Let S be the endomorphism $L \mapsto \bigotimes_{g\in G} g^*L$ of $\mathrm{Pic}(X)$; recall that $\mathrm{H}^1(G, \mathrm{Pic}(X))$ is isomorphic to $\mathrm{Ker}\,\mathsf{S}/\,\mathrm{Im}(1-\sigma^*)$. We have $\pi^*\,\mathrm{Nm}(L) = \mathsf{S}(L)$ for $L \in \mathrm{Pic}(X)$, hence Nm maps $\mathrm{Ker}\,\mathsf{S}$ into $\mathrm{Ker}\,\pi^* \subset \mathrm{Pic}(S)$. Now recall that $\mathrm{Ker}\,\pi^*$ is canonically isomorphic to $\widehat{G}$: to $\chi \in \widehat{G}$ corresponds the subsheaf L_χ of $\pi_*\mathcal{O}_X$ where G acts through the character χ. Since $\mathrm{Nm}\circ(1-\sigma^*) = 0$, the norm induces a homomorphism $\mathrm{H}^1(G, \mathrm{Pic}(X)) \to \mathrm{Ker}\,\pi^* \cong \widehat{G}$. The Proposition will follow from:

Lemma 4.2. *The map $d_2 : \mathrm{H}^1(G, \mathrm{Pic}(X)) \to \widehat{G}$ coincides with the homomorphism induced by the norm.*

Proof. We apply the formalism of [S], Proposition 1.1, where a very close situation is considered. This Proposition, together with property (1) which follows it, tells us that d_2 is given by cup-product with the extension class in $\mathrm{Ext}^2_G(\mathrm{Pic}(X), k^*)$ of the exact sequence of G-modules

$$1 \to k^* \longrightarrow R_X^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0\ ,$$

where R_X is the field of rational functions on X. This means that d_2 is the composition

$$\mathrm{H}^1(G, \mathrm{Pic}(X)) \xrightarrow{\partial} \mathrm{H}^2(G, R_X^*/k^*) \xrightarrow{\partial'} \mathrm{H}^3(G, k^*)$$

where ∂ and ∂' are the coboundary maps associated to the short exact sequences

$$0 \to R_X^*/k^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$$

and

$$0 \to k^* \to R_X^* \to R_X^*/k^* \to 0\ .$$

Let $\lambda \in \mathrm{H}^1(G, \mathrm{Pic}(X))$, represented by $L \in \mathrm{Pic}(X)$ with $\perp_{g\in G} g^*L \cong \mathcal{O}_X$. Let $D \in \mathrm{Div}(X)$ such that $L = \mathcal{O}_X(D)$. Then $\sum_g g^*D$ is the divisor of a rational function $\psi \in R_X^*$, whose class in R_X^*/k^* is well-defined. This class is invariant under G, and defines the element $\partial(\lambda) \in \mathrm{H}^2(G, R_X^*/k^*)$. Since div ψ is invariant under G, there exists a character $\chi \in \widehat{G}$ such that $g^*\psi = \chi(g)\psi$ for each $g \in G$. Then $d_2^{1,1}(\lambda) = \chi$ viewed as an element of $\mathrm{H}^3(G, k^*) = \widehat{G}$.

It remains to prove that $\mathcal{O}_S(\pi_*D) = L_\chi$. Since $\mathrm{div}\,(\psi) = \pi^*\pi_*D$, multiplication by ψ induces a global isomorphism $u : \pi^*\mathcal{O}_S(\pi_*D) \xrightarrow{\sim} \mathcal{O}_X$. Let $\varphi \in R_X$ be a generator of $\mathcal{O}_X(D)$ on an open G-invariant subset U of X. Then $\mathrm{Nm}(\varphi)$ is a generator of $\mathcal{O}_S(\pi_*D)$ on $\pi(U)$, and $\pi^*\,\mathrm{Nm}(\varphi)$ is a generator of $\pi^*\mathcal{O}_S(\pi_*D)$ on U; the function $h := \psi\,\pi^*\mathrm{Nm}(\varphi)$ on U satisfies $g^*h = \chi(g)h$ for all $g \in G$. This proves that the homomorphism $u^\flat : \mathcal{O}_S(\pi_*D) \to \pi_*\mathcal{O}_X$ deduced from u maps $\mathcal{O}_S(\pi_*D)$ onto the subsheaf L_χ of $\pi_*\mathcal{O}_X$, hence our assertion. $\square$

We will need a complement of the Proposition in the complex case:

Corollary 4.3. *Assume $k = \mathbb{C}$, and $\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$. The following conditions are equivalent:*

(i) *The map $\pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is not injective;*

(ii) *there exists $L \in \mathrm{Pic}(X)$ whose class $\lambda = c_1(L)$ in $\mathrm{H}^2(X, \mathbb{Z})$ satisfies $\pi_*\lambda = 0$ and $\lambda \notin (1-\sigma^*)(\mathrm{H}^2(X, \mathbb{Z}))$.*

Observe that the hypotheses of the Corollary are satisfied when S is a complex Enriques surfaces and $\pi : X \to S$ its universal cover.

Proof. By Proposition 4.1 (i) is equivalent to the existence of a line bundle L on X with $\mathrm{Nm}(L)=\mathcal{O}_S$ and $[L]\neq 0$ in $\mathrm{H}^1(G,\mathrm{Pic}(X))$, while (ii) means that there exists such L with $[c_1(L)]\neq 0$ in $\mathrm{H}^1(G,\mathrm{H}^2(X,\mathbb{Z}))$. Therefore it suffices to prove that the map

$$\mathrm{H}^1(c_1):\mathrm{H}^1(G,\mathrm{Pic}(X))\to\mathrm{H}^1(G,\mathrm{H}^2(X,\mathbb{Z}))$$

is injective.

Since $\mathrm{H}^1(X,\mathcal{O}_X)=0$ we have an exact sequence

$$0\to\mathrm{Pic}(X)\xrightarrow{c_1}\mathrm{H}^2(X,\mathbb{Z})\longrightarrow Q\to 0\qquad\text{with } Q\subset\mathrm{H}^2(X,\mathcal{O}_X)\ .$$

Since $\mathrm{H}^2(S,\mathcal{O}_S)=0$, there is no nonzero invariant vector in $\mathrm{H}^2(X,\mathcal{O}_X)$, hence in Q. Then the associated long exact sequence implies that $\mathrm{H}^1(c_1)$ is injective. □

5. More algebraic topology

5.1. As in §3, we denote by S a complex Enriques surface, by $\pi:X\to S$ its universal cover and by σ the corresponding involution of X. We will need some more precise results on the topology of the surfaces X and S. We refer again to [BHPV], ch. VIII.

Let E be the lattice $(-E_8)\oplus H$, where H is the rank 2 hyperbolic lattice. Let $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}$ be the quotient of $\mathrm{H}^2(S,\mathbb{Z})$ by its torsion subgroup $\{0,K_S\}$. We have isomorphisms

$$\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}\cong E\qquad\mathrm{H}^2(X,\mathbb{Z})\cong E\oplus E\oplus H$$

such that $\pi^*:\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tf}}\to\mathrm{H}^2(X,\mathbb{Z})$ is identified with the diagonal embedding $\delta: E\hookrightarrow E\oplus E$, and σ^* is identified with the involution

$$\rho:(\alpha,\alpha',\beta)\mapsto(\alpha',\alpha,-\beta)\quad\text{of}\ \ E\oplus E\oplus H\ .$$

5.2. We consider now the cohomology with values in $\mathbb{Z}/2$. For a lattice M, we will write $M_2:=M/2M$. The scalar product of M induces a product $M_2\otimes M_2\to\mathbb{Z}/2$; if moreover M is *even*, there is a natural quadratic form $q:M_2\to\mathbb{Z}/2$ associated with that product, defined by $q(m)=\frac{1}{2}\tilde{m}^2$, where $\tilde{m}\in M$ is any lift of $m\in M_2$. In particular, H_2 contains a unique element ε with $q(\varepsilon)=1$: it is the class of $e+f$ where (e,f) is a hyperbolic basis of H.

Using the previous isomorphism we identify $\mathrm{H}^2(X,\mathbb{Z}/2)$ with $E_2\oplus E_2\oplus H_2$.

Proposition 5.3. *The image of* $\pi^*:\mathrm{H}^2(S,\mathbb{Z}/2)\to\mathrm{H}^2(X,\mathbb{Z}/2)$ *is* $\delta(E_2)\oplus(\mathbb{Z}/2)\varepsilon$.

Proof. This image is invariant under σ^*, hence is contained in $\delta(E_2)\oplus H_2$; by Proposition 3.6 (i) it is 11-dimensional, hence a hyperplane in $\delta(E_2)\oplus H_2$, containing $\delta(E_2)$ (which is spanned by the classes coming from $\mathrm{H}^2(S,\mathbb{Z})$). So $\pi^*\mathrm{H}^2(S,\mathbb{Z}/2)$ is spanned by $\delta(E_2)$ and a nonzero element of H_2; it suffices to prove that this element is ε. Since the elements of $\mathrm{H}^2(S,\mathbb{Z}/2)$ which do not come from $\mathrm{H}^2(S,\mathbb{Z})$ have square 1 (3.2), this is a consequence of the following lemma. □

Lemma 5.4. *For every* $\alpha\in\mathrm{H}^2(S,\mathbb{Z}/2)$*,* $q(\pi^*\alpha)=\alpha^2$.

Proof. This proof has been shown to me by J. Lannes. The key ingredient is the *Pontryagin square*, a cohomological operation

$$\mathcal{P} : \mathrm{H}^{2m}(M, \mathbb{Z}/2) \longrightarrow \mathrm{H}^{4m}(M, \mathbb{Z}/4)$$

defined for any reasonable topological space M and satisfying a number of interesting properties (see [M-T], ch. 2, exerc. 1). We will state only those we need in the case of interest for us, namely $m = 2$ and M is a compact oriented 4-manifold. We identify $\mathrm{H}^4(M, \mathbb{Z}/4)$ with $\mathbb{Z}/4$; then $\mathcal{P} : \mathrm{H}^2(M, \mathbb{Z}) \to \mathbb{Z}/4$ satisfies:

a) For $\alpha \in \mathrm{H}^2(M, \mathbb{Z}/2)$, the class of $\mathcal{P}(\alpha)$ in $\mathbb{Z}/2$ is α^2;

b) If $\alpha \in \mathrm{H}^2(M, \mathbb{Z}/2)$ comes from $\tilde{\alpha} \in \mathrm{H}^2(M, \mathbb{Z})$, then $\mathcal{P}(\alpha) = \tilde{\alpha}^2$ (mod. 4). In particular, if M is a K3 surface, we have $\mathcal{P}(\alpha) = 2q(\alpha)$ in $\mathbb{Z}/4$.

Coming back to our situation, let $\alpha \in \mathrm{H}^2(S, \mathbb{Z}/2)$. We have in $\mathbb{Z}/4$:

$$\begin{aligned} \mathcal{P}(\pi^*\alpha) &= 2\,\mathcal{P}(\alpha) && \text{by functoriality} \\ &= 2\,\alpha^2 && \text{by a), and} \\ \mathcal{P}(\pi^*\alpha) &= 2\,q(\pi^*\alpha) && \text{by b).} \end{aligned}$$

Comparing the two last lines gives the lemma. □

Corollary 5.5. *The kernel of* $\pi_* : H_2 \to \{0, k_S\}$ *is* $\{0, \varepsilon\}$.

Proof. By Proposition 5.3 ε belongs to $\mathrm{Im}\,\pi^*$, hence $\pi_*\varepsilon = 0$. It remains to check that π_* is nonzero on $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{H}^2(X, \mathbb{Z})) \cong H_2$. We know that there is an element $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$ with $\pi_*\alpha = K_S$ (Prop. 3.6 (ii)); it belongs to $\mathrm{Ker}(1 + \sigma^*)$, hence defines an element $\bar{\alpha}$ of $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{H}^2(X, \mathbb{Z}))$ with $\pi_*\bar{\alpha} \neq 0$. □

Corollary 5.6. *Let* $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$. *The following conditions are equivalent:*

(i) $\pi_*\lambda = 0$ *and* $\lambda \notin (1 - \sigma^*)(\mathrm{H}^2(X, \mathbb{Z}))$;

(ii) $\sigma^*\lambda = -\lambda$ *and* $\lambda^2 \equiv 2$ (mod. 4).

Proof. Write $\lambda = (\alpha, \alpha', \beta) \in E \oplus E \oplus H$; let $\bar{\beta}$ be the class of β in H_2. Both conditions imply $\sigma^*\lambda = -\lambda$, hence $\alpha' = -\alpha$. Since $(\alpha, -\alpha) = (1 - \sigma^*)(\alpha, 0)$ and $2\beta = (1 - \sigma^*)(\beta)$, the conditions of (i) are equivalent to $\pi_*\bar{\beta} = 0$ and $\bar{\beta} \neq 0$, that is, $\bar{\beta} = \varepsilon$ (Corollary 5.5). On the other hand we have $\lambda^2 = 2\alpha^2 + \beta^2 \equiv 2q(\bar{\beta})$ (mod. 4), hence (ii) is also equivalent to $\bar{\beta} = \varepsilon$. □

This allows us to rephrase Corollary 4.3 in a simpler way:

Corollary 5.7. *We have* $\pi^* b_S = 0$ *if and only if there exists a line bundle* L *on* X *with* $\sigma^* L = L^{-1}$ *and* $c_1(L)^2 \equiv 2$ (mod. 4). □

Remark.– My original proof of (5.3-5) was less direct and less general, but still perhaps of some interest. The key point is to show that on H_2 q takes the value 1 exactly on the nonzero element of $\mathrm{Ker}\,\pi_*$, or equivalently that an element $\alpha \in H_2$ with $\pi_*\alpha = k_S$ satisfies $q(\alpha) = 0$. Using deformation theory (see (6.1) below), one can assume that α comes from a class in $\mathrm{Pic}(X)$. To conclude I applied the following lemma:

Lemma 5.8. *Let* L *be a line bundle on* X *with* $\mathrm{Nm}(L) = K_S$. *Then* $c_1(L)^2$ *is divisible by* 4.

Proof. Consider the rank 2 vector bundle $E = \pi_*(L)$. The norm induces a non-degenerate quadratic form $N : \mathrm{Sym}^2 E \to K_S$ ([EGA2], 6.5.5). In particular, N induces an isomorphism $E \xrightarrow{\sim} E^* \otimes K_S$, and defines a pairing

$$\mathrm{H}^1(S,E) \otimes \mathrm{H}^1(S,E) \to \mathrm{H}^2(S,K_S) \cong \mathbb{C}$$

which is skew-symmetric and non-degenerate. Thus $h^1(E)$ is even; since $h^0(E) = h^2(E)$ by Serre duality, $\chi(E)$ is even, and so is $\chi(L) = \chi(E)$. By Riemann-Roch this implies that $\frac{1}{2}c_1(L)^2$ is even. □

6. The vanishing of $\pi^* b_S$ on the moduli space

6.1. We briefly recall the theory of the period map for Enriques surfaces, due to Horikawa (see [BHPV], ch. VIII, or [N]). We keep the notations of (5.1). We denote by L the lattice $E \oplus E \oplus H$, and by L^- the (-1)-eigenspace of the involution $\rho : (\alpha, \alpha', \beta) \mapsto (\alpha', \alpha, -\beta)$, that is, the submodule of elements $(\alpha, -\alpha, \beta)$.

A *marking* of the Enriques surface S is an isometry $\varphi : \mathrm{H}^2(X,\mathbb{Z}) \to L$ which conjugates σ^* to ρ. The line $\mathrm{H}^{2,0} \subset \mathrm{H}^2(X,\mathbb{C})$ is anti-invariant under σ^*, so its image by $\varphi_{\mathbb{C}} : \mathrm{H}^2(X,\mathbb{C}) \to L_{\mathbb{C}}$ lies in $L^-_{\mathbb{C}}$. The corresponding point $[\omega]$ of $\mathbb{P}(L^-_{\mathbb{C}})$ is the *period* $\wp(S,\varphi)$. It belongs to the domain $\Omega \subset \mathbb{P}(L^-_{\mathbb{C}})$ defined by the equations

$$(\omega \cdot \omega) = 0 \qquad (\omega \cdot \bar{\omega}) > 0 \qquad (\omega \cdot \lambda) \neq 0 \quad \text{for all } \lambda \in L^- \text{ with } \lambda^2 = -2\ .$$

This is an analytic manifold, which is the moduli space for marked Enriques surfaces. To each class $\lambda \in L^-$ we associate the hypersurface H_λ of Ω defined by $(\lambda \cdot \omega) = 0$.

Proposition 6.2. *We have $\pi^* b_S = 0$ if and only if $\wp(S,\varphi)$ belongs to one of the hypersurfaces H_λ for some vector $\lambda \in L^-$ with $\lambda^2 \equiv 2$ (mod. 4).*

Proof. The period point $\wp(S,\varphi)$ belongs to H_λ if and only if λ belongs to $c_1(\mathrm{Pic}(X))$; by Corollary 5.7, this is equivalent to $\pi^* b_S = 0$. □

To get a complete picture we want to know which of the H_λ are really needed:

Lemma 6.3. *Let λ be a primitive element of L^-.*

(*i*) *The hypersurface H_λ is non-empty if and only if $\lambda^2 < -2$.*

(*ii*) *If μ is another primitive element of L^- with $H_\mu = H_\lambda \neq \varnothing$, then $\mu = \pm\lambda$.*

Proof. Let W be the subset of $L^-_{\mathbb{C}}$ defined by the conditions $\omega^2 = 0$, $\omega \cdot \bar{\omega} > 0$. If we write $\omega = \alpha + i\beta$ with $\alpha, \beta \in L^-_{\mathbb{R}}$, these conditions translate as $\alpha^2 = \beta^2 > 0$, $\alpha \cdot \beta = 0$. Thus $W \cap \lambda^\perp \neq \varnothing$ is equivalent to the existence of a positive 2-plane in $L^-_{\mathbb{R}}$ orthogonal to λ. Since L^- has signature $(2,10)$, this is also equivalent to $\lambda^2 < 0$.

If $W \cap \lambda^\perp$ is non-empty, $\lambda^\perp$ is the only hyperplane containing it, and $\mathbb{C}\,\lambda$ is the orthogonal of $\lambda^\perp$ in L^-. Then λ and $-\lambda$ are the only primitive vectors of L^- contained in $\mathbb{C}\,\lambda$. In particular λ is determined up to sign by H_λ, which proves (*ii*).

Let us prove (*i*). We have seen that H_λ is empty for $\lambda^2 \geq 0$, and also for $\lambda^2 = -2$ by definition of Ω. Assume $\lambda^2 < -2$ and $H_\lambda = \varnothing$; then H_λ must be contained in one of the hyperplanes H_μ with $\mu^2 = -2$; by (*ii*) this implies $\lambda = \pm\mu$, a contradiction. □

6.4. Let Γ be the group of isometries of L^-. The group Γ acts properly discontinuously on Ω, and the quotient $\mathcal{M} = \Omega/\Gamma$ is a quasi-projective variety. The image in $\mathcal{M}$ of the period $\wp(S,\varphi)$ does not depend on the choice of φ; let us denote it by $\wp(S)$. The map $S \mapsto \wp(S)$ induces a bijection between isomorphism classes of Enriques surfaces and $\mathcal{M}$; the variety $\mathcal{M}$ is a (coarse) moduli space for Enriques surfaces.

Corollary 6.5. *The surfaces S for which $\pi^* b_S = 0$ form an infinite, countable union of (non-empty) hypersurfaces in the moduli space $\mathcal{M}$.*

Proof. Let Λ be the set of primitive elements λ in L^- with $\lambda^2 < -2$ and $\lambda^2 \equiv 2 \pmod 4$. For $\lambda \in \Lambda$, let $\mathcal{H}_\lambda$ be the image of H_λ in $\mathcal{M}$; the argument of [BHPV], ch. VIII, Cor. 20.7 shows that $\mathcal{H}_\lambda$ is an algebraic hypersurface in $\mathcal{M}$. By Proposition 6.2 and Lemma 6.3 the surfaces S with $\pi^*(b_S) = 0$ form the subset $\bigcup_{\lambda\in\Lambda} \mathcal{H}_\lambda$. By Lemma 6.3 (*ii*) we have $\mathcal{H}_\lambda = \mathcal{H}_\mu$ if and only if $\mu = \pm g\lambda$ for some element g of Γ. This implies $\lambda^2 = \mu^2$; but λ^2 can be any number of the form $-2k$ with k odd > 1 (take for instance $\lambda = e - kf$, where (e, f) is a hyperbolic basis of H), so there are infinitely many distinct hypersurfaces among the $\mathcal{H}_\lambda$. $\square$

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