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ON THE DECAY OF SOLUTIONS TO A CLASS OF DEFOCUSING NLS

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Abstract. We consider the following family of Cauchy problems:

$$i\partial_t u = \Delta u - u|u|^\alpha, (t,x) \in \mathbb{R} \times \mathbb{R}^d$$
$$u(0) = \varphi \in H^1(\mathbb{R}^d)$$

where $0 < \alpha < \frac{4}{d-2}$ for $d \geq 3$ and $0 < \alpha < \infty$ for $d = 1, 2$. We prove that the L^r-norms of the solutions decay as $t \to \pm\infty$, provided that $2 < r < \frac{2d}{d-2}$ when $d \geq 3$ and $2 < r < \infty$ when $d = 1, 2$. In particular we extend previous results obtained in [5] for $d \geq 3$ and in [8] for $d = 1, 2$, where the same decay results are proved under the extra assumption $\alpha > \frac{4}{d}$.

This paper is devoted to the analysis of some asymptotic properties of solutions to the following family of defocusing NLS:

$$i\partial_t u = \Delta u - u|u|^\alpha, (t,x) \in \mathbb{R} \times \mathbb{R}^d \tag{0.1}$$
$$u(0) = \varphi \in H^1(\mathbb{R}^d)$$

where

$$0 < \alpha < \frac{4}{d-2} \text{ for } d \geq 3 \text{ and } 0 < \alpha < \infty \text{ for } d = 1, 2. \tag{0.2}$$

A lot of attention has been devoted in the literature to the Cauchy problem (0.1). In particular the questions of local and global well-posedness and scattering theory have been extensively studied. There exists an huge literature on the field and for this reason we cannot be exhaustive in the bibliography, however for the moment we would like to quote the book [1] for an extended description of the topics mentioned above and also for an extended bibliography.

It is well–known from [4] (see also [6] for the more general question of unconditional uniqueness) that, under the assumptions (0.2) on α, for every initial data $\varphi \in H^1(\mathbb{R}^d)$ there exists a unique global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (0.1). Moreover the global solutions of (0.1) satisfy the following conservation laws:

$$\|u(t,x)\|_{L^2(\mathbb{R}^d)} \equiv const \ \forall t \in \mathbb{R} \tag{0.3}$$

and

$$\frac{1}{2}\|\nabla_x u(t,x)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\alpha+2}\|u(t,x)\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^d)} \equiv const \ \forall t \in \mathbb{R}. \tag{0.4}$$

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The main contribution of this paper concerns the decay, in suitable Lebesgue spaces, of the global solutions to (0.1) as $t \to \pm\infty$.

Theorem 0.1. *Assume α as in* (0.2). *Let $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ be the unique global solution to*

$$i\partial_t u = \Delta u - u|u|^\alpha, (t,x) \in \mathbb{R} \times \mathbb{R}^d$$
$$u(0) = \varphi \in H^1(\mathbb{R}^d).$$

Then for every $2 < r < \frac{2d}{d-2}$ when $d \geq 3$ and for every $2 < r < \infty$ when $d = 1, 2$, we have:

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^r(\mathbb{R}^d)} = 0. \tag{0.5}$$

Moreover in the case $d = 1$ we also have:

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^\infty(\mathbb{R})} = 0. \tag{0.6}$$

Remark 0.1. The proof of (0.6) follows easily by combining the conservation law (0.4), (0.5) and the following Gagliardo-Nirenberg inequality:

$$\|u(t,x)\|_{L^\infty(\mathbb{R})} \leq C\|\partial_x u(t,x)\|^{\frac{1}{4}}_{L^2(\mathbb{R})}\|u(t,x)\|^{\frac{3}{4}}_{L^6(\mathbb{R})}.$$

Hence we shall focus in the sequel on the proof of (0.5).

Remark 0.2. Let us underline that the original proof of Theorem 0.1 in the case $\frac{4}{d} < \alpha < \frac{4}{d-2}$ and $d \geq 3$ is given in [5]. In fact this is the basic step on which the scattering theory in the energy space $H^1(\mathbb{R}^n)$ is based. More precisely once the decay of some L^r norm for solutions to (0.1) is known and $\frac{4}{d} < \alpha < \frac{4}{d-2}$, then the estimates in Strichartz spaces follow almost immediately, and in turn this implies easily the asymptotic completeness. Hence the main novelty in our result is that we prove dispersion of solution to NLS also in the case $0 < \alpha \leq \frac{4}{d}$.
In the case $d = 1, 2$ and $\frac{4}{n} < \alpha < \infty$ the content of Theorem 0.1 can be deduced from [8]. However we point out that also in dimensions $d = 1, 2$ Theorem 0.1 covers the range $0 < \alpha \leq \frac{4}{d}$.

Remark 0.3. Theorem 0.1 could be proved by the conformal conservation law provided that the initial data φ belongs to suitable weighted L^2 spaces. In fact in this case it can be deduced also a decay rate of the solution (see Theorem 7.3.1 in [1]). However we emphasize that in general the decay of solutions to (0.1) with initial data in $H^1(\mathbb{R}^d)$ and $0 < \alpha \leq \frac{4}{n}$ was a completelely open question.

Remark 0.4. The proof of Theorem 0.1 follows by a combination of Strichartz estimates with the Interaction Morawetz Estimates (see [3] for the dimension $d \geq 3$ and [2], [9] for the dimensions $d = 1, 2$). In particular, as a consequence of Theorem 0.1, we provide a new proof of the scattering results in [5] and [8].

Remark 0.5. Actually one can show a slightly stronger version of Theorem 0.1. More precisely in the case $d \geq 3$ and $\frac{4}{d} < \alpha < \frac{4}{d-2}$ we also have

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = 0. \tag{0.7}$$

The proof of this fact goes as follows. Once (0.5) is proved and α is as above, then by using classical arguments (see [1]), we can construct the scattering operator in the energy space. More precisely given any $\varphi \in H^1(\mathbb{R}^d)$ there exist $\varphi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t,x) - e^{it\Delta}\varphi_\pm\|_{H^1(\mathbb{R}^n)} = 0, \tag{0.8}$$

where $u(t,x)$ denotes the corresponding solution to (0.1). Hence we get

$$\begin{aligned} \|u(t,x)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} &\leq \|u(t,x) - e^{it\Delta}\varphi_\pm\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \|e^{it\Delta}\varphi_\pm\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \\ &\leq C\|u(t,x) - e^{it\Delta}\varphi_\pm\|_{H^1(\mathbb{R}^d)} + \|e^{it\Delta}\varphi_\pm\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}, \end{aligned} \tag{0.9}$$

where we have used the Sobolev embedding at the last step. On the other by combining the dispersive estimate

$$\|e^{it\Delta}\psi\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq \frac{C}{t}\|\psi\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)},$$

with the Sobolev embedding and with a density argument, we can deduce easily that

$$\lim_{t\to\pm\infty} \|e^{it\Delta}\psi\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = 0 \ \forall \psi \in H^1(\mathbb{R}^d). \tag{0.10}$$

Hence by combining (0.8), (0.9) and (0.10) we get (0.7).

Remark 0.6. Arguing as in remark 0.5 and by using the embedding $L^\infty(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$, we can deduce:

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{BMO(\mathbb{R}^2)} = 0,$$

provided that $u(t,x)$ solves (0.1) with $d = 2$ and $\alpha > 2$.

Along this paper a fundamental role will be played by the Strichartz estimates for the propagator $e^{it\Delta}$, hence for the sake of completeness they will be stated below.
First we need to introduce some notations that will be useful in the sequel.

For any subinterval $I \equiv (0,T)$ of $\mathbb{R}$ and for every $p, q \in [0,\infty]$ we define the mixed space-time norms

$$\|u\|_{L^p_T L^q_x} \equiv \left(\int_I \|u(t,\cdot)\|^p_{L^q(\mathbb{R}^n)} dt\right)^{1/p}. \tag{0.11}$$

In the case $I \equiv \mathbb{R}$ we write $\|u\|_{L^p_t L^q_x}$.
We shall also use the notation $\|\varphi\|_{L^r_x} \equiv \|\varphi\|_{L^r(\mathbb{R}^d)}$ for every $1 \leq r \leq \infty$ and $\|\varphi\|_{H^1_x} \equiv \|\varphi\|_{H^1(\mathbb{R}^d)}$.
Given $d \geq 1$ we say that the pair (p,q) is *d-(Schrödinger) admissible* if

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad p, q \in [2,\infty], \qquad (d,p,q) \neq (2,2,\infty). \tag{0.12}$$

We can now state the Strichartz estimates for the free propagator. For any *d-(Schrödinger) admissible* couples (p,q) and $(\tilde{p},\tilde{q})$ there exists a constant $C(p,\tilde{p})$ such that, for all $T > 0$, for all functions $u_0 \in L^2_x$, and $F(t,x) \in L^{\tilde{p}'}_T L^{\tilde{q}'}_x$ the following inequalities hold:

$$\left\|e^{it\Delta}u_0\right\|_{L^p_T L^q_x} \leq C(p,\tilde{p})\,\|u_0\|_{L^2_x} \tag{0.13}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{L_T^p L_x^q} \leq C(p, \tilde{p}) \, \|F\|_{L_T^{\tilde{p}'} L_x^{\tilde{q}'}}, \tag{0.14}$$

where we have used a prime to denote conjugate indices. Note that the constant in the previous estimates are independent of the interval $T > 0$. For a proof of the Strichartz estimates in the non end–point case see [1], for the general case see [7].

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1. Proof of Theorem 0.1

We shall need the following

Lemma 1.1. *Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a cut–off function and $\varphi_n \in H_x^1$ be a sequence such that $M \equiv \sup_{n\in\mathbb{N}} \|\varphi_n\|_{H_x^1} < \infty$ and $\varphi_n \rightharpoonup \tilde{\varphi}$ in H_x^1. Let $u_n(t,x), \tilde{u}(t,x) \in \mathcal{C}(\mathbb{R}; H_x^1)$ be the corresponding solutions to* (0.1) *with initial data φ_n and $\tilde{\varphi}$ respectively. Then for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ and $\nu(\epsilon) \in \mathbb{N}$ such that*

$$\sup_{t\in(0,T(\epsilon))} \|\chi(x)(u_n(t,x) - \tilde{u}(t,x))\|_{L_x^2} \leq \epsilon \ \forall n > \nu(\epsilon).$$

Proof. By combining (0.3) and (0.4) it is easy to deduce that

$$\sup_{n\in\mathbb{N}, t\in\mathbb{R}} \{\|u_n(t,x)\|_{H_x^1}, \|\tilde{u}(t,x)\|_{H_x^1}\} < \infty. \tag{1.1}$$

By using the Rellich compactness theorem we have

$$\lim_{n\to\infty} \|\chi(x)(\varphi_n - \tilde{\varphi})\|_{L_x^2} = 0. \tag{1.2}$$

Next we introduce the functions

$$v_n(t,x) \equiv \chi(x) u_n(t,x) \text{ and } \tilde{v}(t,x) \equiv \chi(x)\tilde{u}(t,x)$$

that solve the following Cauchy problems:

$$i\partial_t v_n = \Delta v_n - 2\nabla\chi \cdot \nabla u_n - u_n \Delta\chi - \chi u_n |u_n|^\alpha$$
$$v_n(0) = \chi(x)\varphi_n$$

and

$$i\partial_t \tilde{v} = \Delta\tilde{v} - 2\nabla\chi \cdot \nabla\tilde{u} - \tilde{u}\Delta\chi - \chi\tilde{u}|\tilde{u}|^\alpha$$
$$\tilde{v}(0) = \chi(x)\tilde{\varphi}.$$

By using the integral formulation of the previous Cauchy problems we deduce:

$$\begin{aligned} v_n(t,x) - \tilde{v}(t,x) &= e^{it\Delta}[\chi(x)(\varphi_n - \tilde{\varphi})] \\ &+ i\int_0^t e^{i(t-s)\Delta}[2\nabla\chi \cdot \nabla(u_n(s) - \tilde{u}(s)) + (u_n(s) - \tilde{u}(s))\Delta\chi]dx \\ &+ i\int_0^t e^{i(t-s)\Delta}[\chi(x)(u_n(s)|u_n(s)|^\alpha - \tilde{u}(s)|\tilde{u}(s)|^\alpha)]ds. \end{aligned} \tag{1.3}$$

Next we split the proof in two cases.

First case: $d \geq 3$

We fix the following *d-(Schrödinger) admissible* couple

$$(p(d), q(d)) \equiv \left(\frac{8}{\alpha(d-2)}, \frac{4d}{2d - \alpha d + 2\alpha}\right)$$

and by using the estimates (0.13) and (0.14) (where we make the choices $(\tilde{p}, \tilde{q}) = (p(d), q(d))$ and $(\tilde{p}, \tilde{q}) = (\infty, 2)$) we get:

$$\|v_n - \tilde{v}\|_{L_T^{p(d)} L_x^{q(d)}} \leq C\|\chi(x)(\varphi_n - \tilde{\varphi})\|_{L_x^2} + C\|\nabla\chi \cdot \nabla(u_n - \tilde{u})\|_{L_T^1 L_x^2}$$
$$+ C\|(u_n - \tilde{u})\Delta\chi\|_{L_T^1 L_x^2} + C\|\chi(u_n|u_n|^\alpha - \tilde{u}|\tilde{u}|)^\alpha\|_{L_T^{p(d)'} L_x^{q(d)'}},$$

that in conjunction with (1.1) and with the Hölder inequality implies:

$$\|v_n - \tilde{v}\|_{L_T^{p(d)} L_x^{q(d)}} \leq C\|\chi(x)(\varphi_n - \tilde{\varphi})\|_{L_x^2} + CT$$
$$+ C\|\chi(x)(u_n - \tilde{u})\|_{L_T^{p(d)'} L_x^{q(d)}} \sup_{t \in (0,T)} \left(\|u_n(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}} + \|\tilde{u}(t)\|^\alpha_{L_x^{\frac{2d}{d-2}}}\right).$$

By using now the Sobolev embedding $H_x^1 \subset L_x^{\frac{2d}{d-2}}$, (1.1) and the Hölder inequality in the time variable, we get:

$$\|v_n - \tilde{v}\|_{L_T^{p(d)} L_x^{q(d)}} \leq C\|\chi(x)(\varphi_n - \tilde{\varphi})\|_{L_x^2} + CT$$
$$+ CT^{\frac{p(d)-2}{p(d)}} \|v_n - \tilde{v}\|_{L_T^{p(d)} L_x^{q(d)}}.$$

By combining this estimate with (1.2) we deduce that for every $\epsilon > 0$ there exist $\nu(\epsilon)$ and $T(\epsilon)$ such that

$$\|v_n - \tilde{v}\|_{L_{T(\epsilon)}^{p(d)} L_x^{q(d)}} \leq \epsilon \ \forall n > \nu(\epsilon). \tag{1.4}$$

Next we consider again (1.3) and we use again the Strichartz estimates with the choice $(p, q) = (\infty, 2)$ and $(\tilde{p}, \tilde{q})$ as above, and arguing as above we deduce:

$$\|v_n - \tilde{v}\|_{L_{T(\epsilon)}^\infty L_x^2} \leq C\|\chi(x)(\varphi_n - \tilde{\varphi})\|_{L_x^2} + CT(\epsilon)$$
$$+ CT(\epsilon)^{\frac{p(d)-2}{p(d)}} \|v_n - \tilde{v}\|_{L_{T(\epsilon)}^{p(d)} L_x^{q(d)}}.$$

By combining this estimate with (1.2) and (1.4) we deduce the desired result.

Second case: $d = 1, 2$

The proof is similar to the case $d \geq 3$ provided that we make respectively the following choice of 1-*(Schrödinger) admissible* and 2-*(Schrödinger) admissible* couples:

$$(p(1), q(1)) = (\infty, 2) \text{ and } (p(2), q(2)) = (4, 4).$$

$\square$

Proof of Thm 0.1 We shall prove (0.5) for $t \to \infty$ (the case $t \to -\infty$ can be treated in a similar way). We split the proof in two cases.

First case: $d \geq 3$

Notice that by combining (0.3) and (0.4) with the Hölder inequality, it is enough to prove (0.5) for $r = \frac{2d+4}{d}$. Next we recall the following consequence of the Gagliardo–Nirenberg inequality for $d \geq 3$:

$$\|\psi\|_{L_x^{\frac{2d+4}{d}}}^{\frac{2d+4}{d}} \leq C \left(\sup_{x \in \mathbb{R}^d} \|\psi\|_{L^2(Q_1(x))} \right)^{\frac{4}{d}} \|\psi\|_{H_x^1}^2, \tag{1.5}$$

where $Q_r(x)$ denote the cube in $\mathbb{R}^d$ centered in x whose edge has lenght r. Moreover as a consequence of (0.3) and (0.4) we get

$$\sup_{t \in \mathbb{R}} \|u(t,x)\|_{H_x^1} < \infty. \tag{1.6}$$

Next we assume by the absurd that there is a sequence $t_n \to \infty$ such that

$$\|u(t_n,x)\|_{L_x^{\frac{2d+4}{d}}} \geq \epsilon_0 > 0.$$

Then by combining (1.6) with (1.5), where we choose $\psi \equiv u(t_n,x)$, we deduce the existence of a sequence $x_n \in \mathbb{R}^d$ such that

$$\|u(t_n,x)\|_{L^2(Q_1(x_n))} \geq \epsilon_1 > 0. \tag{1.7}$$

Next we introduce the functions

$$\varphi_n(x) \equiv u(t_n, x + x_n). \tag{1.8}$$

which are bounded in H_x^1 by (1.6). By combining the compactness of the Sobolev embedding on bounded set with (1.7), we have that up to subsequence φ_n converges weakly in H_x^1 to a nontrivial function $\tilde{\varphi} \in H_x^1$ such that

$$\|\tilde{\varphi}\|_{L^2(Q_1(0))} \geq \epsilon_1 > 0. \tag{1.9}$$

We are now in condition to apply Lemma 1.1 where we choose the function $\chi(x)$ as any cut–off function supported in $Q_2(0)$ and such that $\chi(x) \equiv 1$ on the cube $Q_1(0)$. We also introduce the functions $u_n(t,x)$ and $\tilde{u}(t,x)$ as the solutions to:

$$i\partial_t u_n = \Delta u_n - u_n|u_n|^\alpha, (t,x) \in \mathbb{R} \times \mathbb{R}^d$$
$$u_n(0) = \varphi_n \in H_x^1$$

and

$$i\partial_t \tilde{u} = \Delta \tilde{u} - \tilde{u}|\tilde{u}|^\alpha, (t,x) \in \mathbb{R} \times \mathbb{R}^d$$
$$\tilde{u}(0) = \tilde{\varphi} \in H_x^1.$$

Notice that by combining (1.9) with a continuity argument we deduce the existence of $\bar{T} > 0$ such that

$$\inf_{t \in (0,\bar{T})} \|\chi(x)\tilde{u}(t,x)\|_{L_x^2} \geq \frac{\epsilon_1}{2} > 0. \tag{1.10}$$

By Lemma 1.1 there exist $\tilde{T} > 0$ and $\nu \in \mathbb{N}$ such that

$$\sup_{t \in (0,\tilde{T})} \|\chi(x)(u_n(t) - \tilde{u}(t))\|_{L_x^2} \leq \frac{\epsilon_1}{4} \ \forall n > \nu, \tag{1.11}$$

and by combining (1.10) and (1.11) we get

$$\|\chi(x)u_n(t,x)\|_{L_x^2} \geq -\|\chi(x)(u_n(t,x) - \tilde{u}(t,x))\|_{L_x^2} + \|\chi(x)\tilde{u}(t,x)\|_{L_x^2} \geq \frac{\epsilon_1}{4} \tag{1.12}$$
$$\forall t < T_0 \equiv \min\{\bar{T}, \tilde{T}\}, \forall n > \nu.$$

Notice that, due to the properties of $\chi(x)$, (1.12) implies

$$\|u_n(t,x)\|_{L^2(Q_2(0))} \geq \frac{\epsilon_1}{4} \ \forall t < T_0, \forall n > \nu. \tag{1.13}$$

On the other hand by the translation invariance of NLS and due to the definition on φ_n (see (1.8)), it is easy to deduce that the previous estimate is equivalent to the following one:

$$\|u(t,x)\|_{L^2(Q_2(x_n))} \geq \frac{\epsilon_1}{4} \ \forall t \in (t_n, t_n + T_0), \forall n > \nu. \tag{1.14}$$

Next we show that (1.14) lead to a contradiction and it will complete the proof of (0.5) when $d \geq 3$.

Choose $a(x) \equiv |x|$ in the inequality written at page 10 in [3]. Then this implies

$$\int_{\mathbb{R}} \int \int_{\mathbb{R}^d_x \times \mathbb{R}^d_y} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} dxdydt < \infty \text{ for } d \geq 4 \tag{1.15}$$

and

$$\|u(t,x)\|_{L^4_{t,x}} < \infty \text{ for } d = 3. \tag{1.16}$$

Notice that since $t_n \to \infty$ we can assume up to subsequence that the sets $(t_n, t_n + T_0)$ are disjoint sets. Next we use (1.14) and we get

$$\int_{\mathbb{R}} \int \int_{\mathbb{R}^d_x \times \mathbb{R}^d_y} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} dtdxdy$$

$$\geq C \sum_{n \in N} \int_{t_n}^{t_n + T_0} \int \int_{Q_2(x_n) \times Q_2(x_n)} |u(t,x)|^2 |u(t,y)|^2 | dtdxdy = \infty$$

and this is in contradiction with (1.15). This complete the proof of (0.5) for $d \geq 4$. Similarly for $d = 3$ we deduce by (1.14) that

$$\|u(t,x)\|^4_{L^4(Q_2(x_n) \times (t_n, t_n + T_0))} \geq C \epsilon_1^4 T_0$$

and also in this case we get easily a contradiction with (1.16).

Second case: $d = 1, 2$

As in the previous case it is sufficient to prove (0.5) for $r = 3$. In order to do that we shall need the following version of (1.5) in dimensions $d = 1, 2$:

$$\|\psi\|^3_{L^3_x} \leq C \left(\sup_{x \in \mathbb{R}^d} \|\psi\|_{L^2(Q_1(x))} \right) \|\psi\|^2_{H^1_x}.$$

Arguing as in the previous case we can deduce that if (0.5) is false with $r = 3$, then there exist $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\|u(t,x)\|_{L^2(Q_2(x_n))} \geq \epsilon_2 > 0 \ \forall t \in (t_n, t_n + T_0). \tag{1.17}$$

It is now easy to deduce (arguing as we did above in the case $d = 3$) that this lead to a contradiction with the following a–priori bounds proved in [2] and [9]:

$$\|u\|_{L^4_t L^8_x} < \infty \text{ for } d = 2 \text{ and } \|u\|_{L^{\alpha+4}_{t,x}} < \infty \text{ for } d = 1,$$

(more precisely the estimates above follow for instance from Theorems 1.1,1.2 in [2]). Hence the proof of (0.5) is complete also for $d = 1, 2$.

$\square$

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