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SYMMETRY FOR A DIRICHLET-NEUMANN PROBLEM ARISING IN WATER WAVES

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Abstract. Given a smooth $u : \mathbb{R}^n \to \mathbb{R}$, say $u = u(y)$, we consider $\overline{u} = \overline{u}(x, y)$ to be a solution of

$$\begin{cases} \Delta \overline{u} = 0 & \text{for any } (x,y) \in (0,1) \times \mathbb{R}^n, \\ \overline{u}(0,y) = u(y) & \text{for any } y \in \mathbb{R}^n, \\ \overline{u}_x(1,y) = 0 & \text{for any } y \in \mathbb{R}^n. \end{cases}$$

We define the Dirichlet-Neumann operator $(\mathscr{L}u)(y) = \overline{u}_x(0, y)$ and we prove a symmetry result for equations of the form $(\mathscr{L}u)(y) = f(u(y))$.

In particular, bounded, monotone solutions in $\mathbb{R}^2$ are proven to depend only on one Euclidean variable.

Introduction

The aim of this paper is to provide a symmetry result for a Dirichlet-Neumann problem.

Our set up is the following. We consider the slab $[0, 1] \times \mathbb{R}^n$, endowed with coordinates $x \in [0, 1]$ and $y \in \mathbb{R}^n$.

We define the operator $\mathscr{L}$ as follows. Given a smooth u, which will be taken to be bounded together with its derivatives, we define $\overline{u}(x, y) \in C^2((0, 1)\times\mathbb{R}^n)\cap C^1([0, 1]\times \mathbb{R}^n)$ to be the solution of

$$\begin{cases} \Delta \overline{u} = 0 & \text{in } (0,1) \times \mathbb{R}^n, \\ \overline{u}(0,y) = u(y), \\ \overline{u}_x(1,y) = 0. \end{cases} \tag{1}$$

As customary, the subscript denotes the partial derivative and $\Delta\overline{u} = \overline{u}_{xx} + \overline{u}_{y_1y_1} + \cdots + \overline{u}_{y_ny_n}$ is the Laplace operator. The problem in (1) is well-posed and it possesses nice regularity properties, due to the elliptic PDE theory (see, e.g., Theorems 6.6 and 6.26 in [23]). Then, we define

$$(\mathscr{L}u)(y) = \overline{u}_x(0, y).$$

The linear operator $\mathscr{L}$ may also be written in the harmonic analysis setting. That is, if $\mathscr{F}$ denotes the Fourier transform in the y variables (and the transformed frequency variables are called $\xi \in \mathbb{R}^n$), we have that

$$\mathscr{L}u = \mathscr{F}^{-1}\Big(|\xi| \frac{e^{-|\xi|} - e^{|\xi|}}{e^{-|\xi|} + e^{|\xi|}} (\mathscr{F}u)(\xi) \Big), \tag{2}$$

up to a normalization factor.

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From (2), we may say that the symbol of the operator $\mathscr{L}$ in Fourier space is

$$|\xi|\frac{e^{-|\xi|}-e^{|\xi|}}{e^{-|\xi|}+e^{|\xi|}}. \tag{3}$$

Though Fourier analysis will not explicitly play much of a role in this paper, it is convenient to keep in mind that, for large frequencies ξ, (3) is asymptotic to the symbol of the square root of the Laplacian.

The operator $\mathscr{L}$ arises in the theory of water waves of irrotational, incompressible, inviscid fluids in the small amplitude, long wave regime [35, 38, 37, 12, 9, 28, 13, 15, 11, 10, 22, 25, 29].

Related nonlocal operators are studied in flame propagation and semipermeable membranes [3], in optimization [17], in relation with the ultrarelativistic limit of quantum mechanics [20], in the theory of quasi-geostrophic flows [27, 7] in inverse spectral and multiple scattering problems [16, 6, 24] and in the thin obstacle problem [4].

Of course, these operators are also a classical topic in harmonic analysis and in singular integral theory [26, 32].

The main result that we prove is the following:

Theorem 1. *Let $f \in C^1(\mathbb{R})$.*

Let u be a bounded solution of $(\mathscr{L}u)(y) = f(u(y))$ for any $y \in \mathbb{R}^n$.

Suppose that

$$u_{y_n}(y) > 0 \text{ for any } y \in \mathbb{R}^n \tag{4}$$

and that there exists $C > 0$ such that

$$\int_{x\in[0,1]}\int_{|y|\le\tau} |\nabla_y \overline{u}(x,y)|^2\,dy\,dx \le C\tau^2 \tag{5}$$

for any $\tau \ge C$.

Then, there exist $u_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in \mathrm{S}^{n-1}$ such that

$$u(y) = u_o(\omega\cdot y) \qquad \text{for any } y \in \mathbb{R}^n. \tag{6}$$

We remark that (6) states that u depends only on one Euclidean variable up to rotation (equivalently, u is constant in the directions orthogonal to ω). In this sense, Theorem 1 is inspired by a celebrated conjecture for monotone, entire solutions of elliptic PDEs in [14].

In particular, as a consequence of Theorem 1, we obtain the following result for $n = 2$:

Corollary 2. *Let $f \in C^1(\mathbb{R})$.*

Let u be a bounded solution of $(\mathscr{L}u)(y) = f(u(y))$ for any $y \in \mathbb{R}^2$, such that

$$u_{y_2}(y) > 0 \text{ for any } y \in \mathbb{R}^2.$$

Then, there exist $u_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in \mathrm{S}^1$ such that

$$u(y) = u_o(\omega\cdot y) \qquad \text{for any } y \in \mathbb{R}^2.$$

The analogy between the result in Corollary 2 and the conjecture for entire, monotone, bounded solutions of semilinear elliptic PDEs in [14] is manifest. We would like to mention that [8] presents rigidity results for nonnegative, localized solitary waves and [36] contains symmetry results for different fluid dynamics problems also inspired by [14].

The proofs of the above results are suitable modifications of the work done in [31] and they are based on a geometric inequality (namely (25) below) which may be seen as an extension of a similar one obtained, in a different setting, by [33, 34].

The idea of using geometric inequalities to derive symmetry results was also used in [18, 19].

We would also like to recall that the first symmetry result for boundary reaction PDEs was obtained, with different methods, in [2] for the halfspace (such setting as a fractional operator, corresponds to the square root of the Laplacian). For related results, see also [30, 5].

Below are the details of the proofs of Theorem 1 and Corollary 2.

Proofs of the main results

In order to prove Theorem 1, we need some preliminary observations:

Lemma 3 (Weak form of the equation)**.** *Let $\overline{u}$ be a solution of* (1).

Then, for any $\phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$-\int_{\{0\}\times\mathbb{R}^n} \phi(\mathscr{L}u) = \int_{[0,1]\times\mathbb{R}^n} \nabla\phi\cdot\nabla\overline{u}. \tag{7}$$

Proof. Given $\phi \in C_0^\infty(\mathbb{R}^{n+1})$, we denote by $\mathscr{D}_\phi$ the intersection between a ball containing the support of ϕ and $[0,1]\times\mathbb{R}^n$. We also denote by ν the exterior normal of $\partial\mathscr{D}_\phi$, which is well-defined almost everywhere.

Then, we have

$$\begin{aligned} 0 &= \int_{[0,1]\times\mathbb{R}^n} \Delta\overline{u}\phi = \int_{\mathscr{D}_\phi} \big(\operatorname{div}(\phi\nabla\overline{u}) - \nabla\phi\cdot\nabla\overline{u}\big) \\ &= \int_{\partial\mathscr{D}_\phi} \phi\nabla\overline{u}\cdot\nu - \int_{\mathscr{D}_\phi} \nabla\phi\cdot\nabla\overline{u} \\ &= -\int_{\{0\}\times\mathbb{R}^n} \phi(\mathscr{L}u) - \int_{[0,1]\times\mathbb{R}^n} \nabla\phi\cdot\nabla\overline{u}. \qquad \square \end{aligned}$$

Lemma 4 (Weak form of the linearized equation)**.** *Let $f \in C^1(\mathbb{R})$ and let u be a solution of $(\mathscr{L}u)(y) = f(u(y))$ for any $y \in \mathbb{R}^n$.*

Assume that $u(y) = \overline{u}(0,y)$, with $\overline{u}$ as in (1).

Given $i = 1,\dots,n$, we have that

$$-\int_{\{0\}\times\mathbb{R}^n} \psi f'(u)u_{y_i} = \int_{[0,1]\times\mathbb{R}^n} \nabla\psi\cdot\nabla\overline{u}_{y_i} \tag{8}$$

for any $\psi \in C_0^\infty(\mathbb{R}^{n+1})$.

Proof. We start with an elementary observation about the integration by parts formula: if $\Psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $\zeta \in C^1((0,1)\times\mathbb{R}^n)$, then, for any fixed $x \in (0,1)$, the map $y \mapsto \Psi(x,y)$ belongs to $C_0^\infty(\mathbb{R}^n)$ and therefore

$$-\int_{\mathbb{R}^n} \Psi(x,y)\zeta_{y_i}(x,y)\,dy = \int_{\mathbb{R}^n} \Psi_{y_i}(x,y)\zeta(x,y)\,dy.$$

Therefore, integrating in x, we obtain

$$-\int_{[0,1]\times\mathbb{R}^n} \Psi\zeta_{y_i} = \int_{[0,1]\times\mathbb{R}^n} \Psi_{y_i}\zeta. \tag{9}$$

Now, we take $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $\phi = \psi_{y_i}$ in (7), we use (9) and we conclude that

$$\begin{aligned} -\int_{\{0\}\times\mathbb{R}^n} \psi f'(u)u_{y_i} &= -\int_{\{0\}\times\mathbb{R}^n} \psi\big(f(u)\big)_{y_i} = \int_{\{0\}\times\mathbb{R}^n} \psi_{y_i} f(u) \\ &= -\int_{[0,1]\times\mathbb{R}^n} \nabla\psi_{y_i}\cdot\nabla\overline{u} = \int_{[0,1]\times\mathbb{R}^n} \nabla\psi\cdot\nabla\overline{u}_{y_i}. \end{aligned}$$ □

Lemma 5 (Sign property)**.** *Let $v \in C^2((0,1)\times\mathbb{R}^n)\cap C^1([0,1]\times\mathbb{R}^n)$, with finite $\|v(0,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n)}$, satisfy*

$$\begin{cases} \Delta v = 0 & \text{in } (0,1)\times\mathbb{R}^n, \\ v_x(1,y) = 0. \end{cases} \tag{10}$$

If $v(0,y) > 0$ for any $y \in \mathbb{R}^n$, then $v(x,y) > 0$ for any $x \in [0,1)$ and any $y \in \mathbb{R}^n$.

Proof. By the strong maximum principle, it is enough to show that $v \geq 0$ in $(0,1)\times\mathbb{R}^n$.

Thus, we argue by contradiction and we suppose that $v(\bar{x},\bar{y}) < 0$ for some $(\bar{x},\bar{y}) \in (0,1)\times\mathbb{R}^n$.

Hence, by the maximum principle,

$$\inf_{(x,y)\in(0,1)\times\mathbb{R}^n} v(x,y) = \inf_{y\in\mathbb{R}^n} v(1,y) < 0.$$

Therefore, we take a sequence y_j such that

$$\lim_{j\to+\infty} v(1,y_j) = \inf_{y\in\mathbb{R}^n} v(1,y) < 0.$$

We define

$$v_j(x,y) = v(x,y_j+y).$$

By elliptic regularity [23], up to an even reflection across $\{x = 1\}$, we have that $\|v\|_{C^{2,\beta}((0,1)\times\mathbb{R}^n)}$ is bounded, for some $\beta \in (0,1)$. So, up to subsequences v_j converges locally uniformly to some w, together with its first two derivatives.

Thus, (10) gives that

$$\begin{cases} \Delta w = 0 & \text{in } (0,1)\times\mathbb{R}^n, \\ w_x(1,y) = 0. \end{cases} \tag{11}$$

Also

$$w(0,y) = \lim_{j\to+\infty} v(0,y_j+y) \geq 0 \tag{12}$$

and

$$w(1,0) = \lim_{j\to+\infty} v(1,y_j) = \inf_{y\in\mathbb{R}^n} v(1,y). \tag{13}$$

From (13), we have that

$$w(1,0) < 0 \tag{14}$$

and that

$$w(1,0) \le v(1, y + y_j) = v_j(1, y) \text{ for any } y. \tag{15}$$

Accordingly, (15) gives that

$$w(1,0) \le w(1,y) \text{ for any } y. \tag{16}$$

Then, making use of (11), (12), (14), (16) and the maximum principle, we have that

$$\inf_{(x,y)\in(0,1)\times\mathbb{R}^n} w(x,y) = \inf_{y\in\mathbb{R}^n} w(1,y) = w(1,0).$$

Consequently, Hopf principle and (11) imply that w is constant.

This constant must be nonnegative, due to (12), but this is in contradiction with (14). □

Corollary 6 (Monotonicity property I)**.** *Let* $\overline{u}$ *be a solution of* (1)*.*

If $\overline{u}_{y_n}(0,y) > 0$ *for any* $y \in \mathbb{R}^n$*, then* $\overline{u}_{y_n}(x,y) > 0$ *for any* $(x,y) \in [0,1) \times \mathbb{R}^n$*.*

Proof. Set $v = \overline{u}_{y_n}$ and employ Lemma 5. □

Lemma 7 (Monotonicity property II)**.** *Let* $\overline{u}$ *be a solution of* (1)*.*

If $\overline{u}_{y_n}(x,y) > 0$ *for any* $(x,y) \in [0,1) \times \mathbb{R}$*, then*

$$\int_{[0,1]\times\mathbb{R}^n} |\nabla\varphi|^2 + \int_{\{0\}\times\mathbb{R}^n} f'(u)\varphi^2 \ge 0 \tag{17}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$*.*

Proof. The following is a variation of a classical argument (see [1]). Possibly after approximation, we may take $i = n$ and $\psi = \varphi^2/\overline{u}_{y_n}$ in (8). Thus, making use of the Cauchy-Schwarz inequality we obtain

$$-\int_{\{0\}\times\mathbb{R}^n} f'(u)\varphi^2 = \int_{[0,1]\times\mathbb{R}^n} \Big(\frac{2\varphi\nabla\varphi\cdot\nabla\overline{u}_{y_n}}{\overline{u}_{y_n}} - \frac{\varphi^2|\nabla\overline{u}_{y_n}|^2}{\overline{u}_{y_n}^2}\Big) \le \int_{[0,1]\times\mathbb{R}^n} |\nabla\varphi|^2. \quad \square$$

With the above observations, we can now complete the

Proof of Theorem 1. We take $\overline{u}$ as in (1), such that $u(y) = \overline{u}(0,y)$. We also write $X = (x,y) \in [0,1] \times \mathbb{R}^n$. Notice that, in this notation

$$\nabla = (\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}) = (\partial_{X_1}, \ldots, \partial_{X_{n+1}}). \tag{18}$$

Given $\eta \in C_0^\infty(\mathbb{R}^n)$, we choose $\psi = \overline{u}_{y_i}\eta^2$ in (8). By summing over the index i, and using the notation in (18), we obtain, after a simple calculation,

$$-\int_{\{0\}\times\mathbb{R}^n} f'(u)|\nabla_y u|^2\eta^2 = \int_{[0,1]\times\mathbb{R}^n} \Big(\eta^2 \sum_{\substack{2\le i\le n+1\\1\le j\le n+1}} (\partial_{X_iX_j}\overline{u})^2 + \frac{1}{2}\nabla\eta^2\cdot\nabla|\nabla_y\overline{u}|^2\Big). \tag{19}$$

Furthermore, by (4) and Corollary 6, we have that $\overline{u}_{y_n}(x,y) > 0$ for any $(x,y) \in [0,1) \times \mathbb{R}^n$.

This and Lemma 7 imply that (17) holds true. Accordingly, given $\eta \in C_0^\infty(\mathbb{R}^{n+1})$, possibly after an approximation argument, we may take $\varphi = |\nabla_y \overline{u}|\eta$ in (17) and conclude that

$$\int_{[0,1]\times\mathbb{R}^n} \left(|\nabla\eta|^2|\nabla_y\overline{u}|^2 + \eta^2\big|\nabla|\nabla_y\overline{u}|\big|^2 + \frac{1}{2}\nabla\eta^2\cdot\nabla|\nabla_y\overline{u}|^2\right) \geq -\int_{\{0\}\times\mathbb{R}^n} f'(u)|\nabla_y\overline{u}|^2\eta^2.$$

As a consequence of this and of (19), some interesting cancellations give that

$$\int_{[0,1]\times\mathbb{R}^n} \eta^2\Big(\sum_{\substack{2\leq i\leq n+1\\ 1\leq j\leq n+1}} (\partial_{X_iX_j}\overline{u})^2 - \big|\nabla|\nabla_y\overline{u}|\big|^2\Big) \leq \int_{[0,1]\times\mathbb{R}^n} |\nabla\eta|^2|\nabla_y\overline{u}|^2. \tag{20}$$

Now, recalling (4), we have that $\nabla_y\overline{u}\neq 0$ in $(0,1)\times\mathbb{R}^n$, and so we write

$$\begin{aligned} &\sum_{\substack{2\leq i\leq n+1\\ 1\leq j\leq n+1}} (\partial_{X_iX_j}\overline{u})^2 - \big|\nabla|\nabla_y\overline{u}|\big|^2 \\ =\;& \sum_{\substack{2\leq i\leq n+1\\ 1\leq j\leq n+1}} (\partial_{X_iX_j}\overline{u})^2 - \big(\partial_x|\nabla_y\overline{u}|\big)^2 - \big|\nabla_y|\nabla_y\overline{u}|\big|^2 \\ =\;& \sum_{\substack{2\leq i\leq n+1\\ 2\leq j\leq n+1}} (\partial_{X_iX_j}\overline{u})^2 + \sum_{2\leq i\leq n+1}(\partial_{xX_i}\overline{u})^2 - \Big(\nabla_y\overline{u}_x\cdot\frac{\nabla_y\overline{u}}{|\nabla_y\overline{u}|}\Big)^2 - \big|\nabla_y|\nabla_y\overline{u}|\big|^2. \end{aligned} \tag{21}$$

Thus, we define

$$\mathscr{Z} = \sum_{2\leq i\leq n+1}(\partial_{xX_i}\overline{u})^2 - \Big(\nabla_y\overline{u}_x\cdot\frac{\nabla_y\overline{u}}{|\nabla_y\overline{u}|}\Big)^2.$$

Using the Cauchy-Schwarz inequality,

$$\Big(\nabla_y\overline{u}_x\cdot\frac{\nabla_y\overline{u}}{|\nabla_y\overline{u}|}\Big)^2 \leq |\nabla_y\overline{u}_x|^2 = \sum_{2\leq i\leq n+1}(\partial_{xX_i}\overline{u})^2,$$

so

$$\mathscr{Z}\geq 0 \tag{22}$$

and

$$\mathscr{Z}=0 \text{ if and only if } \nabla_y\overline{u}_x \text{ is parallel to } \nabla_y\overline{u}\,. \tag{23}$$

From (20), (21) and (22),

$$\int_{[0,1]\times\mathbb{R}^n} \eta^2\Big(|\mathscr{Z}| + \sum_{\substack{2\leq i\leq n+1\\ 2\leq j\leq n+1}} (\partial_{X_iX_j}\overline{u})^2 - \big|\nabla_y|\nabla_y\overline{u}|\big|^2\Big) \leq \int_{[0,1]\times\mathbb{R}^n} |\nabla\eta|^2|\nabla_y\overline{u}|^2 \tag{24}$$

We now introduce some geometric notation on the level set of $\overline{u}$.

Fixed any $x_o\in(0,1)$ and any $c\in\mathbb{R}$, we consider the level set of $\overline{u}$ on the slice $\{x=x_o\}$, that is

$$L = \big\{y\in\mathbb{R}^n \text{ s.t. } \overline{u}(x_o,y)=c\big\}.$$

Due to (4), we have that L is, locally, a smooth $(n-1)$-dimensional manifold, thus we may consider its principal curvatures $\kappa_1,\dots,\kappa_{n-1}$.

We define

$$\mathscr{K} = \sqrt{\kappa_1^2+\cdots+\kappa_{n-1}^2}.$$

Also, we may consider the tangential gradient $\overline{\nabla}$ along L. Namely, given a smooth function $G : \mathbb{R}^n \to \mathbb{R}$, we set

$$\overline{\nabla} G(y) = \nabla_y G(y) - \left(\nabla_y G(y) \cdot \frac{\nabla_y \overline{u}(x_o, y)}{|\nabla_y \overline{u}(x_o, y)|} \right) \frac{\nabla_y \overline{u}(x_o, y)}{|\nabla_y u(x_o, y)|}.$$

From Lemma 2.1 of [33], applied on the slice $\{x = x_o\}$, one has that

$$\sum_{\substack{2 \le i \le n+1 \\ 2 \le j \le n+1}} (\partial_{X_i X_j} \overline{u})^2 - \left| \nabla_y |\nabla_y \overline{u}| \right|^2 = |\nabla_y \overline{u}|^2 \mathscr{K}^2 + \left| \overline{\nabla} |\nabla_y \overline{u}| \right|^2.$$

As a consequence, (24) becomes

$$\int_{[0,1]\times\mathbb{R}^n} \eta^2 \Big(|\mathscr{L}| + |\nabla_y \overline{u}|^2 \mathscr{K}^2 + \left| \overline{\nabla} |\nabla_y \overline{u}| \right|^2 \Big) \le \int_{[0,1]\times\mathbb{R}^n} |\nabla \eta|^2 |\nabla_y \overline{u}|^2 \tag{25}$$

This geometric estimate may be seen as the extension of the weighted Poincaré inequality of [33, 34] that fits our goals.

Since (25) is valid for any $\eta \in C_0^\infty(\mathbb{R}^{n+1})$, by approximation, we have that it is valid for any $\eta \in W_0^{1,\infty}(\mathbb{R}^{n+1})$.

In particular, fixed $R \ge 1$, to be taken large in the sequel, we take

$$\vartheta \in C_0^\infty(B_{2R^2}, [0,1]),$$

with $\vartheta = 1$ in B_{R^2}, and $\eta(x,y) = \vartheta(x,y)\tilde{\eta}(y)$, with

$$\tilde{\eta}(y) = \begin{cases} \log R & \text{if } |y| \le \sqrt{R}, \\ 2\log\big(R/|y|\big) & \text{if } \sqrt{R} < |y| < R, \\ 0 & \text{if } |y| \ge R \end{cases}$$

We observe that, in $(0,1) \times \mathbb{R}^n$,

$$|\nabla \eta(x,y)| \le \frac{2\chi_{[\sqrt{R},R]}\big(|y|\big)}{|y|}$$

as long as R is large enough.

Hence, (25) yields that

$$(\log R)^2 \int_{[0,1]\times B_{\sqrt{R}}} \Big(|\mathscr{L}| + |\nabla_y \overline{u}|^2 \mathscr{K}^2 + \left| \overline{\nabla} |\nabla_y \overline{u}| \right|^2 \Big) \le \int_{[0,1]\times\{|y|\in[\sqrt{R},R]\}} \frac{|\nabla_y \overline{u}|^2}{|y|^2} \tag{26}$$

for large R.

Now, we define, for any $y \in \mathbb{R}^n$,

$$g_\star(y) = \int_{[0,1]} |\nabla_y \overline{u}(x,y)|^2 \, dx$$

and, for any $\tau \ge 0$,

$$\eta_\star(\tau) = \int_{|y|\le\tau} g_\star(y)\, dy.$$

By (5), we know that $\eta_\star(\tau) \le C\tau^2$ as long as $\tau \ge C$.

As a consequence, employing Lemma 3.1 of [21],

$$\frac{1}{2}\int_{x\in[0,1]}\int_{\sqrt{R}\le|y|\le R}\frac{|\nabla_y\overline{u}|^2}{|y|^2}\,dy\,dx=\frac{1}{2}\int_{\sqrt{R}\le|y|\le R}\frac{g_\star(y)}{|y|^2}\,dy$$
$$\le\int_{\sqrt{R}}^{R}\frac{\eta_\star(\tau)}{\tau^3}\,d\tau+\frac{\eta_\star(R)}{R^2}\le C(\log R+1)$$

provided that $R\ge C$.

Therefore, (26) gives that

$$\lim_{R\to+\infty}\int_{[0,1]\times B_{\sqrt{R}}}\Big(|\mathscr{L}|+|\nabla_y\overline{u}|^2\mathscr{K}^2+\big|\overline{\nabla}|\nabla_y\overline{u}|\big|^2\Big)\le\lim_{R\to+\infty}\frac{16C}{\log R}=0.$$

Thus,

$$\mathscr{K}\text{ vanishes identically}\tag{27}$$

$$\text{and so does }\mathscr{L}.\tag{28}$$

From (27), we have that all the principal curvatures of any sliced level set L vanish.

So, there exist $U:(0,1)\times\mathbb{R}\to\mathbb{R}$ and $\omega:(0,1)\to\mathrm{S}^{n-1}$ such that

$$\overline{u}(x,y)=U\big(x,\omega(x)\cdot y\big)$$

for any $x\in(0,1)$ and $y\in\mathbb{R}^n$.

Moreover, $\nabla_y\overline{u}_x$ is parallel to $\nabla_y\overline{u}$, thanks to (28) and (23). This, (4) and Lemma A.1 of [5] imply that ω is constant.

Therefore

$$u(y)=\lim_{x\to0^+}\overline{u}(x,y)=\lim_{x\to0^+}U(x,\omega\cdot y),$$

which completes the proof of Theorem 1. □

With this, we are now ready for the

Proof of Corollary 2. Let $\overline{u}$ be as in (1), and $u(y)=\overline{u}(0,y)$. Since u is bounded, elliptic regularity theory [23] gives that $|\nabla\overline{u}|\in L^\infty([0,1]\times\mathbb{R}^2)$ and so (5) holds true since $n=2$ in this case. Then, Corollary 2 plainly follows from Theorem 1. □

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