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THE GENERALIZED CASSELS-TATE DUAL EXACT SEQUENCE FOR 1-MOTIVES

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Abstract. We establish a generalized Cassels-Tate dual exact sequence for 1-motives over global fields. We thereby extend the main theorem of [4] from abelian varieties to arbitrary 1-motives.

1. Introduction

Let K be a global field and let $M = (Y \to G)$ be a (Deligne) 1-motive over K, where Y is étale-locally isomorphic to $\mathbb{Z}^r$ for some $r \geq 0$ and G is a semiabelian variety over K. Let M^* be the 1-motive dual to M. If B is a topological abelian group, $B^\wedge$ will denote the completion of B with respect to the family of open subgroups of finite index. Let $Ш^1(M)$ (resp. $Ш^1_\omega(M)$) denote the subgroup of $\mathbb{H}^1(K, M)$ of all classes which are locally trivial at all (resp. all but finitely many) primes of K. There exists a canonical exact sequence of discrete torsion groups

$$0 \to Ш^1(M) \to Ш^1_\omega(M) \to \bigoplus_{\text{all } v} \mathbb{H}^1(K_v, M) \to Ч^1(M) \to 0,$$

where, for each prime v of K, K_v denotes the completion of K at v and $Ч^1(M)$ denotes the cokernel of the middle map. Now, for any topological abelian group B, let $B^D = \mathrm{Hom}_{\text{cont.}}(B, \mathbb{Q}/\mathbb{Z})$ and endow it with the compact-open topology, where $\mathbb{Q}/\mathbb{Z}$ carries the discrete topology. Then, by the local duality theorem for 1-motives [7], Theorem 2.3 and Proposition 2.9, the Pontryagin dual of the above exact sequence is an exact sequence

$$0 \to Ч^1(M)^D \to \prod_{\text{all } v} \mathbb{H}^0(K_v, M^*)^\wedge \to Ш^1_\omega(M)^D \to Ш^1(M)^D \to 0,$$

where each group $\mathbb{H}^0(K_v, M^*)$ is endowed with the topology defined in [7] p.99, (for archimedean v, $\mathbb{H}^0(K_v, M)$ denotes the *reduced* 0-th (Tate) hypercohomology group of M_{K_v} [7] p.103). A fundamental problem is to describe $Ч^1(M)^D$. This problem was first addressed in the case of elliptic curves E over number fields K (i.e., $Y = 0$ and $G = E$ above), by J.W.S.Cassels (see [2], Theorem 7.1, and [3], Appendix 2). Cassels showed that $Ч^1(E^*)^D$ is canonically isomorphic to the pro-Selmer group $T\,\mathrm{Sel}(E)$ of E. This result was extended to abelian varieties A over number fields K by J.Tate,

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under the assumption that $Ш^1(A)$ is finite (unpublished). In this case $T\mathrm{Sel}(A)$ is isomorphic to $H^0(K,A)^\wedge$ and $\mathbb{H}^0(K_v,M)^\wedge = H^0(K_v,A)^\wedge = H^0(K_v,A)$ for any v since $H^0(K_v,A)$ is profinite. Further, $Ш^1_\omega(A^*) = H^1(K,A^*)$ and $Ш^1(A^*)^D = Ш^1(A)$. The exact sequence obtained by Tate, now known as the *Cassels-Tate dual exact sequence*, is

$$0 \to H^0(K,A)^\wedge \to \prod_{\text{all } v} H^0(K_v,A) \to H^1(K,A^*)^D \to Ш^1(A) \to 0. \tag{1}$$

Further, the image of $H^0(K,A)^\wedge$ is isomorphic to the closure $\overline{H^0(K,A)}$ of the diagonal image of $H^0(K,A)$ in $\prod_{\text{all } v} H^0(K_v,A)$. See [12], Remark I.6.14(b), p.102. The preceding exact sequence was recently extended to arbitrary 1-motives over number fields by D.Harari and T.Szamuely [8], Theorem 1.2, again under the assumption that $Ш^1(A)$ is finite, where A is the abelian part of M (this implies the finiteness of $Ш^1(M)$). They established the exactness of the sequence

$$0 \to \overline{\mathbb{H}^0(K,M)} \to \prod_{\text{all } v} \mathbb{H}^0(K_v,M) \to Ш^1_\omega(M^*)^D \to Ш^1(M) \to 0,$$

where the middle map is induced by the local pairings of [7], §2. This natural analogue of (1) was used in [op.cit., §6] to study weak approximation on semiabelian varieties over number fields. However, the preceding sequence with M and M^* interchanged does not provide a description of $Ч^1(M)^D$ when $Ш^1(M^*)$ (or, equivalently, $Ш^1(M)$) is finite. Our objective in this paper is to describe $Ч^1(M)^D$ for any K independently of the finiteness assumption on $Ш^1(M)$. In order to state our main result, let

$$\mathrm{Sel}(M^*)_n = \mathrm{Ker}\left[H^1(K, T_{\mathbb{Z}/n}(M^*)) \to \prod_{\text{all } v} \mathbb{H}^1(K_v,M^*)_n\right]$$

be the n-th Selmer group of M^*, where n is any positive integer and $T_{\mathbb{Z}/n}(M^*)$ is the n-adic realization of M^*. Let $T\mathrm{Sel}(M^*) = \varprojlim_n \mathrm{Sel}(M^*)_n$ be the pro-Selmer group of M^*. Our main theorem is the following result.

Theorem 1.1. (The generalized Cassels-Tate dual exact sequence for 1-motives). *Let M be a 1-motive over a global field K. Then there exists a canonical exact sequence of profinite groups*

$$\begin{aligned} 0 \to Ш^2(M)^D &\to T\mathrm{Sel}(M^*)^\wedge \to \prod_{\text{all } v} \mathbb{H}^0(K_v,M^*)^\wedge \\ &\to Ш^1_\omega(M)^D \to Ш^1(M)^D \to 0. \end{aligned}$$

The proof of the theorem depends crucially on Poitou-Tate duality for finite modules ([12], Theorem I.4.10, p.70, and [5], Theorem 4.9).

An immediate corollary of the theorem is the existence of a canonical exact sequence

$$0 \to Ч^1(M) \to (T\mathrm{Sel}(M^*)^\wedge)^D \to Ш^2(M) \to 0.$$

When $M = (0 \to T)$ is a torus, it seems likely that the above exact sequence is the same as the toric case of an exact sequence obtained by J.Oesterlé in [14], Theorem 2.7(d), p.52. When $M = (0 \to A)$ is an abelian variety, $Ш^2(M) = 0$ and our main theorem reduces to the main theorem of [4] (properly corrected. See below).

Applications of Theorem 1.1 will be given in [6].

Remark 1.2. We take this opportunity to correct the statement of the main theorem of [4]. For it to be valid, for each prime v of K the field K_v appearing there must be taken to be equal to the *completion* (rather than the henselization) of K at v. Since the only application of the main theorem of [4] that we are aware of [15] makes use of this corrected version, no harm appears to have resulted from the authors' error.

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2. Preliminaries

Let K be a global field, i.e. K is a finite extension of $\mathbb{Q}$ (the "number field case") or is finitely generated and of transcendence degree 1 over a finite field of constants k (the "function field case"). For any prime v of K, K_v will denote the completion of K at v and $\mathcal{O}_v$ will denote the corresponding ring of integers. Thus $\mathcal{O}_v$ is a complete discrete valuation ring. Further, X will denote either the spectrum of the ring of integers of K (in the number field case) or the unique smooth complete curve over k with function field K (in the function field case).

All cohomology groups below are flat (fppf) cohomology groups.

If n is any positive integer, B/n will denote B/nB with the quotient topology. Let $B_\wedge = \varprojlim_{n\in\mathbb{N}} B/n$ with the inverse limit topology. Further, define $B^\wedge = \varprojlim_{U\in\mathcal{U}} B/U$, where $\mathcal{U}$ denotes the family of open subgroups of finite index in B. If $B_\sim := \varprojlim_{n\in\mathbb{N}} B\big/\overline{nB}$, where $\overline{nB}$ denotes the closure of nB in B, then there exists a canonical isomorphism $(B_\sim)^\wedge = B^\wedge$. Consequently, there exists a canonical map $B_\wedge \to B^\wedge$. If nB is closed in B for every n (i.e., B/n is Hausdorff), then $B_\sim = B_\wedge$ and therefore $(B_\wedge)^\wedge = B^\wedge$. We also note that $B^\wedge = B$ if B is profinite (see, e.g., [16], Theorem 2.1.3, p.22). For any positive integer n, B_n will denote the n-torsion subgroup of B and $TB = \varprojlim_{n\in\mathbb{N}} B_n$ is the total Tate module of B. Note that $TB = 0$ if B is finite.

Let $M = (Y \to G)$ be a Deligne 1-motive over K, where Y is étale-locally isomorphic to $\mathbb{Z}^r$ for some r and G is a semiabelian variety (for basic information on 1-motives over global fields, see [7] §1, or [5], §3. Let n be a positive integer. The *n-adic realization of* M is a finite and flat K-group scheme $T_{\mathbb{Z}/n}(M)$ which fits into an exact sequence

$$0 \to G_n \to T_{\mathbb{Z}/n}(M) \to Y/n \to 0.$$

There exists a perfect pairing

$$T_{\mathbb{Z}/n}(M) \times T_{\mathbb{Z}/n}(M^*) \to \mu_n,$$

where μ_n is the sheaf of n-th roots of unity. Further, given positive integers n and m with $n \mid m$, there exist canonical maps $T_{\mathbb{Z}/n}(M) \to T_{\mathbb{Z}/m}(M)$ and $T_{\mathbb{Z}/m}(M) \to T_{\mathbb{Z}/n}(M)$. Let $T(M)_{\text{tors}} = \varinjlim T_{\mathbb{Z}/n}(M)$. Further, for any $i \geq 0$, define

$$H^i(K, T(M)) = \varprojlim_n H^i(K, T_{\mathbb{Z}/n}(M)).$$

The groups $H^i(K, T_{\mathbb{Z}/n}(M))$ will be endowed with the discrete topology. If v is archimedean and $i \geq -1$, $\mathbb{H}^i(K_v, M)$ will denote the (finite, 2-torsion) *reduced* (Tate) hypercohomology groups of M_{K_v} defined in [7], p.103. All groups $\mathbb{H}^i(K_v, M)$ will be given the discrete topology, except for $\mathbb{H}^0(K_v, M)$ for non-archimedean v. The latter group will be given the topology defined in [7], p.99. Thus, there exists an exact sequence $0 \to I \to \mathbb{H}^0(K_v, M) \to F \to 0$, where F is finite and I is an open subgroup of $\mathbb{H}^0(K_v, M)$ which is isomorphic to $G(K_v)/L$ for some finitely generated subgroup L of $G(K_v)$. If n is a positive integer, $G(K_v)/n$ is profinite (see [5], beginning of §5). Thus the exactness of

$$L/n \to G(K_v)/n \to I/n \to 0$$

shows that I/n is profinite as well. Now the exactness of

$$F_n \to I/n \to \mathbb{H}^0(K_v, M)/n \to F/n \to 0$$

shows that $\mathbb{H}^0(K_v, M)/n$ is profinite (see [16], Proposition 2.2.1(e), p.28). The latter also holds if v is archimedean. We conclude that $\mathbb{H}^0(K_v, M)_\wedge$ is profinite for every v (see [16], Proposition 2.2.1(d), p.28).

The groups $\mathbb{H}^i(K, M)$ will be endowed with the discrete topology.

For each $i \geq 0$, let $\mathbb{P}^i(M)$ be the restricted direct product over all primes of K of the groups $\mathbb{H}^i(K_v, M)$ with respect to the subgroups

$$\mathbb{H}^i_{\text{nr}}(K_v, M) = \operatorname{Im}\left[\mathbb{H}^i(\mathcal{O}_v, \mathcal{M}) \to \mathbb{H}^i(K_v, M)\right]$$

for $v \in U$, where U is any nonempty open subscheme of X such that M extends to a 1-motive $\mathcal{M}$ over U. The groups $P^i(F)$ are defined similarly for any abelian fppf sheaf F on $\operatorname{Spec} K$. By [7], Lemma 5.3[1], for any positive integer n the group $\mathbb{P}^0(M)/n$ is the restricted direct product of the profinite groups $\mathbb{H}^0(K_v, M)/n$ with respect to the subgroups $\mathbb{H}^0_{\text{nr}}(K_v, M)/n$. It is therefore Hausdorff and locally compact (see [9], 6.16(c), p.57). In particular, $(\mathbb{P}^0(M)_\wedge)^\wedge = \mathbb{P}^0(M)^\wedge$. Further, since $\mathbb{H}^0(K_v, M)/n$ and $\mathbb{H}^0(K_v, M)^\wedge/n$ have the same continuous dual for every n and v, [7], Theorems 2.3 and 2.10, show that the dual of $\mathbb{P}^0(M)_\wedge$ is $\mathbb{P}^1(M^*)_{\text{tors}}$. Therefore the dual of the profinite group $\mathbb{P}^0(M)^\wedge$ is the discrete torsion group $\mathbb{P}^1(M^*)_{\text{tors}}$.

Recall that a morphism $f\colon A \to B$ of topological groups is said to be *strict* if the induced map $A/\operatorname{Ker} f \to \operatorname{Im} f$ is an isomorphism of topological groups. Equivalently, f is strict if it is open onto its image [1], §III.2.8, Proposition 24(b), p.236. Every continuous homomorphism from a compact group to a Hausdorff group is strict [1], §III.2.8, p.237. We will need the following

Lemma 2.1. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of abelian topological groups and strict morphisms. If $C \to C^\wedge$ is injective, then $A^\wedge \xrightarrow{\widehat{f}} B^\wedge \xrightarrow{\widehat{g}} C^\wedge$ is also exact.*

Proof. The map $A \to \operatorname{Im} f$ induced by f is an open surjection, so $A^\wedge \to (\operatorname{Im} f)^\wedge$ is surjective as well. Further, since $B \to \operatorname{Im} g$ is an open surjection, the sequence $(\operatorname{Im} f)^\wedge \to B^\wedge \to (\operatorname{Im} g)^\wedge \to 0$ is exact [7], Appendix. Finally, since C injects into $C^\wedge$, $(\operatorname{Im} g)^\wedge$ is the closure of $\operatorname{Im} g$ in $C^\wedge$, whence $(\operatorname{Im} g)^\wedge \to C^\wedge$ is injective. □

[1]This result and its proof are also valid in the function field case, using the fact that $H^1_v(\mathcal{O}_v, T_{\mathbb{Z}/p^m}(\mathcal{M})) = 0$ for any m by [12], beginning of §7, p.349.

3. The Poitou-Tate exact sequence for 1-motives over function fields

For any positive integer n, there exists a canonical exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}^0(K,M)/n & \longrightarrow & H^1(K,T_{\mathbb{Z}/n}(M)) & \longrightarrow & \mathbb{H}^1(K,M)_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{P}^0(M)/n & \longrightarrow & P^1(T_{\mathbb{Z}/n}(M)) & \longrightarrow & \mathbb{P}^1(M)_n & \longrightarrow & 0, \end{array} \tag{2}$$

whose vertical maps are induced by the canonical morphisms $\operatorname{Spec} K_v \to \operatorname{Spec} K$. For the exactness of the rows, see [7], p.109. Now, for any $i \geq -1$, set

$$Ш^i(M) = \operatorname{Ker}\left[\,\mathbb{H}^i(K,M) \to \mathbb{P}^i(M)\,\right].$$

Further, define

$$\operatorname{Sel}(M)_n = \operatorname{Ker}\left[H^1(K,T_{\mathbb{Z}/n}(M)) \to \mathbb{P}^1(M)_n\right],$$

where the map involved is the composite

$$H^1(K,T_{\mathbb{Z}/n}(M)) \to P^1(T_{\mathbb{Z}/n}(M)) \to \mathbb{P}^1(M)_n\,.$$

Diagram (2) yields an exact sequence

$$0 \to \mathbb{H}^0(K,M)/n \to \operatorname{Sel}(M)_n \to Ш^1(M)_n \to 0 \tag{3}$$

and a map

$$\theta_{0,n}\colon \operatorname{Sel}(M)_n \to \mathbb{P}^0(M)/n.$$

Now the group $\mathbb{H}^0(K,M)$ is countable (this follows by devissage from the Mordell-Weil theorem, the finiteness of $H^1(K,Y)$ and the fact that $H^0(K,T)$ is a subgroup of $(L^*)^d$ for some finite extension L of K and some positive integer d). On the other hand, by [13] and devissage again, $Ш^1(M)_n$ is finite. Thus (3) shows that $\operatorname{Sel}(M)_n$ is discrete and countable, hence locally compact and σ-compact.

Lemma 3.1. *The canonical map $H^1(K,T_{\mathbb{Z}/n}(M))) \twoheadrightarrow \mathbb{H}^1(K,M)_n$ appearing in diagram (2) induces an isomorphism*

$$H^1(K,T_{\mathbb{Z}/n}(M))/\operatorname{Sel}(M)_n \simeq \mathbb{H}^1(K,M)_n/\ Ш^1(M)_n.$$

Proof. This is immediate from (2) and the definitions of $Ш^1(M)$ and $\operatorname{Sel}(M)_n$. □

The above lemma shows that there exists a canonical exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Sel}(M)_n & \longrightarrow & H^1(K,T_{\mathbb{Z}/n}(M)) & \twoheadrightarrow & \mathbb{H}^1(K,M)_n/\ Ш^1(M)_n \\ & & \downarrow{\scriptstyle\theta_{0,n}} & & \downarrow{\scriptstyle\theta_n} & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^0(M)/n & \longrightarrow & P^1(T_{\mathbb{Z}/n}(M)) & \twoheadrightarrow & \mathbb{P}^1(M)_n\,. \end{array} \tag{4}$$

We conclude that $\operatorname{Ker}\theta_{0,n} = \operatorname{Ker}\theta_n =\ Ш^1(T_{\mathbb{Z}/n}(M))$, which is finite by [12], Theorem I.4.10, p.70, and [5], Proposition 4.7.

Lemma 3.2. *$\theta_{0,n}$ is a strict morphism.*

Proof. By [9], Theorem 5.29, p.42, and [17], Theorem 4.8, p.45, it suffices to check that $\operatorname{Im}\theta_{0,n}$ is a closed subgroup of the locally compact Hausdorff group $\mathbb{P}^0(M)/n$. The image of the map θ_n in diagram (4) can be identified with the kernel of the map

$$P^1(T_{\mathbb{Z}/n}(M)) \to H^1(K, T_{\mathbb{Z}/n}(M^*))^D$$

coming from the Poitou-Tate exact sequence for $T_{\mathbb{Z}/n}(M)$ ([12], Theorem I.4.10, p.70, and [5], Theorem 4.12). Consequently $\operatorname{Im}\theta_{0,n}$ can be identified with the kernel of the continuous composite map

$$\mathbb{P}^0(M)/n \to P^1(T_{\mathbb{Z}/n}(M)) \to H^1(K, T_{\mathbb{Z}/n}(M^*))^D.$$

Thus $\operatorname{Im}\theta_{0,n}$ is indeed closed in $\mathbb{P}^0(M)/n$. □

Now set

$$\begin{aligned} T\mathrm{Sel}(M) &= \varprojlim_n \mathrm{Sel}(M)_n \\ P^1(T(M)) &= \varprojlim_n P^1(T_{\mathbb{Z}/n}(M)). \end{aligned}$$

Since $(\mathbb{H}^0(K,M)/n)$ is an inverse system with surjective transition maps, the inverse limit of (3) is an exact sequence

$$0 \to \mathbb{H}^0(K,M)_\wedge \to T\,\mathrm{Sel}(M) \to T\,Ш^1(M) \to 0. \tag{5}$$

Thus, if $Ш^1(M)$ is finite, then $T\,\mathrm{Sel}(M)$ is canonically isomorphic to $\mathbb{H}^0(K,M)_\wedge$. In particular, $T\,\mathrm{Sel}(M)^\wedge = (\mathbb{H}^0(K,M)_\wedge)^\wedge = \mathbb{H}^0(K,M)^\wedge$.

Now consider the map

$$\theta_0 = \varprojlim_n \theta_{0,n} \colon T\mathrm{Sel}(M) \to \mathbb{P}^0(M)_\wedge\,.$$

Proposition 3.3. *There exists a perfect pairing*

$$\operatorname{Ker}\theta_0 \times Ш^2(M^*) \to \mathbb{Q}/\mathbb{Z}\,,$$

where the first group is profinite and the second is discrete and torsion.

Proof. By Poitou-Tate duality for finite modules ([12], Theorem I.4.10, p.70, and [5], Theorem 4.9), $\operatorname{Ker}\theta_0 = \varprojlim_n Ш^1(T_{\mathbb{Z}/n}(M))$ is canonically dual to $Ш^2(T(M^*)_{\mathrm{tors}}) := \varinjlim_n Ш^2(T_{\mathbb{Z}/n}(M^*))$. Now [5], proof of Lemma 5.8(a), shows the last group to be isomorphic to $Ш^2(M^*)$, which completes the proof. □

Remark 3.4. In the number field case, $Ш^2(M^*)$ has been shown to be finite by P.Jossen [11]. Further, by [op.cit., proof of Theorem 9.4], the finite group $\operatorname{Ker}\theta_0 = \varprojlim_n Ш^1(T_{\mathbb{Z}/n}(M))$ is canonically isomorphic to

$$\operatorname{Ker}\left[\mathbb{H}^0(K,M) \to \prod_{\text{all } v} \mathbb{H}^0(K_v,M)_\wedge\right],$$

which conjecturally is the same as $Ш^0(M)$.

Lemma 3.5. θ_0 *is a strict morphism.*

Proof. This follows from the fact that, by Lemma 3.2, θ_0 is an inverse limit of strict morphisms with finite kernel from an abelian discrete group to an abelian topological group. See [7], Complement to the Appendix, for the details. □

There exists a natural commutative diagram

$$\begin{array}{ccc} T\,\mathrm{Sel}(M) & \xrightarrow{\theta_0} & \mathbb{P}^0(M)_\wedge \\ \downarrow & & \downarrow \\ T\,\mathrm{Sel}(M)^\wedge & \xrightarrow{\beta_0} & \mathbb{P}^0(M)^\wedge\,, \end{array} \tag{6}$$

where $\beta_0 = \widehat{\theta}_0$.

Lemma 3.6. *The vertical maps in the preceding diagram are injective.*

Proof. (Cf. [7], proof of Proposition 5.4, p.119) Let $\xi = (\xi_n) \in T\,\mathrm{Sel}(M)$ be nonzero. Then, for some n, $\xi_n \in \mathrm{Sel}(M)_n$ is nonzero. Since the canonical map $\mathrm{Sel}(M)_n \to \mathrm{Sel}(M)_n^\wedge$ is injective by [7], Lemma 5.5, we conclude that the image of ξ_n in $\mathrm{Sel}(M)_n^\wedge$ is nonzero. Consequently, there exists a subgroup N of $\mathrm{Sel}(M)_n$, of finite index, such that $\xi_n \notin N$. It follows that ξ is not contained in the inverse image of N under the canonical map $T\,\mathrm{Sel}(M) \to \mathrm{Sel}(M)_n$, which is an open subgroup of finite index in $T\,\mathrm{Sel}(M)$. We conclude that the image of ξ in $T\,\mathrm{Sel}(M)^\wedge$ is nonzero. This proves the injectivity of the left-hand vertical map in diagram (6). To prove the injectivity of the right-hand vertical map, let $x = (x_v) \in \mathbb{P}^0(M)_\wedge$ be nonzero. Then $x \notin n\mathbb{P}^0(M)$ for some n, whence $x_v \notin n\mathbb{H}^0(K_v, M)$ for some v (see [7], Lemma 5.3, p.118). Thus the image of x under the canonical map

$$\mathbb{P}^0(M)_\wedge \to \mathbb{H}^0(K_v, M)/n = (\mathbb{H}^0(K_v, M)/n)^\wedge$$

is nonzero, where the equality comes from the fact that $\mathbb{H}^0(K_v, M)/n$ is profinite. But the preceding map factors through $\mathbb{P}^0(M)^\wedge$, so the image of x in $\mathbb{P}^0(M)^\wedge$ is nonzero. □

Proposition 3.7. *The map* $\mathrm{Ker}\,\theta_0 \to \mathrm{Ker}\,\beta_0$ *induced by diagram (6) is an isomorphism.*

Proof. The injectivity of the above map is immediate from Lemma 3.6. Now, by Lemmas 2.1, 3.5 and 3.6, the exact sequence

$$\mathrm{Ker}\,\theta_0 \to T\,\mathrm{Sel}(M) \xrightarrow{\theta_0} \mathbb{P}^0(M)_\wedge$$

induces an exact sequence

$$(\mathrm{Ker}\,\theta_0)^\wedge \to T\,\mathrm{Sel}(M)^\wedge \xrightarrow{\beta_0} \mathbb{P}^0(M)^\wedge\,.$$

But $(\mathrm{Ker}\,\theta_0)^\wedge = \mathrm{Ker}\,\theta_0$ since $\mathrm{Ker}\,\theta_0$ is profinite by Proposition 3.3, so $\mathrm{Ker}\,\theta_0 \to \mathrm{Ker}\,\beta_0$ is indeed surjective. □

For each v and any $n \geq 1$, there exists a canonical pairing

$$(-,-)_v \colon \mathbb{H}^0(K_v, M)/n \times \mathbb{H}^1(K_v, M^*)_n \to \mathbb{Q}/\mathbb{Z}$$

which vanishes on $\mathbb{H}^0_{\mathrm{nr}}(K_v, M)/n \times \mathbb{H}^1_{\mathrm{nr}}(K_v, M^*)_n$. See [7], p.99 and proof of Theorem 2.10, p.104. Let $\gamma'_{0,n} \colon \mathbb{P}^0(M)/n \to (\mathbb{H}^1(K, M^*)_n)^D$ be defined as follows. For $x = (x_v) \in \mathbb{P}^0(M)/n$ and $\xi \in \mathbb{H}^1(K, M^*)_n$, set

$$\gamma'_{0,n}(x)(\xi) = \sum_{\text{all } v} (x_v, \xi|_{K_v})_v,$$

where $\xi|_{K_v}$ is the image of ξ under the canonical map $\mathbb{H}^1(K, M^*)_n \to \mathbb{H}^1(K_v, M^*)_n$ (the sum is actually finite since $x_v \in \mathbb{H}^0_{\rm nr}(K_v, M)/n$ and $\xi\,|_{K_v}\!\in \mathbb{H}^1_{\rm nr}(K_v, M)_n$ for all but finitely many primes v). Consider the map

$$\gamma_0' := \varprojlim_n \gamma_{0,n}' \colon \mathbb{P}^0(M)_\wedge \to \mathbb{H}^1(K, M^*)^D.$$

By [7], p.122, the sequence

$$TSel(M) \xrightarrow{\theta_0} \mathbb{P}^0(M)_\wedge \xrightarrow{\gamma_0'} \mathbb{H}^1(K, M^*)^D \tag{7}$$

is a complex.

Lemma 3.8. *The sequence (7) is exact.*

Proof. As noted in the proof of Lemma 3.2, $\operatorname{Im}\theta_{0,n}$ can be identified with the kernel of the composite map

$$\mathbb{P}^0(M)\,/n \to P^1(T_{\mathbb{Z}/n}(M)) \to H^1(K, T_{\mathbb{Z}/n}(M^*))^D.$$

Now, using the fact that $\varprojlim_n^{(1)}\text{Ш}^1(T_{\mathbb{Z}/n}(M)) = 0$ (see [10], Proposition 2.3, p.14), we conclude that $\operatorname{Im}\theta_0$ can be identified with the kernel of the continuous composite map

$$\mathbb{P}^0(M)_\wedge \to P^1(T(M)) \to H^1(K, T(M^*)_{\rm tors})^D.$$

Further, there exists a canonical commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^0(M)_\wedge & \longrightarrow & P^1(T(M)) \\
\big\downarrow{\scriptstyle \gamma_0'} & & \big\downarrow \\
\mathbb{H}^1(K, M^*)^D & \hookrightarrow & H^1(K, T(M^*)_{\rm tors})^D,
\end{array}$$

where the bottom map is the dual of the surjection $H^1(K, T(M^*)_{\rm tors}) \to \mathbb{H}^1(K, M^*)$ (the latter map being the direct limit over n of the surjections appearing on the top row of diagram (2) for M^*). We conclude that $\operatorname{Im}\theta_0 = \operatorname{Ker}\gamma_0'$, as claimed. □

Lemma 3.9. γ_0' *is a strict morphism.*

Proof. By Lemma 3.8, γ_0' induces a continuous map $\overline{\gamma}_0'\colon \operatorname{Coker}\theta_0 \to \mathbb{H}^1(K, M^*)^D$, where $\operatorname{Coker}\theta_0$ is endowed with the quotient topology. On the other hand, there exists a canonical exact commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & TSel(M) & \longrightarrow & H^1(K, T(M)) & \twoheadrightarrow & T\mathbb{H}^1(K, M)/T\ \text{Ш}^1(M) \\
& \big\downarrow{\scriptstyle \theta_0} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow \\
0 \longrightarrow & \mathbb{P}^0(M)_\wedge & \longrightarrow & P^1(T(M)) & \twoheadrightarrow & T\mathbb{P}^1(M)\,,
\end{array}$$

which shows that $\operatorname{Coker}\theta_0$ (with the quotient topology) injects as a closed subgroup of $\operatorname{Coker}\theta$. Now the Poitou-Tate exact sequence for finite modules ([12], Theorem I.4.10, p.70 and [5], Theorem 4.12) shows that $\operatorname{Coker}\theta$ is a closed subgroup of $H^1(K, T(M^*)_{\rm tors})^D$, which is profinite. We conclude that $\operatorname{Coker}\theta_0$ is profinite, whence $\overline{\gamma}_0'$ is strict [1], §III.2.8, p.237. It follows that γ_0' is strict, as claimed. □

Now consider

$$\gamma_0 = (\gamma_0')^\wedge \colon \mathbb{P}^0(M)^\wedge \to \big(\mathbb{H}^1(K, M^*)^D\big)^\wedge = \mathbb{H}^1(K, M^*)^D.$$

Proposition 3.10. *The sequence*

$$T\mathrm{Sel}(M)^\wedge \xrightarrow{\beta_0} \mathbb{P}^0(M)^\wedge \xrightarrow{\gamma_0} \mathbb{H}^1(K, M^*)^D,$$

is exact.

Proof. This follows by applying Lemma 2.1 to the exact sequence (7) using Lemmas 3.5 and 3.9. □

The following is the main result of this Section. It extends [7], Theorem 5.6, p.120, to the function field case.

Theorem 3.11. *Let K be a global function field and let M be a 1-motive over K. Assume that $\text{Ш}^1(M)$ is finite. Then there exists a canonical 12-term exact sequence*

$$\begin{array}{ccccc}
\mathbb{H}^{-1}(K,M)^\wedge & \overset{\gamma_2^D}{\hookrightarrow} & \prod_{\text{all } v} \mathbb{H}^2(K_v, M^*)^D & \xrightarrow{\beta_2^D} & \mathbb{H}^2(K, M^*)^D \\
 & & & & \downarrow \\
\mathbb{H}^1(K, M^*)^D & \xleftarrow{\gamma_0} & \mathbb{P}^0(M)^\wedge & \xleftarrow{\beta_0} & \mathbb{H}^0(K, M)^\wedge \\
\downarrow & & & & \\
\mathbb{H}^1(K, M) & \xrightarrow{\beta_1} & \mathbb{P}^1(M)_{\mathrm{tors}} & \xrightarrow{\gamma_1} & (\mathbb{H}^0(K, M^*)^D)_{\mathrm{tors}} \\
 & & & & \downarrow \\
\mathbb{H}^{-1}(K, M^*)^D & \overset{\gamma_2}{\twoheadleftarrow} & \bigoplus_{\text{all } v} \mathbb{H}^2(K_v, M) & \xleftarrow{\beta_2} & \mathbb{H}^2(K, M),
\end{array}$$

where the maps β_i are canonical localization maps, the maps γ_i are induced by local duality and the unlabeled maps are defined in the proof.

Proof. The exactness of the first line follows as in [7], p.122, using [5], Theorem 4.12, and noting that [7], Lemma 5.8, remains valid (with the same proof) in the function field case. The top right-hand vertical map $\mathbb{H}^2(K, M^*)^D \to \mathbb{H}^0(K, M)^\wedge$ is the composite

$$\begin{array}{rcl}
\mathbb{H}^2(K, M^*)^D \twoheadrightarrow \text{Ш}^2(M^*)^D & \xrightarrow{\sim} & \operatorname{Ker} \theta_0 \xrightarrow{\sim} \operatorname{Ker} \beta_0 \\
 & \hookrightarrow & T\mathrm{Sel}(M)^\wedge = \mathbb{H}^0(K, M)^\wedge,
\end{array}$$

where the isomorphisms come from Propositions 3.3 and 3.7 and the equality is a consequence of the finiteness hypothesis on $\text{Ш}^1(M)$. The exactness of the second line of the sequence of the theorem is the content of Proposition 3.10 (again using the equality $T\mathrm{Sel}(M)^\wedge = \mathbb{H}^0(K, M)^\wedge$). Since γ_0 is the dual of the natural map $\mathbb{H}^1(K, M^*) \to \mathbb{P}^1(M^*)_{\mathrm{tors}}$ and $\text{Ш}^1(M^*)^D \simeq \text{Ш}^1(M)$ by [7], Corollary 4.9 and

Remark 5.10, and [5], corollary 6.7, we conclude that there exists an exact sequence

$$
\begin{array}{ccccc}
0 \longrightarrow \mathbb{H}^{-1}(K,M)^{\wedge} & \xhookrightarrow{\gamma_2^D} & \displaystyle\prod_{\text{all } v} \mathbb{H}^2(K_v,M^*)^D & \xrightarrow{\beta_2^D} & \mathbb{H}^2(K,M^*)^D \\
 & & & & \downarrow \\
\mathbb{H}^1(K,M^*)^D & \xleftarrow{\gamma_0} & \mathbb{P}^0(M)^{\wedge} & \xleftarrow{\beta_0} & \mathbb{H}^0(K,M)^{\wedge} \\
\downarrow & & & & \\
\text{Ш}^1(M). & & & &
\end{array}
$$

The above is an exact sequence of profinite groups and continuous homomorphisms, so each morphism is strict. Consequently, the dual of the preceding sequence is also exact [17], Theorem 23.7, p.19. Exchanging the roles of M and M^* in this dual exact sequence and noting that $(\mathbb{H}^0(K,M^*)^{\wedge})^D = (\mathbb{H}^0(K,M^*)^D)_{\text{tors}}$ and $(\mathbb{H}^{-1}(K,M^*)^{\wedge})^D = \mathbb{H}^{-1}(K,M^*)^D$ (since $\mathbb{H}^{-1}(K,M^*)$ is finitely generated by [7], Lemma 2.1, p.98), we obtain an exact sequence

$$
\begin{array}{ccccc}
\text{Ш}^1(M) & & & & \\
\big\downarrow & & & & \\
\mathbb{H}^1(K,M) & \longrightarrow & \mathbb{P}^1(M)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K,M^*)^D)_{\text{tors}} \\
 & & & & \downarrow \\
0 \longleftarrow \mathbb{H}^{-1}(K,M^*)^D & \longleftarrow & \displaystyle\bigoplus_{\text{all } v} \mathbb{H}^2(K_v,M) & \longleftarrow & \mathbb{H}^2(K,M).
\end{array}
$$

The sequence of the theorem may now be obtained by splicing together the preceding two exact sequences. □

4. The generalized Cassels-Tate dual exact sequence

For $i = 1$ or 2, define

$$
\text{Ш}^i(T(M)) = \operatorname{Ker}\left[H^i(K,T(M)) \to \prod_{\text{all } v} H^i(K_v,T(M)) \right]
$$

and

$$
\text{Ш}^i(M) = \operatorname{Ker}\left[\mathbb{H}^i(K,M) \to \prod_{\text{all } v} \mathbb{H}^i(K_v,M) \right],
$$

where the v-component of each of the maps involved is induced by the natural morphism $\operatorname{Spec} K_v \to \operatorname{Spec} K$.

Proposition 4.1. *There exists a perfect pairing*

$$
\text{Ш}^1(T(M^*)) \times \text{Ш}^2(M) \to \mathbb{Q}/\mathbb{Z},
$$

where the first group is profinite and the second is discrete and torsion.

Proof. The proof is similar to the proof of Proposition 3.3. □

Let S be any finite set of primes of K and define, for $i = 1$ or 2,

$$Ш^i_S(T(M)) = \operatorname{Ker}\left[H^i(K, T(M)) \to \prod_{v \notin S} H^i(K_v, T(M))\right]$$

and

$$Ш^i_S(M) = \operatorname{Ker}\left[\mathbb{H}^i(K, M) \to \prod_{v \notin S} \mathbb{H}^i(K_v, M)\right].$$

Thus $Ш^i_\emptyset(T(M)) = Ш^i(T(M))$ and $Ш^i_\emptyset(M) = Ш^i(M)$. Now partially order the family of finite sets S by defining $S \leq S'$ if $S \subset S'$. Then $Ш^1_S(M) \subset Ш^1_{S'}(M)$ for $S \leq S'$. Define

$$Ш^1_\omega(M) = \varinjlim_S Ш^1_S(M) = \bigcup_S Ш^1_S(M) \subset \mathbb{H}^1(K, M),$$

where the transition maps in the direct limit are the inclusion maps. Thus $Ш^1_\omega(M)$ is the subgroup of $\mathbb{H}^1(K, M)$ of all classes which are locally trivial at all but finitely many places of K. Clearly, for each S as above, there exists an exact sequence of discrete torsion groups

$$0 \to Ш^1(M^*) \to Ш^1_S(M^*) \to \prod_{v \in S} \mathbb{H}^1(K_v, M^*)$$

whose dual is an exact sequence of profinite groups

$$\prod_{v \in S} \mathbb{H}^0(K_v, M)^\wedge \xrightarrow{\widehat{\theta}_S} Ш^1_S(M^*)^D \to Ш^1(M^*)^D \to 0. \tag{8}$$

The map $\widehat{\theta}_S$ is given by

$$\widehat{\theta}_S((m_v))(\xi) = \sum_{v \in S} (m_v, \xi|_{K_v})_v$$

where, for each $v \in S$, $(-,-)_v$ is the pairing of [7], Theorem 2.3(2) and Proposition 2.9, and $\xi|_{K_v}$ is the image of $\xi \in Ш^1_S(M^*) \subset \mathbb{H}^1(K, M^*)$ in $\mathbb{H}^1(K_v, M^*)$ under the map induced by $\operatorname{Spec} K_v \to \operatorname{Spec} K$. We define $\widehat{\theta} = \varprojlim_S \widehat{\theta}_S \colon \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge \to Ш^1_\omega(M^*)^D$.

Proposition 4.2. *There exists a canonical exact sequence*

$$T\mathrm{Sel}(M)^\wedge \xrightarrow{\widehat{\phi}} \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge \xrightarrow{\widehat{\theta}} Ш^1_\omega(M^*)^D.$$

Further, the map $\widehat{\phi}$ factors as

$$T\mathrm{Sel}(M)^\wedge \xrightarrow{\beta_0} \mathbb{P}^0(M)^\wedge \to \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge,$$

where the second map is the canonical one.

Proof. The sequence of Proposition 3.10 is an exact sequence of profinite groups and strict morphisms, so its dual

$$\mathbb{H}^1(K, M^*) \to \mathbb{P}^1(M^*)_{\text{tors}} \xrightarrow{\beta_0^D} (T\mathrm{Sel}(M)^\wedge)^D$$

is also exact (cf. proof of Theorem 3.11). The above sequence induces an exact sequence of discrete groups

$$Ш^1_S(M^*) \to \prod_{v\in S} \mathbb{H}^1(K_v, M^*) \to (T\mathrm{Sel}(M)^\wedge)^D$$

whose dual is an exact sequence

$$T\mathrm{Sel}(M)^\wedge \to \prod_{v\in S} \mathbb{H}^0(K_v, M)^\wedge \xrightarrow{\widehat{\theta}_S} Ш^1_S(M^*)^D.$$

Taking the inverse limit over S above and noting that the inverse limit functor is exact on the category of profinite groups [16], Proposition 2.2.4, p.32, we obtain the exact sequence of the proposition. That $\widehat{\phi}$ has the stated factorization follows from the proof. □

Proposition 4.3. *There exists a canonical isomorphism*

$$\mathrm{Ker}\left[T\mathrm{Sel}(M)^\wedge \xrightarrow{\widehat{\phi}} \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge\right] = Ш^2(M^*)^D,$$

where $\widehat{\phi}$ is the map of Proposition 4.2.

Proof. Since $\mathrm{Ker}\,\beta_0 = \mathrm{Ker}\,\theta_0 = Ш^2(M^*)^D$ by Propositions 3.3 and 3.7, it suffices to check, by Proposition 4.2, that the canonical map $\mathbb{P}^0(M)^\wedge \to \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge$ is injective. The argument is similar to that used in the proof of Lemma 3.6. Let $x \in \mathbb{P}^0(M)^\wedge$ be nonzero. There exists an open subgroup $U \subset \mathbb{P}^0(M)$ of finite index n (say) such that the U-component of x, $x_U + U \in \mathbb{P}^0(M)/U$ is nonzero, i.e., $x_U \notin U$. Then $x_U \notin n\mathbb{P}^0(M)$, whence $(x_U)_v \notin n\mathbb{H}^0(K_v, M)$ for some v (see [7], Lemma 5.3, p.118). Thus the image of x in $\mathbb{H}^0(K_v, M)/n = (\mathbb{H}^0(K_v, M)/n)^\wedge$ (recall that $\mathbb{H}^0(K_v, M)/n$ is profinite) is nonzero. Since the map $\mathbb{P}^0(M)^\wedge \to (\mathbb{H}^0(K_v, M)/n)^\wedge$ factors through $\mathbb{H}^0(K_v, M)^\wedge$, the image of x in $\mathbb{H}^0(K_v, M)^\wedge$ is nonzero. □

As noted earlier, the inverse limit functor is exact on the category of profinite groups, so the inverse limit over S of (8) is an exact sequence

$$\prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge \xrightarrow{\widehat{\theta}} Ш^1_\omega(M^*)^D \to Ш^1(M^*)^D \to 0.$$

We now use Propositions 4.2 and 4.3 to extend the above exact sequence to the left. We obtain

Theorem 4.4. (The generalized Cassels-Tate dual exact sequence) *There exists a canonical exact sequence of profinite groups*

$$\begin{aligned} 0 \to Ш^2(M^*)^D &\to T\mathrm{Sel}(M)^\wedge \to \prod_{\text{all } v} \mathbb{H}^0(K_v, M)^\wedge \\ &\to Ш^1_\omega(M^*)^D \to Ш^1(M^*)^D \to 0. \quad \square \end{aligned}$$

Corollary 4.5. *There exists a canonical exact sequence of discrete torsion groups*

$$\begin{aligned} 0 \to Ш^1(M) &\to Ш^1_\omega(M) \to \bigoplus_{\text{all } v} \mathbb{H}^1(K_v, M) \\ &\to (T\mathrm{Sel}(M^*)^\wedge)^D \to Ш^2(M) \to 0. \quad \square \end{aligned}$$

We conclude this paper with the following result, which extends [8], Theorem 1.2, to the function field case.

Theorem 4.6. *Let K be a global function field and let M be a 1-motive over K. Assume that $Ш^1(M)$ is finite. Then there exists an exact sequence*

$$0 \to \overline{\mathbb{H}^0(K,M)} \to \prod_{\text{all } v} \mathbb{H}^0(K_v,M) \to Ш^1_\omega(M^*)^D \to Ш^1(M) \to 0,$$

where $\overline{\mathbb{H}^0(K,M)}$ is the closure of the image of $\mathbb{H}^0(K,M)$ in $\prod_{\text{all } v} \mathbb{H}^0(K_v,M)$ under the diagonal map.

Proof. The proof is essentially the same as that of [8], Theorem 1.2, noting that $T\,\mathrm{Sel}(M)^\wedge = \mathbb{H}^0(K,M)^\wedge$ if $Ш^1(M)$ is finite and using Proposition 4.2 in place of [8], Proposition 5.3(1). □

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