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Π_2 CONSEQUENCES OF BMM + NS_{ω_1} IS PRECIPITOUS AND THE SEMIPROPERNESS OF STATIONARY SET PRESERVING FORCINGS

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Abstract. We investigate which Π_2 sentences (over H_{ω_2}) that are consequences of MM also follow from $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. It turns out that admissible club guessing (acg), $\underset{\sim}{\delta}^1_2 = \omega_2$, the club bounding principle (CBP), and ψ_{AC} as well as ϕ_{AC} follow from this weaker theory. This was known for $\underset{\sim}{\delta}^1_2 = \omega_2$ and ψ_{AC} but not for ϕ_{AC} and acg. Additionally we show that if for all regular $\theta \geq \omega_2$ there is a semiproper partial ordering that adds a generic iteration of length ω_1 with last model H_θ, then all stationary set preserving forcings are semiproper.

1. Introduction

By NS_{ω_1} we denote the nonstationary ideal on ω_1. A V-generic G for the forcing $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}, \subset)$ is an ultrafilter on V that extends the club filter. Hence we can form the ultrapower $j : V \to \mathrm{Ult}(V, G)$ in $V[G]$. We will always assume the well-founded part of such an ultrapower to be transitive. Clearly j has critical point ω_1. If every condition $S \in \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$ forces that $\mathrm{Ult}(V, G)$ is well-founded, then we call NS_{ω_1} precipitous. Since the precipitousness of an ideal can be recast as a first order statment, the model $\mathrm{Ult}(V, G)$ has a precipitous nonstationary ideal if V has one. One can now pick a $\mathrm{Ult}(V, G)$-generic for $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}, \subset)^{\mathrm{Ult}(V,G)}$ and form another ultrapower. This leads to the notion of generic iterations.

Definition 1.1 Let M be a transitive model of ZFC^*+“ω_1 exists” and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ “I is a uniform and normal ideal on ω_1^M.” Let $\gamma \leq \omega_1$. Then

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle \in V$$

is called a *putative generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma + 1$) iff the following hold true.

1. (1) $M_0 = M$ and $I_0 = I$.
2. (2) For all $i \leq j \leq \gamma$, $\pi_{i,j} \colon \langle M_i; \in, I_i\rangle \to \langle M_j; \in, I_j\rangle$ is elementary, $I_i = \pi_{0,i}(I)$, and $\kappa_i = \pi_{0,i}(\omega_1^M) = \omega_1^{M_i}$.
3. (3) For all $i < \gamma$, M_i is transitive and G_i is $(\mathcal{P}(\kappa_i) \setminus I_i, \subset)$-generic over M_i.
4. (4) For all $i + 1 \leq \gamma$, $M_{i+1} = \mathrm{Ult}(M_i; G_i)$ and $\pi_{i,i+1}$ is the associated ultrapower map.
5. (5) $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$ for $i \leq j \leq \gamma$.
6. (6) If $\lambda \leq \gamma$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}, i < \lambda\rangle$ is the direct limit of $\langle M_i, \pi_{i,j}, i \leq j < \lambda\rangle$.

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We call

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

a *generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma+1$) iff it is a putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma+1$ and M_γ is transitive. $\langle M; \in, I\rangle$ is *generically* $\gamma+1$ *iterable* iff every putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma+1$ is an iteration.

The theory ZFC^* is defined in [10, 3.1]. Notice that we want (putative) iterations of a given model $\langle M; \in, I\rangle$ to exist in V, which amounts to requiring that the relevant generics G_i may be found in V.

In [2] the notion of forcing $\mathbb{P}(\theta, \mathsf{NS}_{\omega_1})$ was defined for regular $\theta \geq \omega_2$. Granted the precipitousness of nonstationary ideal NS_{ω_1} the forcing is nonempty and preserves stationary subsets of ω_1. Forcing with $\mathbb{P}(\theta, \mathsf{NS}_{\omega_1})$ adds a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that all M_i with countable index are countable and the last model M_{ω_1} equals H_θ of V. Here I_i is M_i's nonstationary ideal and the $\kappa_i = \omega_1^{M_i}$ are the critical points of the generic ultrapowers $\pi_{i,i+1} : M_i \to M_{i+1} \simeq \mathrm{Ult}(M_i, G_i)$. It is also possible to produce iterations as above with generically iterable M_0. This fact is used in [2] to show that $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous implies $\underset{\sim}{\delta}^1_2 = \omega_2$. Note that $\underset{\sim}{\delta}^1_2 = \omega_2$ is a Π_2 statment in H_{ω_2}. In this paper we use generic iterations as above to analyse which Π_2 sentences in H_{ω_2} that are consequences of $\mathsf{ZFC} + \mathsf{MM}$ are also consequences of the weaker theory $\mathsf{ZFC} + \mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. Note that MM implies that NS_{ω_1} is ω_2-saturated [3] but by [10, 10.103, 10.99] $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous does not[1]. We consider two Π_2 statements in H_{ω_2}. Both are known to hold in H_{ω_2} if MM holds.

Definition 1.2

(1) We call the following principle *admissible club guessing* (acg). For all clubs $C \subseteq \omega_1$ there exists a real x such that

$$A_x := \{\alpha < \omega_1\,;\, L_\alpha[x] \text{ is admissible}\} \subset C.$$

(2) Let $S \subset \omega_1$. Then we set

$$\tilde{S} := \{\alpha < \omega_2\,;\, \omega_1 \leq \alpha \wedge \mathbf{1}_{\mathbb{B}} \Vdash \check{\alpha} \in j(\check{S})\},$$

where $\mathbb{B} = \mathrm{ro}(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})$ and j is a name for the corresponding generic elementary embedding $V \to (M, E) \subset V^{\mathbb{B}}$. Note that $\alpha \in \tilde{S}$ if and only if for all (one) canonical function(s) f_α for α, there is a club C such that if $\beta \in C$ then $f_\alpha(\beta) \in S$.

Let $\vec{S} = \langle S_i\,;\, i \in \omega\rangle$, $\vec{T} = \langle T_i\,;\, i \in \omega\rangle$ be sequences of pairwise disjoint subsets of ω_1, such that all S_i are stationary and

$$\omega_1 = \bigcup\{T_i\,;\, i \in \omega\}.$$

$\varphi_{AC}(\vec{S}, \vec{T})$ is the conjunction of the following two statements:

[1]In the situation of [10, 10.103] one considers a ${}^2\mathbb{P}_{\max}$ extension; there NS_{ω_1} is not saturated but one can check that it is precipitous using the ${}^2\mathbb{P}_{\max}$ analysis in [10, 6.14].

(a) There is an ω_1 sequence of distinct reals.[2]
(b) There is $\gamma < \omega_2$ and a continuous increasing function $F : \omega_1 \to \gamma$ with range cofinal in γ such that for all $i \in \omega$

$$F``T_i \subset \tilde{S}_i.$$

$\varphi_{AC}(\vec{S}, \vec{T})$ is clearly $\Sigma_1(\{\vec{S}, \vec{T}\})$ in $\langle H_{\omega_2}; \in\rangle$. We set

$$\phi_{AC} :\equiv \forall \vec{S} \forall \vec{T} \varphi_{AC}(\vec{S}, \vec{T}).$$

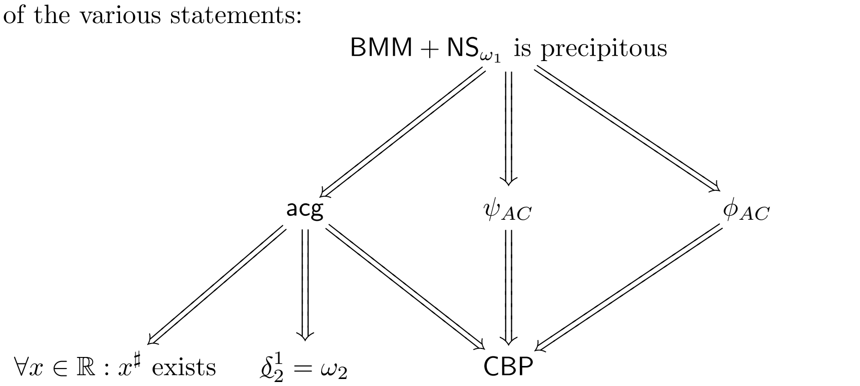
Note that ϕ_{AC} is equivalent to a Π_2 statement in $\langle H_{\omega_2}; \in\rangle$.

Remark 1.3 The principle acg was isolated by Woodin. If MM holds, then the universe is closed under the sharp operation (this is already a consequence of BMM). So by [10, 3.17] $\underset{\sim}{\delta}^1_2 = \omega_2$ and hence by [10, 3.16, 3.19] acg holds.
The axiom ϕ_{AC} is due to Woodin. By [10, 5.9] MM implies ϕ_{AC}. Note that by an observation of Larson $\mathsf{MM}(c)$ already suffices, see [10, p.200].

We now state our results.

Theorem 1.4 *If* BMM *holds and additionally* NS_{ω_1} *is precipitous, then* acg *and* ϕ_{AC} *hold.*

We will prove the above theorem using (a variant of) $\mathbb{P}(\theta, \mathsf{NS}_{\omega_1})$. The technology developed to show ϕ_{AC} can also be used to yield ψ_{AC}. We sketch such a construction only since Woodin has shown that $\mathsf{BMM}+\mathsf{NS}_{\omega_1}$ is precipitous implies ψ_{AC} using more straightforward methods, see [10, 10.95]. The following diagram illustrates the logical structure of the various statements:



Here CBP is the club bounding principle, i.e. the statement that every function $f : \omega_1 \to \omega_1$ is bounded by a canonical function for some ordinal $< \omega_2$ on a club. The implication from ψ_{AC} to CBP is due to Aspero and Welch, see [1]. The implication from ϕ_{AC} to CBP follows easily from the next lemma. All implications from acg are due to Woodin, see [10, (proof of) 3.19].

Lemma 1.5 (Folklore?) *The following are equivalent:*

[2] We are working in models of ZFC so this will trivially hold. It is more interesting if working in models of $\mathsf{ZF} + \mathsf{DC}$.

(1) CBP
(2) *For every club $C \subset \omega_1$ there is some $\alpha \in \tilde{C}$ such that $\omega_1 < \alpha$.*

Note that $\tilde{C}$ contains ω_1 if and only if C is club.

Proof. We assume CBP. Let $C \subset \omega_1$ be club. Inductively we construct a sequence $\langle f_i; i < \omega \rangle$ and a sequence $\langle \alpha_i; i < \omega \rangle$ of ordinals $< \omega_2$ such that $\mathrm{ran}(f_i) \subset C$ and such that there is a club D_i such that $f_i(\xi) < f_{\alpha_i}(\xi) < f_{i+1}(\xi)$ for all $\xi \in D_i$ where f_{α_i} is a canonical function for α_i. Set $f_0(\xi) = \min(C \setminus (\xi + 1))$ for $\xi < \omega_1$. Then by CBP there is some $\alpha_0 < \omega_2$, a canonical function f_{α_0} for α_0 and a club D_0 such that $f_{\alpha_0}(\xi) > f_0(\xi)$ for $\xi \in D_0$. In the induction step set $f_{i+1}(\xi) = \min(C \setminus (f_{\alpha_i}(\xi) + 1)$ for $\xi < \omega_1$. Note that by the choice of f_0 every α_i is $> \omega_1$. Set $D = \cap_{i<\omega} D_i$. Then for all $\xi \in D$

$$\sup_{i<\omega} f_i(\xi) = \sup_{i<\omega} f_{\alpha_i}(\xi) \in C,$$

since $\mathrm{ran}(f_i) \subset C$ by construction. It is easy to see that $f : \omega_1 \to \omega_1$; $f(\xi) := \sup_{i<\omega} f_{\alpha_i}(\xi)$ is a canonical function for $\alpha := \sup_{i<\omega} \alpha_i$. Hence $f(\xi) \in C$ for all $\xi \in D$ which instantly yields $\alpha \in \tilde{C}$.
It remains to show the converse; let $f : \omega_1 \to \omega_1$ be a function and let $C := \{\beta < \omega_1 \,;\, f``\beta \subset \beta\}$. By the hypothesis there is an $\alpha > \omega_1$ such that $\alpha \in \tilde{C}$. Unraveling the definition of $\tilde{C}$ yields a club D and a canonical function $f_\alpha : \omega_1 \to \omega_1$ for α such that $f_\alpha(\beta) \in C$ for all $\beta \in D$. Since $\alpha > \omega_1$ the set of points where $\beta \geq f_\alpha(\beta)$ is nonstationary. Hence we can assume without loss of generality that $f_\alpha(\beta) > \beta$ for $\beta < \omega_1$. If $\beta \in D$, then $f_\alpha(\beta) \in C$. Hence $f``f_\alpha(\beta) \subset f_\alpha(\beta) > \beta$ for $\beta \in D$. So especially $f(\beta) < f_\alpha(\beta)$ for all $\beta \in D$. □

The second part of this paper deals with the semiproperness of $\mathbb{P}(\theta, \mathsf{NS}_{\omega_1})$ for all regular $\theta \geq \omega_2$ (or more general the semiproperness of any class of forcings that adds generic iterations like above). We will show:

Theorem 1.6 *The following are equivalent:*

(1) *For arbitrarily large $\theta \geq \omega_2$ there is a semiproper partial order $\mathbb{P}$ that adds a generic iteration*

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle.$$

such that $H_\theta \subset M_{\omega_1}$ and all M_i are countable.
(2) *All stationary set preserving forcings are semiproper.*

2. The principle acg

In this section, we shall clean up [2] by showing the following.

Lemma 2.1 $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ *is precipitous* $\implies$ acg.

Proof. Fix some club C. We show that admissible club guessing holds under BMM if the nonstationary ideal is precipitous. The forcing $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ from [2] adds a countable generically iterable M_0 generically iterating in ω_1^V many steps to $\langle (H_{\omega_2}^V)^\sharp, \in, \mathsf{NS}_{\omega_1}\rangle$[3], i.e. an iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle.$$

Note that $(H_{\omega_2}^V)^\sharp$ exists because BMM holds, see [6] or [7]. For brevity we write π_α instead of π_{α,ω_1}. So there is some $\alpha_0 < \omega_1$ such that $C \cap \omega_1^{M_{\alpha_0}} \in M_{\alpha_0}$ and $\pi_{\alpha_0}(C \cap \omega_1^{M_{\alpha_0}}) = C$. We can assume w.l.o.g. by changing some indices that $0 = \alpha_0$. We now show that in the extension by $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ there is a real y such that $A_y \subset C$. Let x be a real that codes M_0 and let y code $x^\sharp$.
Writing $C_\alpha = C \cap \omega_1^{M_\alpha}$ we have $C_\alpha \in M_\alpha$ and $\pi_\alpha(C_\alpha) = C$ for all $\alpha < \omega_1$. By elementarity, C_α is unbounded in $\omega_1^{M_\alpha}$. So by the closedness of C we have $\omega_1^{M_\alpha} \in C$.

Claim 1. If α is an x-indiscernible and

$$\langle\langle M_i', \pi_{i,j}', I_i', \kappa_i'; i \leq j \leq \alpha\rangle, \langle G_i'; i < \alpha\rangle\rangle$$

is an *arbitrary* generic iteration of $M = {M_0}'$ then $\alpha = \omega_1^{M_\alpha'}$.

Proof of Claim 1. First note that M is generically $\omega_1 + 1$ iterable, by Theorem 18 of [2]. Fix an x-indiscernible α and an iteration as above. Every x-indiscernible is inacessible in $L[x]$, so for all $\beta < \alpha$

$$L[x]^{\mathrm{Col}(\omega,\beta)} \models \alpha \text{ is inacessible.}$$

Let $g \subset \mathrm{Col}(\omega, \beta)$ be $L[x]$-generic. Assume w.l.o.g. that g is a real. Then, by [10, 3.15] (compare Lemma 19 in [2]), $M_\beta' \cap OR < \omega_1^{L[x,g]}$. Hence $\omega_1^{M_\beta'} < \alpha$. This implies $\omega_1^{M_\alpha'} \leq \alpha$. So it follows easily that $\omega_1^{M_\alpha'} = \alpha$. □(Claim 1)

If α is $x^\sharp$-admissible, then α is x-indiscernible. Hence by the above claim it follows that each y-admissible $< \omega_1$ is in C. Hence $A_{x^\sharp} \subset C$. Since the existence of a real y such that $A_y \subset C$ can be recast as a Σ_1-statement over H_{ω_2} with C as a parameter, BMM implies that it is already true in V. □

3. Obtaining ϕ_{AC}

We modify the forcing $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ from [2] to show an arbitrary instance of ϕ_{AC} in the generic extension. An application of BMM will then give us the desired result.

3.1. Hitting many regular cardinals. The following lemma states that for a generically iterable $\langle M, I\rangle$ there is a generic iteration that realizes many regular cardinals.

[3] For a set X we denote by $X^\sharp$ the least X-mouse. Note that the universe of $X^\sharp$ is a model of ZFC^*+"ω_1 exists." See (the proof of) [2, Theorem 18] for more details on sharps and generic iterability.

Lemma 3.1 (Hitting many regular cardinals lemma) *Let $\langle M, I\rangle$ be a countable model of ZFC^*+ "ω_1 exists" and let I be a precipitous ideal on ω_1^M. Assume that $\mathcal{P}(\mathcal{P}(\omega_1))$ exists in M. Let $\theta, \alpha \in M$ be such that*

$$M \models (2^{2^{\omega_1}})^+ = \theta = \aleph_\alpha,$$

furthermore assume that

$$M \models (\aleph_{\alpha+\omega_1})^M \text{ exists.}$$

Let $\theta' := (\aleph_{\alpha+\omega_1})^M$. Then a genericity iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta'\rangle, \langle G_i; i < \theta'\rangle\rangle$$

of $M_0 = M$ exists such that for all $\beta < \omega_1^M$

$$\pi_{0,\aleph^M_{\alpha+\beta+1}}(\omega_1^M) = \aleph^M_{\alpha+\beta+1}.$$

Proof. Let $g \subset \mathrm{Col}(\omega, <\theta')$ be generic over M. Since M is countable in V the generic g can be chosen in V. Let $\mathbb{P} := \mathcal{P}(\omega_1^M)^M \setminus I$. For $\beta < \omega_1^M$ we set

$$g_{\alpha+\beta+1} := g \cap \mathrm{Col}(\omega, < \aleph^M_{\alpha+\beta+1}).$$

Clearly all the g_i defined in this fashion are generic over M. Recursively we construct a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta'\rangle, \langle G_i; i < \theta'\rangle\rangle$$

such that for $\beta < \omega_1^M$ the sequence $\langle G_i; i < \aleph^M_{\alpha+\beta+1}\rangle$ is in $M[g_{\alpha+\beta+1}]$. We inductively maintain the following:

- For $\beta < \omega_1^M$ and $i < \aleph^M_{\alpha+\beta+1}$ the set
$$D_i = \{d \in M_i\,;\, d \subset \pi_{0,i}(\mathbb{P}) \wedge M_i \models d \text{ is dense in } \pi_{0,i}(\mathbb{P})\}$$
is countable in $M[g_{\alpha+\beta+1}]$.

Set $M_0 = M$, $I_0 = I$ and $\kappa_0 = \omega_1^M$. Assume we are at stage $i < \theta'$ of the construction. Let $\beta < \omega_1^M$ be least such that $i < \aleph^M_{\alpha+\beta+1}$. Inductively we have that D_i is countable in $M[g_{\alpha+\beta+1}]$. Choose a D_i-generic G_i in $M[g_{\alpha+\beta+1}]$. At limit stages form direct limits.

Let us check our inductive hypotheses in the successor case, the limit case being an easy consequence of the fact that the sequence $\langle G_i; i < \aleph^M_{\alpha+\beta+1}\rangle$ is in $M[g_{\alpha+\beta+1}]$. For the successor case note that an appropriate hull of

$$\pi_{0,i+1}``(H_\theta)^{M_0} \cup \{\kappa_j\,;\, j < i+1\}$$

is $(H_{\theta_{i+1}})^{M_{i+1}}$ where $\theta_{i+1} = \pi_{0,i+1}(\theta)$. This hull can be calculated in $M[g_{\alpha+\beta+1}]$. Hence $D_{i+1} \subset (H_{\theta_{i+1}})^{M_{i+1}}$ is also countable in $M[g_{\alpha+\beta+1}]$. It is trivial to maintain that the sequence $\langle G_j; j < i+1\rangle$ is in $M[g_{\alpha+\beta+1}]$.

Now we need that $\aleph^M_{\alpha+\beta+1}$ is regular in M. Hence

$$\omega_1^{M[g_{\alpha+\beta+1}]} = \aleph^M_{\alpha+\beta+1}.$$

So an easy calculation shows that for all $\beta < \omega_1^M$

$$\pi_{0,\aleph^M_{\alpha+\beta+1}}(\omega_1^M) = \aleph^M_{\alpha+\beta+1}.$$

□

Clearly the previous lemma can be generalized further. Since we only need the case above, we refrained to state it in a more general fashion. Note that we have a lot of freedom when choosing the generics of the iteration; the only true restriction is that they come from small generic extensions. We will make use of this later.
We define a set of ordinals relative to a generic iteration. This set will come in handy in the proof of the main result of this section.

Definition 3.2 Let $\langle M, I\rangle$ be a model of $\mathsf{ZFC}^* +$ "ω_1 exists," such that $M \models I$ is precipitous. Let θ be a cardinal in M. Let

$$\mathcal{J} := \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \rho\rangle, \langle G_i; i < \rho\rangle\rangle$$

be a generic iteration of $\langle M_0, I_0\rangle = \langle M, I\rangle$. We inductively define the *important ordinals of $\mathcal{J}$ relative to θ.*

(1) 0 is an important ordinal.
(2) If α is an important ordinal then the least ordinal γ such that $\pi_{0,\alpha}(\theta) \leq \gamma = \kappa_\gamma$ is the next important ordinal.
(3) Limits of important ordinals are important.

Remark 3.3 Let $\langle M, I\rangle$ be countable and as in the previous definition and let $\mathcal{J}$ as in the previous defintion and $\rho = \omega_1$. Then clearly the set of important ordinals of $\mathcal{J}$ relative to θ is a club in ω_1. Also, if α is important, then $\kappa_\alpha = \alpha$.

3.2. Forcing ϕ_{AC}. We will show the following theorem:

Theorem 3.4 *Let $\aleph_\alpha = 2^{2^{\omega_1}}$. Let $\theta := \aleph_{\alpha+\omega_1}$. Let NS_{ω_1} be precipitous and suppose $H_\theta^\sharp$ exists. Let $F : \omega_1 \to \theta$ defined by*

$$F(\beta) = \aleph_{\alpha+\beta+1}.$$

Let $\vec{S} = \langle S_k\,;\, k \in \omega\rangle$, $\vec{T} = \langle T_k\,;\, k \in \omega\rangle$ be sequences of pairwise disjoint subsets of ω_1, such that all S_k are stationary and $\omega_1 = \bigcup\{T_k\,;\, k \in \omega\}$. There exists a forcing construction $\mathbb{P} = \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$ that preserves stationary subsets of ω_1 such that if G is $\mathbb{P}$-generic over V, then in $V[G]$ there is generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$. In particular, M_0 is generically ω_1-iterable. Additionally the following holds in $V[G]$ for all $k \in \omega$:

$$F``T_k \subset \tilde{S}_k.$$

We use a similar setup as [2], i.e. we assume:

$$\theta = 2^{<\theta} < 2^\theta < \rho = 2^{<\rho},$$

for some cardinal ρ. For reasons of convenience we like to think of $\aleph_\alpha = 2^{2^{\omega_1}}$ as $\aleph_3$. This eases notation considerably. Note that we can force $\aleph_3 = 2^{2^{\omega_1}}$ with stationary set preserving forcing. If $2^{\omega_1} = \aleph_2$, then the precipitousness of NS_{ω_1} is preserved by forcing with $\mathrm{Col}(\omega_3, 2^{2^{\omega_1}})$, since no new subsets of 2^{ω_1} are added, see [4, 22.19]. Nevertheless the reader will gladly verify that all of the following arguments go through

for an arbitrary $\aleph_\alpha$ instead of $\aleph_3$. If $\aleph_\alpha = \aleph_3$, then clearly $\theta = \aleph_{\omega_1}$.
At this point a remark is in order. In [2] θ is supposed to be regular. Nevertheless it is straightforward to check that if one can add generic iterations like in in [2] with last model H_η for arbitrarily large regular η you can also add generic iterations with last model H_θ. We can hence work with a singular θ and use the theory of [2].
Fix a well-order $<$ of H_ρ as in [2]. We now fix $\vec{S} = \langle S_k\,;\, k \in \omega\rangle$, $\vec{T} = \langle T_k\,;\, k \in \omega\rangle$ sequences of pairwise disjoint subsets of ω_1, such that all S_k are stationary and $\omega_1 = \bigcup\{T_k\,;\, k \in \omega\}$. We use

$$\mathcal{H} = \langle H_\rho; \in, H_\theta^\sharp, \mathsf{NS}_{\omega_1}, <\rangle$$

and

$$\mathcal{M} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}, <\rangle$$

since we are defining a variant of $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1})$. We will now define our modified forcing construction $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$.

Definition 3.5 Conditions p in $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$ are triples

$$p = \langle\langle \kappa_i^p; i \in \mathrm{dom}(p)\rangle, \langle \pi_i^p; i \in \mathrm{dom}(p)\rangle, \langle \tau_i^p; i \in \mathrm{dom}_-(p)\rangle\rangle$$

such that the following conditions hold:

(1) Both $\mathrm{dom}(p)$ and $\mathrm{dom}_-(p)$ are finite, and $\mathrm{dom}_-(p) \subset \mathrm{dom}(p) \subset \omega_1$.
(2) $\langle \kappa_i^p\,;\, i \in \mathrm{dom}(p)\rangle$ is a sequence of countable ordinals.
(3) $\langle \pi_i^p; i \in \mathrm{dom}(p)\rangle$ is a sequence of finite partial maps from ω_1 to $H_\theta^\sharp \cap \mathsf{OR}$.
(4) $\langle \tau_i^p; i \in \mathrm{dom}_-(p)\rangle$ is a sequence of complete $\mathcal{H}$-types over H_θ, i.e., for each $i \in \mathrm{dom}_-(p)$ there is some $x \in H_\rho$ such that, having φ range over $\mathcal{H}$-formulae with free variables $u, \vec{v}$,

$$\tau_i^p = \{\langle \ulcorner\varphi\urcorner, \vec{z}\rangle\ ;\ \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[x, \vec{z}]\}.$$

(5) If $i, j \in \mathrm{dom}_-(p)$, where $i < j$, then there is some $n < \omega$ and some $\vec{u} \in \mathrm{ran}(\pi_j^p)$ such that

$$\tau_i^p = \{(m, \vec{z})\ ;\ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau_j^p\}.$$

(6) In $V^{\mathrm{Col}(\omega,\theta)}$, there is a *model which certifies p with respect to* $\mathcal{M}$, i.e. a model $\mathfrak{A}$ such that $H_\theta^\sharp \in \mathrm{wfp}(\mathfrak{A})$, $\mathfrak{A} \models \mathsf{ZFC}^-$, for all stationary $S \subset \omega_1$in V we have $\mathfrak{A} \models$ "S is stationary", and inside $\mathfrak{A}$ there is a generic iteration

$$\mathcal{J}^{\mathfrak{A}} := \langle\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, I_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \le j \le \omega_1\rangle, \langle G_i^{\mathfrak{A}}; i < \omega_1\rangle\rangle$$

such that
   (a) if $i < \omega_1$, then $M_i^{\mathfrak{A}}$ is countable,
   (b) if $i < \omega_1$ and if $\xi < \theta$ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$, then $\xi \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,
   (c) $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$,
   (d) if $i \in \mathrm{dom}(p)$, then $\kappa_i^p = \kappa_i^{\mathfrak{A}}$ and $\pi_i^p \subset \pi_{i,\omega_1}^{\mathfrak{A}}$,
   (e) if $i \in \mathrm{dom}_-(p)$, then for all $n < \omega$ and for all $\vec{z} \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

$$\exists y \in H_\theta\ (n, y^\frown \vec{z}) \in \tau_i^p \Longrightarrow \exists y \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\ (n, y^\frown \vec{z}) \in \tau_i^p.$$

(f) Let $D^{\mathfrak{A}}$ be the set of important ordinals of $\mathcal{J}^{\mathfrak{A}}$ relative to $(\pi^{\mathfrak{A}}_{0,\omega_1})^{-1}(\theta)$. If $\gamma \in D^{\mathfrak{A}}$ then for all $\beta < \gamma = \kappa^{\mathfrak{A}}_\gamma$ and all $k \in \omega$.

$$\aleph^{M^{\mathfrak{A}}_\gamma}_{3+\beta+1} \in S_k \iff \beta \in T_k.$$

If $p,\ q \in \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$, then we write $p \leq q$ iff $\mathrm{dom}(q) \subset \mathrm{dom}(p)$, $\mathrm{dom}_-(q) \subset \mathrm{dom}_-(p)$, for all $i \in \mathrm{dom}(q)$, $\kappa^p_i = \kappa^q_i$ and $\pi^q_i \subset \pi^p_i$, and for all $i \in \mathrm{dom}_-(q)$, $\tau^q_i = \tau^p_i$.

Note that the only difference between the above forcing $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1})$ in [2] is condition (6)(f) which is not required to hold for elements of $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1})$.
We now show Theorem 3.4. First we show that $\mathbb{P} := \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T}) \neq \emptyset$, i.e. the analog of Lemma 5 in [2]. Then we proceed as in [2] but we will skip all lemmata and theorems that are literally the same and have literally the same proof.

Lemma 3.6 $\mathbb{P} \neq \emptyset$.

Proof. We need to verify, that in $V^{\mathrm{Col}(\omega,\theta)}$ there is a model which certifies the trivial condition with respect to $\mathcal{M}$. Let g be $\mathrm{Col}(\omega, <\rho)$-generic over V. We work in $V[g]$ until further notice. So $\langle V; \in, \mathsf{NS}_{\omega_1}\rangle$ is $\rho+1$ iterable, by [10, 3.10, 3.11]. Hence $\langle H^\sharp_\theta; \in, \mathsf{NS}_{\omega_1}\rangle$ is also $\rho+1$ iterable. We prepare a book-keeping device: pick a bijection $g : [\rho]^{<\rho} \to \rho$ and a family $\langle U_\nu, \nu < \rho\rangle$ of pairwise disjoint stationary subsets of ρ. Now define $f : \rho \to [\rho]^{<\rho}$ by

$$f(i) = u \iff i \in U_{g(u)}.$$

Note that each u is enumerated stationarily often. We recursively construct a generic iteration

$$\mathcal{J} := \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \rho\rangle, \langle G_i; i < \rho\rangle\rangle$$

of $M_0 = \langle V; \in, \mathsf{NS}_{\omega_1}\rangle$ togethter with a set of local generics g_i. Later the restriction of this iteration to $\langle H^\sharp_\theta; \in, \mathsf{NS}_{\omega_1}\rangle$ will be of interest. For each important ordinal of the iteration a local generic g_i will be picked. Suppose we have already constructed $\mathcal{J}$ to some $i < \rho$. Note that we can calculate the important ordinals of $\mathcal{J}$ relative to θ while we construct $\mathcal{J}$. The following three clauses define the iteration.

(1) If i is an important ordinal of $\mathcal{J}$ relative to θ, then pick some $g_i \subset \mathrm{Col}(\omega, < \pi_{0,i}(\theta))$ in $V[g]$ that is generic over M_i. Then pick G_i in $M_i[g_i]$ such that if for a (unique) j the set $\pi_{j,i}(f(i))$ is a stationary subset of $\omega_1^{M_i}$ in M_i then $\pi_{j,i}(f(i)) \in G_i$. Note that j is unique because $f(i)$ can only be stationary in M_j if $\sup f(i) = \omega_1^{M_j}$.
(2) If i is not important and γ is the largest important ordinal below i, then we already have chosen some $g_\gamma \subset \mathrm{Col}(\omega, < \pi_{0,\gamma}(\theta))$ in $V[g]$ that is generic over M_γ. In the case that $i = \omega^{M_\gamma}_{3+\beta+1}$ for some $\beta < \kappa_\gamma = \gamma$ we pick some G_i in $M_\gamma[g_\gamma \cap \mathrm{Col}(\omega, < \omega^{M_\gamma}_{3+\beta+2})]$ such that

$$\beta \in \pi_{0,\gamma}(T_k) \iff \pi_{0,i}(S_k) \in G_i.$$

Note that since $\vec{T}$ is a partition of ω_1, there is a unique k such that $\beta \in \pi_{0,i}(T_k)$.

(3) If the first and second clause do not hold and γ is the largest important ordinal below i, then we already have chosen some $g_\gamma \subset \mathrm{Col}(\omega, < \pi_{0,\gamma}(\theta))$ in $V[g]$ that is generic over M_γ. In the case that i is not a successor cardinal $< \pi_{0,\gamma}(\theta)$ in M_γ there is a least $\beta < \kappa_\gamma$ such that $i < \omega_{3+\beta+1}^{M_\gamma}$. We pick some arbitrary G_i in $M_\gamma[g_\gamma \cap \mathrm{Col}(\omega, < \omega_{3+\beta+1}^{M_\gamma})]$. Else we pick a completely arbitrary generic.

Fix some important $\gamma > 0$. So $\mathcal{J}$ restricted to $[\gamma, \pi_{0,\gamma}(\theta)[$ is an iteration like in the "Hitting many regular cardinals lemma" 3.1. Hence we know that the iteration is well defined and additionally we have for $\beta < \kappa_\gamma = \gamma$ and $i := \aleph_{3+\beta+1}^{M_\gamma}$

$$i = \pi_{\gamma,i}(\kappa_\gamma) = \kappa_i.$$

By the second clause of the iteration we hence have for i as above and $k \in \omega$:

$$\beta \in \pi_{0,\gamma}(T_k) \iff \pi_{0,i}(S_k) \in G_i \iff \kappa_i \in \pi_{0,i+1}(S_k) \iff i \in \pi_{0,\rho}(S_k).$$

Let D denote the club of important ordinals and let U be a stationary subset of $\omega_1^{M_\rho} = \rho$. Let $j < \rho$ and u be such that $\pi_{j,\rho}(u) = U$. If $i \in D \setminus j$ and $f(i) = u$, then $\pi_{j,i}(u) \in G_i$. This shows that

$$D \cap U_{g(u)} \setminus j \subset \{i < \rho\, ;\, \kappa_i \in U\},$$

so that in fact U is stationary in $V[g]$.
Hence in $M_\rho^{\mathrm{Col}(\omega,\pi_{0,\rho(\theta)})}$ there is a model that certifies the empty condition with respect to $\pi_{0,\rho}(\langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle)$. Now we can literally complete our proof by following the last paragraph of the proof Lemma 5 in [2]. $\square$

We can now literally adopt lemmata 6 through 15 of [2]. So we have, using the notation of [2]:

Lemma 3.7 *Let $G \subset \mathbb{P}$ is V-generic. Let $\kappa_i = \kappa_i^p$ for some $p \in G$. Then in $V[G]$*

$$H_\theta^\sharp \cap \mathsf{OR} = \cup\{\mathrm{ran}(\pi_i)\, ;\, i < \omega_1\}$$

and

$$\mathcal{J}_G := \langle\langle M_i, \tilde{\pi}_{i,j}, I_i, \kappa_i; i \le j \le \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

is a generic iteration of M_0 such that if $i < \omega_1$, then M_i is countable, and $M_{\omega_1} = \langle H_\theta^\sharp; \in, I\rangle$.

Let D_G denote the important ordinals of $\mathcal{J}_G$. We can assume without loss of generality that there are $\vec{s}$, $\vec{t} \in M_0$ such that $\tilde{\pi}_{0,\omega_1}(\langle \vec{s}, \vec{t}\rangle) = \langle \vec{S}, \vec{T}\rangle$.

Lemma 3.8 *D_G is club and for all $\gamma \in D_G$ the following holds: if $\beta < \kappa_\gamma$ then for all $k \in \omega$*

$$\beta \in \pi_{0,\gamma}(t_k) \iff \aleph_{3+\beta+1}^{M_\gamma} \in \pi_{0,\omega_1}(s_k),$$

which by the choice of $\vec{s}$ and $\vec{t}$ means

$$\beta \in T_k \iff \aleph_{3+\beta+1}^{M_\gamma} \in S_k$$

Proof. That D_G is club is obvious.

Claim 1. $p \Vdash \check{\gamma} \in D_{\dot{G}}$ if and only if for all $\mathfrak{A}$ which certify p, $\gamma \in D^{\mathfrak{A}}$.

Proof of Claim 1. Fix p such that $p \Vdash \check{\gamma} \in D_{\dot{G}}$ and some structure $\mathfrak{A}$ which certifies p. Towards a contradiction suppose $\gamma \notin D^{\mathfrak{A}}$. Then there is some $\gamma' < \gamma$, $\gamma' \in D^{\mathfrak{A}}$ with

$$(\pi^{\mathfrak{A}}_{\gamma',\omega_1})^{-1}(\theta) > \gamma.$$

We can extend p to p' also certified by $\mathfrak{A}$ such that $\mathrm{dom}(p')$ contains all the relevant points. Then

$$p' \Vdash \check{\gamma} \notin D_{\dot{G}}.$$

Contradiction! The other direction is easy. □(Claim 1)

Now if $\beta \in \pi_{0,\gamma}(t_k)$ and $\gamma \in D_G$ there is some $p \in G$ with $p \Vdash \check{\gamma} \in D_{\dot{G}}$ and $\beta \in (\pi^p_\gamma)^{-1} \circ \pi^p_0(t_k)$ (Note the following subtlety: π^p_0 is only defined on the ordinals, but using the well ordering $<$ on $H^\sharp_\theta$ we can assume that $\mathrm{dom}(\pi^p_0)$ contains t_k). Let $p' \le p$ be arbitrary and let $\mathfrak{A}$ certify p'. Then $\aleph^{M^{\mathfrak{A}}_\gamma}_{3+\beta+1} \in S_k$ by the above claim and the fact that $\mathfrak{A}$ certifies p'. So we may extend p' to p'' making sure

$$p'' \Vdash \aleph^{M_\gamma}_{3+\beta+1} \in \tilde{\pi}_{0,\omega_1}(s_k).$$

Hence the set of p'' forcing the desired result is dense below p. The other direction is similar. □

We can now literally adopt lemmata 16 and 17 of [2] and their proofs; i.e. it is clear that $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ is stationary set preserving.

To finish the proof of 3.4 we have to show that in $V[G]$ for all $k \in \omega$

$$F``T_k \subset \tilde{S}_k.$$

For this fix $k \in \omega$ and some $\beta \in T_k$. By 3.8 we have for all $\gamma \in D_G \setminus (\beta+1)$

$$\beta \in T_k \iff \aleph^{M_\gamma}_{3+\beta+1} \in S_k.$$

Lemma 3.9 *The function $f : D_G \setminus (\beta+1) \to \omega_1$*

$$\gamma \mapsto \aleph^{M_\gamma}_{3+\beta+1}$$

is a canonical function for $\aleph^V_{3+\beta+1} < \omega_2^{V[G]}$ in $V[G]$.

Proof. Let

$$\mathcal{J}_G := \langle\langle M_i, \tilde{\pi}_{i,j}, I_i, \kappa_i; i \le j \le \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

denote the iteration that is added by G. Set $\eta := \aleph^V_{3+\beta+1}$. Fix some bijection $g : \omega_1 \to \eta$ in $V[G]$. Let $\langle X_i\,;\, i \in \omega_1\rangle$ be a continuous elementarty chain of countable submodels of $H^{V[G]}_{\omega_2}$ such that $g, H^V_\theta \in X_0$. So clearly $H^V_\theta \subset \cup\{X_i\,;\, i \in \omega\}$. So for all $i \in \omega_1$ we have

$$X_i \cap \eta = g``(X_i \cap \omega_1).$$

Clearly $\langle X_i \cap H_\theta^V ; i \in \omega_1\rangle$ is club in $[H_\theta^V]^\omega$. Since the set $\{\mathrm{ran}(\tilde\pi_{i,\omega_1}) \cap H_\theta ; i \in \omega_1\}$ is also a club in $[H_\theta^V]^\omega$ there is a club $C \subset \omega_1$ such that for all $i \in C$

$$X_i \cap \eta = \mathrm{ran}(\tilde\pi_{i,\omega_1}) \cap \eta.$$

So for all $i \in C$ we have

$$i = \mathrm{ran}(\tilde\pi_{i,\omega_1}) \cap \omega_1 = X_i \cap \omega_1$$

and thus

$$\mathrm{otp}(g``i) = \mathrm{otp}(\mathrm{ran}(\tilde\pi_{i,\omega_1}) \cap \eta) = \aleph_{3+\beta+1}^{M_i} = f(i).$$

Hence f is a canonical function. $\square$

So the club $D_G \setminus (\beta+1)$ and f from the previous lemma witness that in $V[G]$

$$\mathbf{1}_{\mathbb{B}} \Vdash \aleph_{3+\beta+1}^V \in j(S_i),$$

where $\mathbb{B}$ is $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{V[G]}$ and j is a name for the generic embedding added by forcing with $\mathbb{B}$. Hence $\aleph_{3+\beta+1}^V \in \tilde S_i$. This finishes the proof of 3.4.

Observe that the single instance of ϕ_{AC} that holds in $V^{\mathbb{P}'(\theta,\mathsf{NS}_{\omega_1},\vec S,\vec T)}$ is a Σ_1 statement in H_{ω_2} in the parameters $\vec S$ and $\vec T$. Since $\mathbb{P}'(\theta,\mathsf{NS}_{\omega_1},\vec S,\vec T)$ preserves stationary subsets an application of BMM yields the following corollary.

Corollary 3.10 *If* NS_{ω_1} *is precipitous*+BMM *then* ϕ_{AC}.

4. Obtaining ψ_{AC}

Definition 4.1 (Woodin) ψ_{AC} : Let $S \subset \omega_1$ and $T \subset \omega_1$ be stationary, costationary sets. Then there exists a canonical function f for some $\eta < \omega_2$ such that for some club $C \subset \omega_1$

$$\{\alpha < \omega_1 ; f(\alpha) \in T\} \cap C = S \cap C.$$

Note the following reformulation of the above definition in terms of generic ultrapowers: let j be a name for the embedding induced by some generic $G \subset \mathcal{P}(\omega_1)\setminus\mathsf{NS}_{\omega_1}$, with S, T as above we have

$$\mathbf{1}_{\mathbb{P}(\omega_1)\setminus\mathsf{NS}_{\omega_1}} \Vdash \check S \in \dot G \iff \eta \in j(T).$$

Woodin has shown:

Theorem 4.2 ([10, 10.95]) *If* BMM + NS_{ω_1} *is precipitous then* ψ_{AC}.

With the technology from the previous section on ϕ_{AC} it is possible to give a different proof of 4.2. Since this is very similar to the section on ϕ_{AC}, we shall only state the required results. The proofs are very similar to the ϕ_{AC} case.

Lemma 4.3 (Hitting regular cardinals lemma) *Let* $\langle M, I\rangle$ *be a countable model of* ZFC^* *and let* I *be a precipitous ideal on* ω_1^M. *Assume that* $\mathcal{P}(\mathcal{P}(\omega_1))$ *exists in* M. *Let* $\theta \in M$ *be such that*

$$M \models \mathrm{Card}(\mathcal{P}(\mathcal{P}(\omega_1)))^+ = \theta,$$

and let $\theta' \geq \theta$ such that θ' is a regular cardinal in M. Then a genericity iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta'\rangle, \langle G_i; i < \theta'\rangle\rangle$$

of $M_0 = M$ exists in V such that $\pi_{0,\theta'}(\omega_1^M) = \theta'$. □

We again modify the forcing $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ to show a weak form of ψ_{AC} in the generic extension. An application of BMM will then give us the desired result.

Theorem 4.4 *Let NS_{ω_1} be precipitous and suppose $H_\theta^\sharp$ exists, where $\theta = 2^{2^{\aleph_1+}}$. For all S, T stationary and costationary there exists a forcing construction $\mathbb{P} = \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ that preserves stationary subsets, such that if G is $\mathbb{P}$-generic over V, then in $V[G]$ there is generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$. In particular, M_0 is generically ω_1-iterable. Additionally the following holds in $V[G]$: there is a club $C \subset \omega_1$, such that for all $\alpha \in C$

$$\omega_1^{M_\alpha} \in S \iff \theta_\alpha \in T,$$

where $\theta_\alpha = \pi_{\alpha,\omega_1}^{-1}(\theta)$.

We will now define our modified forcing construction $\mathbb{P} := \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$.

Definition 4.5 Conditions p in $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ are triples

$$p = \langle\langle \kappa_i^p; i \in \mathrm{dom}(p)\rangle, \langle \pi_i^p; i \in \mathrm{dom}(p)\rangle, \langle \tau_i^p; i \in \mathrm{dom}_-(p)\rangle\rangle$$

such that the following conditions hold:

- **:** Conditions i.,ii.,iii.,iv.,v. as in definition 3.5 hold. We replace condition vi. as follows:
- **:** vi. In $V^{\mathrm{Col}(\omega,\theta)}$, there is a *model which certifies p with respect to $\mathcal{M}$*, i.e. a model $\mathfrak{A}$ such that $H_\theta^\sharp \in \mathrm{wfp}(\mathfrak{A})$, $\mathfrak{A} \models \mathsf{ZFC}^-$, for all stationary S, $\mathfrak{A} \models$ "S is stationary", and inside $\mathfrak{A}$ there is a generic iteration

  $$\mathcal{J}^{\mathfrak{A}} := \langle\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, I_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \leq j \leq \omega_1\rangle, \langle G_i^{\mathfrak{A}}; i < \omega_1\rangle\rangle$$

  such that conditions (a),(b),(c),(d) and (e) as in definition 3.5 hold. We replace (f).

  - **:** (f) Let $D^{\mathfrak{A}}$ be the club of *limits* of important ordinals of $\mathcal{J}^{\mathfrak{A}}$ relative to $\pi_{0,\omega_1}^{\mathfrak{A}-1}(\theta)$. Let $\alpha \in D^{\mathfrak{A}}$. Let β be the next important ordinal above α. Then

    $$\omega_1^{M_\alpha^{\mathfrak{A}}} \in S \iff \pi_{\alpha,\omega_1}^{\mathfrak{A}-1}(\theta) = \omega_1^{M_\beta^{\mathfrak{A}}} \in T.$$

If p, q are conditions, then we write $p \leq q$ iff $p \leq_{\mathbb{P}'(\theta,\mathsf{NS}_{\omega_1})} q$.

Applying the "Hitting regular cardinals lemma" 4.3 one can show that certifying structures exist. Hence one has:

Lemma 4.6 $\mathbb{P} \neq \emptyset$. □

We can now literally adopt lemmata 6 through 15 of [2]. So we have, using the definitions for $\pi_i, M_i, I_i, \kappa_i, G_i, \tilde{\pi}_{i,j},$ of [2]:

Lemma 4.7 *Let $G \subset \mathbb{P}$ is V-generic. Let $\kappa_i = \kappa_i^p$ for some $p \in G$. Then*

$$H_\theta^\sharp \cap \mathsf{OR} = \cup\{\mathrm{ran}(\pi_i)\,;\, i < \omega_1\}$$

and

$$\mathcal{J}_G := \langle\langle M_i, \tilde{\pi}_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

is a generic iteration of M_0 such that if $i < \omega_1$, then M_i is countable, and $M_{\omega_1} = \langle H_\theta^\sharp; \in, I\rangle$. □

We set

$$\theta_i := \tilde{\pi}_{i,\omega_1}^{-1}(\theta),$$

an we let D_G denote the club of limits of important ordinal of $\mathcal{J}$ relative to θ_0. A densitiy argument shows:

Lemma 4.8 *D_G is club and for all $i \in D_G$*

$$\omega_1^{M_i} \in S \iff \theta_i \in T.$$

□

Since the sequence $\langle\theta_i\,;\, i \in D_G\rangle$ is a canonical function for θ in the forcing extension, we have

$$\mathbf{1}_{\mathbb{P}(\omega_1)\setminus\mathsf{NS}_{\omega_1}} \Vdash \check{S} \in \dot{G} \iff \theta \in \check{j}(T).$$

We can now literally adopt lemmata 16 and 17 of [2] and their proofs; i.e. it is clear that $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ is stationary set preserving. Hence theorem 4.4 follows. The referee observed that it is possible to formulate an ad hoc combinatorial principle that implies ϕ_{AC} as well as ψ_{AC}. Such a principle could then be shown to follow from BMM and the precipitousness of NS_{ω_1} by the methods for ϕ_{AC} and ψ_{AC}.

5. The semiproperness of $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$

In [2] it was shown that $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ preserves stationary subsets of ω_1 provided that NS_{ω_1} is precipitous. Since it is consistent relative to large cardinals that all stationary set preserving forcings are semiproper, the forcing $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ can clearly be semiproper. We show that the semiproperness of the forcings $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ implies a generalization of Chang's Conjecture which in turn implies the semiproperness of all stationary set preserving forcings.
Recall the definition of semiproperness.

Definition 5.1 A notion of forcing $\mathbb{P}$ is semiproper if for every sufficiently large λ, every well-ordering $<$ of H_λ and every countable elementary submodel $X \prec \langle H_\lambda; \in, <\rangle$ the following holds:

$$\forall p \in X \cap \mathbb{P}\ \exists q \leq p : q \text{ is } (X, \mathbb{P})\text{-semigeneric},$$

where q is $(X, \mathbb{P})$-semigeneric if for every name $\dot\alpha \in X$ for a countable ordinal

$$q \Vdash \exists \beta \in X : \dot\alpha = \check\beta.$$

Definition 5.2 ([8, XIII. 1.5])

- Let x, y be countable. We write $x \sqsubset y$ if $x \cap \omega_1 = y \cap \omega_1$ and $x \subset y$.
- A set $S \subset [W]^\omega$ is *semistationary in* $[W]^\omega$ if $\{y \in [W]^\omega\, ;\, \exists x \in S : x \sqsubset y\}$ is stationary in $[W]^\omega$.
- Let $\lambda \geq \omega_2$. We denote by $\mathrm{SSR}([\lambda]^\omega)$ the following principle: For every S semistationary in $[\lambda]^\omega$ there is $W \subset \lambda$, $\mathrm{Card}(W) = \omega_1 \subset W$ and $S \cap [W]^\omega$ is semistationary in $[W]^\omega$.
- If $\mathrm{SSR}([\lambda]^\omega)$ holds for all cardinals $\lambda \geq \omega_2$ then we will say that *Semistationary Reflection* (SSR) holds.

Note that [8] has a more general notation for the above reflection principles. In [8] the principle $\mathrm{SSR}([\lambda]^\omega)$ is called $\mathrm{Rss}(\aleph_2, \lambda)$ and SSR is called $\mathrm{Rss}(\aleph_2)$.

Lemma 5.3 ([8, XIII.1.7(3)]) *Semistationary Reflection implies that all stationary set preserving forcings are semiproper.*

Definition 5.4 ([3]) (†) is an abbreviation for: every stationary set preserving forcing is semiproper.

Foreman, Magidor and Shelah have shown:

Lemma 5.5 ([3, Theorem 26]) *If* (†) *holds, then* NS_{ω_1} *is precipitous.*

We will consider a generalization of Chang's Conjecture that we call CC^{**}.

Definition 5.6 Let $\lambda \geq \omega_2$. $\mathrm{CC}^*(\lambda)$ is the following axiom: There are arbitrarily large regular cardinals $\theta > \lambda$ such that for all well-orderings $<$ of H_θ and for all $a \in [\lambda]^{\omega_1}$ and for all countable $X \prec \langle H_\theta; \in, < \rangle$ there is a countable $Y \prec \langle H_\theta; \in, < \rangle$ such that $X \sqsubset Y$ and there is some $b \in Y \cap [\lambda]^{\omega_1}$ such that $a \subset b$.
CC^{**} is $\mathrm{CC}^*(\lambda)$ for all cardinals $\lambda \geq \omega_2$.

Note that $\mathrm{CC}^*(\omega_2)$ implies Todorčević's CC^*; in the case of CC^* one only requires for an X as above that there is $Y \prec \langle H_\theta; \in, < \rangle$ such that $X \sqsubset Y$ and $X \cap \omega_2 \neq Y \cap \omega_2$, see [9]. Note that $\mathrm{CC}^*(\omega_2)$ (and also CC^*) implies the usual Chang Conjecture by building a continuous chain of countable elementary submodels of length ω_1; at each successor stage apply CC^{**}. So the countable ordinals of the last model of the chain are the same as the first model's.
The next theorem answers a question of Todorčević who asked the second author under which cirumstances $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is semiproper.
The authors would like to thank Daisuke Ikegami for communicating valuable results about the relationship of CC^{**}, SSR and (†).

Theorem 5.7 *The following are equivalent:*

(1) NS_{ω_1} *is precipitous and for all regular* $\theta \geq \omega_2$ *the partial ordering* $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ *is semiproper.*
(2) *For arbitrarily large* $\theta \geq \omega_2$ *there is a semiproper partial order* $\mathbb{P}$ *that adds a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle.$$

such that $H_\theta \subset M_{\omega_1}$ *and all* M_i *are countable.*
(3) *CC***
(4) *SSR*
(5) (†)

Before we prove the above theorem note that the Namba-like forcing in [5] is stationary set preserving (cf. [11]) and hence $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is not the only example witnessing the consistency of *2*.

Proof. (1) $\implies$ (2) is trivial and (4) $\implies$ (5) is Lemma 5.3.
(5) $\implies$ (1) is clear since by 5.5, NS_{ω_1} is precipitous in this case and so by [2] the forcing $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ exists for all regular $\theta \leq \omega_2$ and preserves stationary subsets of ω_1. It remains to show (2) $\implies$ (3) and (3) $\implies$ (4) For the first implication we assume that CC** does not hold and work toward a contradiction. So there is a least cardinal $\lambda_0 \geq \aleph_2$ for which CC** fails. Since (2) holds there is a least $\theta_0 > \lambda_0$ such that a semiproper $\mathbb{P}$ exists that adds an iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $H_{\theta_0} \subset M_{\omega_1}$ and all M_i are countable. Let $\theta > \theta_0$ large enough so that a name for an iteration as above and $\mathcal{P}(\mathbb{P})$ are both in H_θ. Let $<$ be some well-ordering of H_θ. Now fix some arbitrary $X \prec \langle H_\theta; \in, <\rangle$ and some $a \in [\lambda_0]^{\omega_1}$. Our aim is now to construct a $Y \prec \langle H_\theta; \in, <\rangle$ like in CC**. For this we first show that it suffices to do so in a generic extension:

Claim 1. If there is some generic extension of V that contains some $Y \prec \langle H_\theta; \in, <\rangle$ such that $X \sqsubset Y$ and there is some $b \in Y \cap [\lambda_0]^{\omega_1} \cap V$ such that $a \subset b$ then there is already some $Z \in V$ with $Z \prec \langle H_\theta; \in, <\rangle$, $X \sqsubset Z$ and $b \in Z$.

Proof of Claim 1. If Y is in some generic extension W of V, then by $b \in V$ there is a tree $T \in V$ searching for a countable $Z \prec \langle H_\theta; \in, <\rangle$ such that $b \in Z$ and $X \sqsubset Z$. So T has a branch in W, this is clearly witnessed by Y. By the absoluteness of wellfoundedness we have a branch through T in V and hence there is some countable $Z \prec \langle H_\theta; \in, <\rangle$ with $X \sqsubset Z$ and $b \in Z$ in V. □(Claim 1)

By the minimality of λ_0 and θ_0 some semiproper forcing and some name for an iteration as above exist in X. Let us call this forcing $\mathbb{P}$ again. Let $G \subset \mathbb{P}$ be generic over V.

Claim 2. $X[G] \prec H_\theta[G]$.

This claim is part of the folklore. For the readers convenience we give a
Proof of Claim 2. An induction along the first order formulae will yield the desired result: let ϕ be a formula and let $\sigma \in X$ denote some name such that

$$H_\theta[G] \models \exists y \phi(y, \sigma^G).$$

Then by the fullness of the forcing names we have

$$H_\theta \models \exists \tau \forall p \in \mathbb{P}(p \Vdash \exists y \phi(y, \sigma) \implies p \Vdash \phi(\tau, \sigma)).$$

So by elementarity such a τ exists in X. By the inductive hypothesis we have

$$H_\theta[G] \models \phi(\tau^G, \sigma^G) \iff X[G] \models \phi(\tau^G, \sigma^G).$$

□(Claim 2)

By our hypothesis we can force the existence of a generic iteration

$$\dot{\mathcal{J}}^G = \langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle.$$

with $M_{\omega_1} \supset H_\theta$. So by the regularity of θ we have $a \in M_{\omega_1}$. Note that $X[G]$ can calculate M_0.

Claim 3. Let $\beta < \alpha \leq \omega_1$. All elements of M_α are of the form $\pi_{\beta,\alpha}(f)(\vec{\xi})$ for some $f : \kappa_\beta^n \to M_\beta$, $f \in M_\beta$ and ordinals $\xi_1, ..., \xi_n < \omega_1^{M_\alpha}$.

This claim is also part of the folklore. Nevertheless we give a proof for the readers convenience.
Proof of Claim 3. Fix $\beta < \omega_1$. We show this by induction on α. Let $\alpha = \gamma + 1$. Then M_α is isomorph to $\mathrm{Ult}(M_\gamma, G_\gamma)$. Hence every element of M_α has the form $\pi_{\gamma,\alpha}(f)(\kappa_\gamma)$ for some $f : \kappa_\gamma \to M_\gamma$, $f \in M_\gamma$. By the inductive hypothesis f is of the form $\pi_{\beta,\gamma}(g)(\vec{\xi})$ for some $g : \kappa_\beta^n \to M_\beta, g \in M_\beta$ and $\vec{\xi} \in \kappa_\gamma^n$. Then

$$\pi_{\gamma,\alpha}(f)(\kappa_\gamma) = \pi_{\gamma,\alpha}(\pi_{\beta,\gamma}(g)(\vec{\xi}))(\kappa_\gamma) = \pi_{\beta,\alpha}(g)(\vec{\xi})(\kappa_\gamma),$$

since the critical point of $\pi_{\gamma,\alpha}$ is κ_γ.
The case $\mathrm{Lim}(\alpha)$ simply uses the fact that M_α is the direct limit of all M_γ for $\gamma < \alpha$: if $x \in M_\alpha$, then $x = \pi_{\gamma,\alpha}(\bar{x})$ for some $\gamma < \alpha$ and some $\bar{x} \in M_\gamma$. Without loss of generality we may assume $\beta < \gamma$. Then $\bar{x}$ is of the form $\pi_{\beta,\gamma}(g)(\vec{\xi})$ for some $g : \kappa_\beta^n \to M_\beta, g \in M_\beta$ and ordinals $\vec{\xi} \in \kappa_\gamma^n$. Then

$$x = \pi_{\gamma,\alpha}(\bar{x}) = \pi_{\gamma,\alpha}(\pi_{\beta,\gamma}(g)(\vec{\xi})) = \pi_{\beta,\alpha}(g)(\vec{\xi}).$$

□(Claim 3)

By setting $\beta = 0$ and $\alpha = \omega_1$ in the above claim, we have that there is some $f \in M_0$, $f : \kappa_0^n \to M_0$ and $\vec{\xi} = \xi_1, ..., \xi_n < \omega_1$ such that

$$a = \pi_{0,\omega_1}(f)(\vec{\xi}).$$

This f is in $X[G]$. We set

$$b := \bigcup \{ \pi_{0,\omega_1}(f)(\vec{\alpha}) \,;\, \vec{\alpha} \in \omega_1^n \wedge \pi_{0,\omega_1}(f)(\vec{\alpha}) \in ([H_\theta]^{\omega_1})^V \}.$$

Clearly $a \subset b$ and $\mathrm{Card}(b) = \omega_1$. Since the parameters $\pi_{0,\omega_1}(f), [H_\theta]^{\omega_1}$ used in the definition of b are in V we have that $b \in V$. Also $b \in X[G]$. By the semiproperness of $\mathbb{P}$ $X \sqsubset X[G]$. So $X[G]$ witnesses that in some generic extension of V there is some Y as desired. This suffices to show by claim 1.
We now show that 3. $\implies$ 4. This implication is a slight generalization of [9, Lemma 6]. Fix an ordinal $\lambda \geq \omega_2$ and a semistationary $S \subset [\lambda]^\omega$. We set

$$\mathcal{W} := \{W \subset \lambda \,;\, \mathrm{Card}(W) = \omega_1 \subset W\}$$

and

$$T := \{y \in [\lambda]^\omega \,;\, \exists x \in S : x \sqsubset y\}.$$

By the very definition of semistationarity T is stationary. Let us assume that SSR does not hold and work toward a contradiction. For all $W \in \mathcal{W}$

$$S_W := \{y \in [W]^\omega \,;\, \exists x \in S \cap [W]^\omega : x \sqsubset y\}$$

is nonstationary. For each $W \in \mathcal{W}$ we may hence pick a function

$$f_W : [W]^{<\omega} \to W$$

such that

$$S_W \cap \{x \in [W]^\omega \,;\, f_W``[x]^{<\omega} \subset x\} = \emptyset.$$

Let $\mathcal{F}$ denote the collection of these f_W. Let $\theta > \lambda$ be regular large enough such that $\mathcal{F}, \mathcal{W}, S, T \in H_\theta$ and such that the implications of $\mathrm{CC}^*(\lambda)$ hold for this θ. Let $<$ be a well-ordering of H_θ. Pick a countable $M \prec \langle H_\theta; \in, <\rangle$ such that $\mathcal{F}, \mathcal{W}, S, T, \lambda \in M$ and

$$M \cap \lambda \in T.$$

Let

$$a := (M \cap \lambda) \cup \omega_1.$$

Since $\mathrm{CC}^*(\lambda)$ holds for θ, there is a countable $M^* \prec H_\theta$ and some $W \in [\lambda]^{\omega_1}$ such that $M \sqsubset M^*$, $a \subset W$ and $W \in M^*$. So $f_W \in M^*$. Then by elementarity of M^*

$$f_W``[W \cap M^*]^{<\omega} \subset W \cap M^*.$$

By the choice of a and the properties of M^* we have

$$M \cap \lambda \sqsubset W \cap M^*.$$

Since we have $M \cap \lambda \in T$ there is some $x \in S$ such that $x \sqsubset M \cap \lambda$. Note that $x \in [W]^\omega$. By the transitivity of $\sqsubset$,

$$x \sqsubset W \cap M^*.$$

This implies $W \cap M^* \in S_W$. We thus have a contradiction to the choice of f_W. This finishes the proof. $\square$

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