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MAXIMAL INEQUALITIES FOR DUAL SOBOLEV SPACES $W^{-1,p}$ AND APPLICATIONS TO INTERPOLATION

Frédéric Bernicot

Abstract. We firstly describe a maximal inequality for dual Sobolev spaces $W^{-1,p}$. This one corresponds to a "Sobolev version" of usual properties of the Hardy-Littlewood maximal operator in Lebesgue spaces. Even in the Euclidean space, this one seems to be new and we develop arguments in the general framework of Riemannian manifold. Then we present an application to obtain interpolation results for Sobolev spaces.

The first maximal inequality in Lebesgue spaces, is described by the L^p-boundedness of the Hardy-Littlewood maximal function. This result holds in a space of homogeneous type (X, d, μ): for $p \in (1,\infty]$, $s \in [1,p)$ and $f \in L^p(X)$

$$\|f\|_{L^p(X)} \lesssim \left\| x \to \sup_{\substack{Q \text{ ball} \\ Q \ni x}} \frac{1}{\mu(Q)^{1/s}} \|f\|_{L^s(Q)} \right\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

Here the left inequality is due to the following "regularity property": for almost every $x \in X$

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu = |f(x)|. \tag{1}$$

The right one corresponds to the L^p boundedness of the maximal operator.

Applying this result to a function and its gradient, we obtain the same result for the Sobolev spaces on a doubling Riemannian manifold M: for $p \in (1,\infty]$, $s \in [1,p)$ and $f \in W^{1,p}$

$$\|f\|_{W^{1,p}} \lesssim \left\| x \to \sup_{\substack{Q \text{ ball} \\ Q \ni x}} \frac{1}{\mu(Q)^{1/s}} \|f\|_{W^{1,s}(Q)} \right\|_{L^p} \lesssim \|f\|_{W^{1,p}}.$$

Therefore the Sobolev norm can easily be described by the corresponding Lebesgue norm of a maximal operator (which is a "Sobolev version" of the Hardy-Littlewood maximal function). Such a property is important because the norms in Lebesgue spaces are specific and satisfy for example the "lattice property" which is not the case of the norms in Sobolev spaces. Then a natural question arises: do we have similar results for the dual Sobolev spaces $W^{-1,p}$?

Recently in [12, 13] the authors have used maximal operators (and duality) to describe interpolation results between Hardy spaces and Lebesgue spaces. To extend

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this theory for Sobolev spaces, we need such maximal inequalities for negative Sobolev spaces. That is why, we study this problem. Despite this objective, the above inequality studied in the current paper may be of independent interest.

We define maximal operators and then prove the following result: under classical assumptions on the Riemannian manifold M, there are implicit constants such that for all functions $f \in W^{-1,p}$

$$\|f\|_{W^{-1,p}} \underset{(a)}{\lesssim} \left\| \sup_{\substack{Q \text{ ball} \\ x \in Q}} \frac{1}{\mu(Q)^{1/s}} \|f\|_{W^{-1,s}(Q)} \right\|_{L^p} \underset{(b)}{\lesssim} \|f\|_{W^{-1,p}} \tag{2}$$

under some restrictions on p. The second inequality (b) is quite easy to obtain and corresponds to a boundedness of the maximal operator. The first one (a) is more difficult to prove. Such property as (1) is not sufficient to conclude.
For example in the Euclidean space, we get:

Theorem 0.1. *On $\mathbb{R}^n$ equipped with the Euclidean structure, (2) holds for every exponents $p, r \in (1,\infty)$.*

We emphasize that even in the Euclidean space $\mathbb{R}^n$, such inequalities are not obvious. In this particular case, we know that the operator $(I+\Delta)^{-1/2}$ defines an isomorphism from $L^p(\mathbb{R}^n)$ to $W^{-1,p}(\mathbb{R}^n)$. However such a description is not sufficiently precise to obtain the inequality (a).
This result seems to be new and does not exist in the litterature. We think that it will permit to better understand the structure of dual Sobolev spaces and above all the interactions with restriction and localization operators. Indeed, we point out that the quantity $\|f\|_{W^{-1,s}(Q)}$ is far more localized than $\left\|(I+\Delta)^{-1/2} f\right\|_{L^s(Q)}$, as it takes information only from $f\mathbf{1}_Q$.
We believe in the interest of such inequalities and we give a first application about interpolation of Sobolev spaces (Section 3). For example we will prove the following result.

Theorem 0.2. *Let M be a doubling Riemannian manifold satisfying a* Reverse Riesz inequality*:*

$$\left\|(1+\Delta)^{1/2}(f)\right\|_{L^r} \lesssim \|f\|_{W^{1,r}},$$

for an exponent $r \in (1,2)$. Then for all $p_0 \in (1,2)$ and $\theta \in (0,1)$ such that

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{2} < \frac{1}{r},$$

we have

$$\left(W^{1,p_0}, W^{1,2}\right)_{\theta,p_\theta} = W^{1,p_\theta}.$$

This result is interesting as we do not require Poincaré inequality as in the work of N. Badr (see [10, 9]). This is the first result of interpolation for Sobolev spaces, which permits to get around the use of Poincaré inequalities. Due to the work of P. Auscher and T. Coulhon (see [5]), our assumed *Reverse Riesz inequality* is weaker than the Poincaré inequality (P_r).

We refer the reader to a forthcoming work (joined with N. Badr, see [11]), where we use these maximal inequalities for Sobolev spaces in order to describe an interpolation theory for abstract Hardy-Sobolev spaces. In this case, they will play a crucial role.

1. Preliminaries

Throughout this paper we will denote by $\mathbf{1}_E$ the characteristic function of a set E and E^c the complement of E. If X is a metric space, Lip will be the set of real Lipschitz functions on X and Lip_0 the set of real, compactly supported Lipschitz functions on X. For a ball Q in a metric space, λQ denotes the ball co-centered with Q and with radius λ times that of B. Finally, C will be a constant that may change from an inequality to another and we will use $u \lesssim v$ to say that there exists two constants C such that $u \leq Cv$ and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$.

In all this paper M denotes a Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_{L^p}$ for the norm on $L^p := L^p(M,\mu)$, $1 \leq p \leq +\infty$. We denote by $Q(x,r)$ the open ball of center $x \in M$ and radius $r > 0$. We will use the positive Laplace-Beltrami operator Δ defined by

$$\forall f, g \in C_0^\infty(M), \qquad \langle \Delta f, g\rangle = \langle \nabla f, \nabla g\rangle.$$

1.1. The doubling property.

Definition 1.1. *Let M be a Riemannian manifold. One says that M is* doubling *or satisfies the (global) doubling property (D) if there exists a constant $C > 0$, such that for all $x \in M$, $r > 0$ we have*

$$(D) \qquad \mu(Q(x,2r)) \leq C\mu(Q(x,r)).$$

Observe that if M satisfies (D) then

$$diam(M) < \infty \Leftrightarrow \mu(M) < \infty \text{ (see [1]).}$$

Therefore if M is a complete Riemannian manifold satisfying (D) then $\mu(M) = \infty$.

Theorem 1.2 (Maximal theorem)**.** *([14]) Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) := \sup_{\substack{Q \ ball \\ x \in Q}} \frac{1}{\mu(Q)} \int_Q |f| d\mu.$$

Then for every $p \in (1,\infty]$, $\mathcal{M}$ is L^p bounded and moreover of weak type $(1,1)$. Consequently for $s \in (0,\infty)$, the operator $\mathcal{M}_s$ defined by

$$\mathcal{M}_s f(x) := \left[\mathcal{M}(|f|^s)(x)\right]^{1/s}$$

is of weak type (s,s) and L^p bounded for all $p \in (s,\infty]$.

1.2. Poincaré inequality.

Definition 1.3 (Poincaré inequality on M). *We say that a Riemannian manifold M admits **a Poincaré inequality** (P_q) for some $q \in [1,\infty)$ if there exists a constant $C>0$ such that, for every function $f \in \mathrm{Lip}_0(M)$ and every ball Q of M of radius $r>0$, we have*

$$(P_q) \qquad \left(\fint_Q \left| f - \fint_Q f \right|^q d\mu \right)^{1/q} \leq Cr \left(\fint_Q |\nabla f|^q d\mu \right)^{1/q}.$$

Remark 1.4. *By density of $C_0^\infty(M)$ in $Lip_0(M)$, we can replace $\mathrm{Lip}_0(M)$ by $C_0^\infty(M)$.*

Let us recall some known facts about Poincaré inequalities with varying q.
It is known that (P_q) implies (P_p) for $p \geq q$ (see [17]). Thus if the set of q such that (P_q) holds is not empty, then it is an interval unbounded on the right. A recent result of S. Keith and X. Zhong (see [19]) asserts that this interval is open in $[1,+\infty)$:

Theorem 1.5. *Let M be a complete doubling Riemannian manifold admitting a Poincaré inequality (P_q), for some $1<q<\infty$. Then there exists $\epsilon>0$ such that M admits (P_p) for every $p>q-\epsilon$.*

2. Maximal characterization of dual Sobolev spaces

From now on, we always assume that the Riemannian manifold satisfies the doubling property (D).

2.1. New maximal operators. First, we recall the "duality-properties" of the Sobolev spaces.

Definition 2.1. *For $p \in [1,\infty]$ and O an open set of M, we define $W^{1,p}(O)$ as following*

$$W^{1,p}(O) := \overline{C_0^\infty(O)}^{\|\ \|_{W^{1,p}(O)}} \quad \text{with} \quad \|f\|_{W^{1,p}(O)} := \| |f| + |\nabla f| \|_{L^p(O)}.$$

Then we denote $W^{-1,p'}(O)$ the dual space of $W^{1,p}(O)$ defined as the set of distributions $f \in \mathcal{D}'(M)$ such that

$$\|f\|_{W^{-1,p'}(O)} = \sup_{g \in C_0^\infty(M)} \frac{|\langle f,g\rangle|}{\|g\|_{W^{1,p}(O)}}.$$

Proposition 2.2. *Let $p \in [1,\infty)$. Then for all open set O of M, we have*

$$\begin{aligned}\|f\|_{W^{-1,p'}(O)} &= \inf_{f=\phi-div(\psi)} \|\phi\|_{L^{p'}(O)} + \|\psi\|_{L^{p'}(O)} \\ &\simeq \inf_{f=\phi-div(\psi)} \| |\phi| + |\psi| \|_{L^{p'}(O)}.\end{aligned}$$

Here we take the infimum over all the decompositions $f=\phi-div(\psi)$ on M with $\phi \in L^{p'}(O)$ and $\psi \in \mathcal{D}'(O,\mathbb{R}^n)$ such that $div(\psi) \in L^{p'}(O)$.

The proof is left to the reader (it is essentially written in [8], Proposition 33).

We introduce the following maximal operators:

Definition 2.3. *Let $s \geq 1$. According to the standard maximal "Hardy-Littlewood" operator $\mathcal{M}_s$, we define two "Sobolev versions":*

$$M_{S,s}(f)(x) := \sup_{\substack{Q\, ball \\ Q \ni x}} \frac{1}{\mu(Q)^{1/s}} \|f\|_{W^{-1,s}(Q)}$$

and

$$M_{S,*,s}(f)(x) := \inf_{f=\phi - div(\psi)} \mathcal{M}_s\left(|\phi| + |\psi|\right)(x).$$

Remark 2.4. *Thanks to Proposition 2.2, it is easy to check that we can compare them pointwisely:*

$$M_{S,s}(f) \leq M_{S,*,s}(f).$$

We dedicate the next subsection to the study of these maximal operators. Mainly we want to describe the dual Sobolev norms by the corresponding Lebesgue norms of these operators.

2.2. First properties of the maximal operators. We begin proving some useful and general properties for the new maximal operators $M_{S,s}$ and $M_{S,*,s}$. These operators can be thought as being equivalent to $\mathcal{M}_s((I+\Delta)^{-1/2})$, where Δ is the positive Laplace-Beltrami operator on the manifold M.

Proposition 2.5. *For $p \in [1,\infty)$, $M_{S,p}$ and $M_{S,*,p}$ are of "weak type (p,p)": there exists an implicit constant such that for all $f \in W^{-1,p}$*

$$\|M_{S,p}(f)\|_{L^{p,\infty}} \leq \|M_{S,*,p}(f)\|_{L^{p,\infty}} \lesssim \|f\|_{W^{-1,p}}. \tag{3}$$

Moreover, they are of "strong type (q,q)" for every $q > p$: there exists an implicit constant such that for all $f \in W^{-1,q}$

$$\|M_{S,p}(f)\|_{L^q} \leq \|M_{S,*,p}(f)\|_{L^q} \lesssim \|f\|_{W^{-1,q}}. \tag{4}$$

Proof. The first inequalities in (3) and (4) are due to Remark 2.4. We only check the second one for (3). Using Fatou's lemma in weak Lebesgue spaces, it yields

$$\|M_{S,*,p}(f)\|_{L^{p,\infty}} \leq \inf_{f=\phi-div(\psi)} \|\mathcal{M}_p(|\phi|+|\psi|)\|_{L^{p,\infty}}.$$

Then using the weak type (p,p) of the Hardy-Littlewood maximal operator it comes

$$\|M_{S,*,p}(f)\|_{L^{p,\infty}} \lesssim \inf_{f=\phi-div(\psi)} \||\phi|+|\psi|\|_{L^p}.$$

Finally Proposition 2.2 finishes the proof of (3) and we similarly prove (4). □

Now we look for reverse inequalities. First we describe an easy fact:

Remark 2.6. *Let $r_1, r_2 \in [1,\infty)$ with $r_1 \leq r_2$. Then*

$$M_{S,*,r_1} \leq M_{S,*,r_2} \quad and \quad M_{S,r_1} \leq M_{S,r_2}.$$

Proposition 2.7. *Let $p \in [1,\infty)$. The two maximal operators $M_{S,*,p}$ and $M_{S,p}$ "control the Sobolev norm in $W^{-1,p}$". That is*

$$\forall f \in W^{-1,p}, \qquad \|f\|_{W^{-1,p}} \lesssim \|M_{S,p}(f)\|_{L^p} \leq \|M_{S,*,p}(f)\|_{L^p}. \tag{5}$$

Proof. Thanks to Remark 2.4, we just have to prove the first inequality. In order to show this one, we choose a collection of balls $(B_i)_i$ of radius 1, which corresponds to a bounded covering of M. Let $(\phi_i)_i$ be a partition of unity associated to this covering. Then we know that there exists a function $g \in C_0^\infty$ such that

$$\|f\|_{W^{-1,p}} \leq 2\langle f, g\rangle = 2\sum_i \langle f, g\phi_i\rangle$$

and $\|g\|_{W^{1,p'}} = 1$. We use the fact that

$$\langle f, g\phi_i\rangle \leq \|f\|_{W^{-1,p}(Q_i)} \|g\phi_i\|_{W^{1,p'}(Q_i)}.$$

Since the balls B_i are of radius 1, the functions ϕ_i can be chosen as uniformly bounded in the Sobolev space $W^{1,p'}$ and so we have

$$\langle f, g\phi_i\rangle \lesssim \|f\|_{W^{-1,p}(Q_i)} \|g\|_{W^{1,p'}(Q_i)} \lesssim \mu(Q_i)^{1/p} \inf_{Q_i} M_{S,p}(f) \|g\|_{W^{1,p'}(Q_i)}.$$

Using Hölder inequality we obtain

$$\|f\|_{W^{-1,p}} \lesssim \left(\sum_i \mu(Q_i) \inf_{Q_i} M_{S,p}(f)^p\right)^{1/p} \left(\sum_i \|g\|^{p'}_{W^{1,p'}(Q_i)}\right)^{1/p'}.$$

The first term is bounded by $\|M_{S,p}(f)\|_{L^p}$. The second term is bounded by $\|g\|_{W^{1,p'}} = 1$ since the collection $(Q_i)_i$ forms a bounded covering. Therefore the proposition follows. □

We also would like to prove a similar result as in Proposition 2.7 with a maximal operator $M_{S,r}$, given by another exponent $r \leq p$. Such a result for $r \geq p$ holds combining Remark 2.6 and Proposition 2.7. For $r < p$ this fact does not seem to be obvious and we do not know if it is true in a general case. That is why, we define the following assumption:

Assumption 2.8. *Take two exponents s_0, s_1 with $1 \leq s_0 < s_1 < \infty$. Then we call (H_{s_0,s_1}) the following assumption: there exists an implicit constant such that for all functions $f \in W^{-1,s_1}$*

$$(H_{s_0,s_1}) \qquad \|f\|_{W^{-1,s_1}} \lesssim \|M_{S,*,s_0}(f)\|_{L^{s_1}}.$$

Remark 2.9. *If $s_0 \geq s_1$, we have seen that (H_{s_0,s_1}) is always satisfied.*

We finish this subsection, by comparing the two maximal operators $M_{S,p}$ and $M_{S,*,p}$. We have already seen in Remark 2.4 that we have a pointwise inequality. We describe here a global reverse inequality.

Proposition 2.10. *Let $p \in (1,\infty)$ and $r \in [1,\infty)$. Assume that the Riemannian manifold M satisfies $\mu(M) = \infty$[1]. Then we have*

$$\|M_{S,*,r}(f)\|_{L^p} \simeq \|M_{S,r}(f)\|_{L^p}. \tag{6}$$

The implicit constants can be chosen independently with respect to any function $f \in W^{-1,p}$.

[1] which is true if we assume M complete since here the Riemannian measure is doubling.

Proof. Using Remark 2.4, we just have to prove that

$$\|M_{S,*,r}(f)\|_{L^p} \lesssim \|M_{S,r}(f)\|_{L^p}. \tag{7}$$

The proof is based on a "good lambdas" inequality. By classical arguments (see [7]), we just need to show the following inequality for any small enough γ and a large enough numerical constant $K > 1$:

$$\begin{aligned} \mu\left(\{x, M_{S,*,r}(f)(x) > K\lambda,\ M_{S,r}(f)(x) \leq \gamma\lambda\}\right) \lesssim \\ \gamma\mu\left(\{x, M_{S,*,r}(f)(x) > \lambda\}\right). \end{aligned} \tag{8}$$

We consider the sets

$$B_\lambda := \{M_{S,*,r}(f) > K\lambda,\ M_{S,r}(f) \leq \gamma\lambda\}$$

and

$$E_\lambda := \{M_{S,*,r}(f) > \lambda\}.$$

First we remark that $B_\lambda \subset E_\lambda$. We choose $(Q_j)_j$ a Whitney decomposition of E_λ and write x_j for a point in $4Q_j \cap E_\lambda^c$. Let x be a point in $B_\lambda \cap Q_j$. We have

$$\inf_{f=\phi-div(\psi)} \sup_{\substack{Q \text{ball} \\ Q \ni x}} \frac{1}{\mu(Q)^{1/r}} \||\phi| + |\psi|\|_{L^r(Q)} \geq K\lambda. \tag{9}$$

However for all ball Q containing x and satisfying $Q \cap (8Q_j)^c \neq \emptyset$, the point x_j belongs to $4Q$. Hence

$$\inf_{f=\phi-div(\psi)} \sup_{\substack{Q \ni x \\ Q\cap(8Q_j)^c \neq \emptyset}} \frac{1}{\mu(4Q)^{1/r}} \||\phi| + |\psi|\|_{L^r(4Q)} \leq M_{S,*,r}(f)(x_j) \leq \lambda.$$

Therefore using (D), we obtain

$$\inf_{f=\phi-div(\psi)} \sup_{\substack{x \ni Q \\ Q\cap(8Q_j)^c \neq \emptyset}} \frac{1}{\mu(Q)^{1/r}} \||\phi| + |\psi|\|_{L^r(Q)} \lesssim \lambda.$$

Taking K large enough (larger than the implicit constant in the previous inequality), it comes

$$\inf_{f=\phi-div(\psi)} \sup_{\substack{Q \text{ball} \\ x \in Q \subset 8Q_j}} \frac{1}{\mu(Q)^{1/r}} \|(|\phi| + |\psi|)\,\mathbf{1}_{8Q}\|_{L^r(Q)} \geq K\lambda.$$

Now we choose ϕ_j and ψ_j such that

$$\||\phi_j| + |\psi_j|\|_{L^r(8Q_j)} \simeq \|f\|_{W^{-1,r}(8Q_j)}. \tag{10}$$

This is possible due to Proposition 2.2. We thus obtain

$$\mathcal{M}_r\left((|\phi_j| + |\psi_j|)\,\mathbf{1}_{8Q_j}\right)(x) \geq K\lambda.$$

So we have proved that

$$B_\lambda \cap Q_j \subset \left\{x,\ \mathcal{M}_r\left((|\phi_j| + |\psi_j|)\,\mathbf{1}_{8Q_j}\right)(x) \geq K\lambda\right\}.$$

Using the weak type (r,r) of the Hardy-Littlewood maximal operator, we deduce that

$$\mu\left(B_\lambda \cap Q_j\right) \lesssim \frac{1}{\lambda^r} \||\phi_j| + |\psi_j|\|_{L^r(8Q_j)}^r \lesssim \frac{1}{\lambda^r} \|f\|_{W^{-1,r}(8Q_j)}^r.$$

The last inequality is due to (10). Then by definition of $M_{S,r}$, we have

$$\|f\|_{W^{-1,r}(8Q_j)} \lesssim \mu(Q_j)^{1/r} \inf_{8Q_j} M_{S,r}(f) \lesssim \gamma \mu(Q_j)^{1/r}\lambda.$$

We conclude that

$$\mu\left(B_\lambda \cap Q_j\right) \lesssim \gamma^r \mu(Q_j).$$

Therefore summing over j, the proof of (8) is completed. $\square$

Corollary 2.11. *Let $1 \leq s_0, s_1 < \infty$. Then under the assumptions (H_{s_0,s_1}) and $\mu(M)=\infty$, the maximal operator M_{S,s_0} "controls the Sobolev norm in W^{-1,s_1}": that is*

$$\|f\|_{W^{-1,s_1}} \lesssim \|M_{S,s_0}(f)\|_{s_1}. \tag{11}$$

It is difficult to check the assumption (H_{s_0,s_1}), some technical details create problems. We are going to check that the assumption (H_{s_0,s_1}) has really a sense and is satisfied under more classical assumptions. The next subsection is devoted to prove that (H_{s_0,s_1}) holds under usual assumptions on the manifold M. This is the main result of this section.

2.3. Some hypotheses insuring (H_{s_0,s_1}). We first define some properties to describe our main result.

Definition 2.12. *We use the second order operator $L := (I + \Delta)$ defined with the positive Laplace-Beltrami operator. We recall that the two operators Δ and L are self-adjoint.*
According to [5]*, we say that for $p \in (1,\infty)$ we have the non-homogeneous property (nhR_p) if*

$$\|f\|_{W^{1,p}} \lesssim \left\|L^{1/2}(f)\right\|_{L^p} \tag{nhR_p}$$

for all $f \in C_0^\infty(M)$. This is equivalent to the L^p boundedness of the local Riesz transform $\nabla L^{-1/2}$. And we have the non-homogeneous reverse property ($nhRR_p$) if

$$\left\|L^{1/2}(f)\right\|_{L^p} \lesssim \|f\|_{W^{1,p}} \tag{$nhRR_p$}$$

for all $f \in C_0^\infty(M)$.

Definition 2.13. *Let $p, q \in [1,\infty)$. We say that the collection $(T_t)_{t>0} = (e^{-t\Delta})_{t>0}$ or $(T_t)_{t>0} = (\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfy "($L^p - L^q$)-off-diagonal estimates", if there exists γ such that for all ball Q of radius r_Q, every function f supported on Q and all index $j \geq 0$*

$$\left(\frac{1}{\mu(2^jQ)}\int_{S_j(Q)} \left|T_{r_Q^2}(f)\right|^q d\mu\right)^{1/q} \lesssim e^{-\gamma 4^j}\left(\frac{1}{\mu(Q)}\int_Q |f|^p\, d\mu\right)^{1/p}.$$

We used $S_j(Q)$ for the dyadic corona around the ball

$$S_j(Q) := \left\{y, 2^j \leq 1 + \frac{d(y,Q)}{r_Q} < 2^{j+1}\right\}.$$

Note that $S_0(Q) = 2Q$. These "off-diagonal estimates" are closely related to "Gaffney estimates" of the semigroup.

We now come to our main result:

Theorem 2.14. *Let $1 < s < r' < \sigma$. Assume that the Riemannian manifold M satisfies $(nhRR_r)$ and $(nhR_{s'})$. Moreover assume that the semigroup $(e^{-t\Delta})_{t>0}$ satisfies "$(L^{\sigma'} - L^{s'})$-off-diagonal estimates" and that the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfies "$(L^{s'} - L^{s'})$-off-diagonal estimates".*
Then there is a constant $c = c(s, r, \sigma)$ such that

$$\forall f \in W^{-1,r'}, \qquad \|f\|_{W^{-1,r'}} \leq c \, \|M_{S,*,s}(f)\|_{L^{r'}} . \tag{12}$$

Therefore (H_{s_0,s_1}) is satisfied for all exponents s_0, s_1 satisfying $s_0 \geq s$ and $s_1 = r'$.

Proof. Thanks to Proposition 2.7, this result is interesting only for $s < r'$, which will be assumed.
The proof is quite technical, we deal with the case where the manifold is of infinite measure $\mu(M) = \infty$. We explain in Remark 2.15, the modifications one has to do in the other case.
Let us take a function $f \in W^{-1,r'}$. By definition, $(nhRR_r)$ implies that

$$\begin{aligned} \|f\|_{W^{-1,r'}} &:= \sup_{\substack{g \in C_0^\infty \\ \|g\|_{W^{1,r}} \leq 1}} |\langle f, g \rangle| \\ &= \sup_{\substack{g \in C_0^\infty \\ \|g\|_{W^{1,r}} \leq 1}} \langle L^{-1/2} f, L^{1/2} g \rangle \\ &\lesssim \sup_{\|h\|_{L^r} \lesssim 1} \langle L^{-1/2} f, h \rangle \\ &\simeq \|L^{-1/2} f\|_{L^{r'}} . \end{aligned} \tag{13}$$

Now we have to use a "Fefferman-Stein" inequality adapted to our operator $L^{-1/2}$. We use the results of [13]. Let us first recall some notations.
We set

$$M_\sigma(f)(x) := \sup_{Q \ni x} \frac{1}{\mu(Q)^{1/\sigma}} \left\| e^{-r_Q^2 \Delta}(f) \right\|_{L^\sigma(Q)}$$

and

$$M_s^\sharp(f)(x) := \sup_{Q \ni x} \frac{1}{\mu(Q)^{1/s}} \left\| f - e^{-r_Q^2 \Delta}(f) \right\|_{L^s(Q)} .$$

The assumed "$(L^{\sigma'} - L^{s'})$-off-diagonal estimates" for $(e^{-t\Delta})_{t>0}$ gives (see [13], Theorem 5.11)

$$M_\sigma(f) \lesssim \mathcal{M}_s(f).$$

Moreover from [13], Proposition 7.1 (which proves that the associated atomic Hardy space is included in L^1) and Corollary 5.8, it comes that for all $q \in (s, \sigma)$

$$\|.\|_{L^q} \simeq \left\| {M_s}^\sharp(.) \right\|_{L^q} . \tag{14}$$

We have used here that $\mu(M)=\infty$ (see Remark 2.15).
Thus applying (13) and (14) with $q=r'$, we obtain

$$\|f\|_{W^{-1,r'}} \lesssim \left\| M_s^\sharp(L^{-1/2}f)\right\|_{L^{r'}}.$$

It remains to prove the following pointwise inequality:

$$M_s^\sharp(L^{-1/2}f) \lesssim M_{S,*,s}(f). \tag{15}$$

Fix an $x_0\in M$. It is first possible to chose a ball $Q\ni x_0$ such that

$$M_s^\sharp(L^{-1/2}f)(x_0) \leq 2\frac{1}{\mu(Q)^{1/s}} \left\|(1-e^{-r_Q^2\Delta})L^{-1/2}f\right\|_{L^s(Q)}.$$

Then, we can find a function $g\in C_0^\infty(Q)$ with $\|g\|_{L^{s'}}\leq 1$ such that

$$\begin{aligned} M_s^\sharp(L^{-1/2}f)(x_0) &\leq \frac{4}{\mu(Q)^{1/s}}\langle (1-e^{-r_Q^2\Delta})L^{-1/2}f,g\rangle \\ &\leq \frac{4}{\mu(Q)^{1/s}}\langle f, L^{-1/2}(1-e^{-r_Q^2\Delta})g\rangle. \end{aligned}$$

Using a decomposition $f=\phi-div(\psi)$, with $\phi\in L^p$ and $\psi\in\mathcal{D}'(M)$, we finally get:

$$\begin{aligned} &M_s^\sharp(L^{-1/2}f)(x_0)\leq \\ &\qquad \frac{4}{\mu(Q)^{1/s}}\left[\langle \phi, L^{-1/2}(1-e^{r_Q^2\Delta})g\rangle + \langle \psi, \nabla L^{-1/2}(1-e^{-r_Q^2\Delta})g\rangle\right]. \end{aligned}$$

Let us study the first term $\langle \phi, L^{-1/2}(1-e^{-r_Q^2\Delta})g\rangle$. We follow ideas of [3] (section 4, Lemma 4.4), using the following representation of the square root:

$$L^{-1/2}(h)=\int_0^\infty e^{-t}e^{-t\Delta}(h)\frac{dt}{\sqrt{t}}.$$

Now using the $(L^{s'}-L^{s'})$-"off-diagonal" decays (implied by the $(L^{\sigma'}-L^{s'})$- ones) of the semigroup $(e^{-t\Delta})_{t>0}$, we claim that for all $j\geq 0$:

$$\frac{1}{\mu(2^jQ)^{1/s'}}\|L^{-1/2}(1-e^{-r_Q^2\Delta})g\|_{L^{s'}(S_j(Q))} \lesssim 2^{-j}\frac{1}{\mu(Q)^{1/s'}}\|g\|_{L^{s'}(Q)}. \tag{16}$$

For $j\in\{0,1\}$, we only use the uniform $L^{s'}$-boundedness of the semigroup (let us recall that g is supported on Q):

$$\begin{aligned} &\frac{1}{\mu(2^jQ)^{1/s'}}\|L^{-1/2}(1-e^{-r_Q^2\Delta})g\|_{L^{s'}(4Q)} \\ &\qquad\lesssim \frac{1}{\mu(Q)^{1/s'}}\int_0^\infty e^{-t}\|e^{-t\Delta}(1-e^{-r_Q^2\Delta})g\|_{L^{s'}(4Q)}\frac{dt}{\sqrt{t}} \\ &\qquad\lesssim \frac{1}{\mu(Q)^{1/s'}}\|g\|_{L^{s'}(Q)}\int_0^\infty e^{-t}\frac{dt}{\sqrt{t}} \lesssim \frac{1}{\mu(Q)^{1/s'}}\|g\|_{L^{s'}(Q)}. \end{aligned}$$

For $j\geq 2$, we do not detail the proof and refer the reader to [3], Lemma 4.4 in the Euclidean case.
Then with the normalization of g, from (16) we finally get

$$\frac{1}{\mu(Q)^{1/s}}\left|\langle \phi, L^{-1/2}(1-e^{-r_Q^2\Delta})g\rangle\right| \lesssim \mathcal{M}_s(\phi)(x_0).$$

Similarly by the "off-diagonal" decays of $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$, we obtain for $j\geq 2$:

$$\frac{1}{\mu(2^jQ)^{1/s'}}\|\nabla L^{-1/2}(1-e^{-r_Q^2\Delta})g\|_{L^{s'}(S_j(Q))}\lesssim 2^{-j}\frac{1}{\mu(Q)^{1/s'}}\|g\|_{L^{s'}(Q)}.$$

When $j\in\{0,1\}$, we use the $(nhR_{s'})$ inequality (which is equivalent to the $L^{s'}$-boundedness of the Riesz transform) and the $L^{s'}$-boundedness of the semigroup as follows:

$$\begin{aligned}\frac{1}{\mu(2^jQ)^{1/s'}}\|\nabla L^{-1/2}(1-e^{-r_Q^2\Delta})g\|_{L^{s'}(4Q)} &\lesssim \frac{1}{\mu(2^jQ)^{1/s'}}\|(1-e^{-r_Q^2\Delta})g\|_{L^{s'}} \\ &\lesssim \frac{1}{\mu(Q)^{1/s'}}\|g\|_{L^{s'}(Q)}.\end{aligned}$$

As previously, we deduce the following estimate

$$\frac{1}{\mu(Q)^{1/s}}\left|\langle\psi,\nabla L^{-1/2}(1-e^{-r_Q^2\Delta})g\rangle\right|\lesssim \mathcal{M}_s(\psi)(x_0),$$

and so we conclude that

$$M_s^\sharp(L^{-1/2}f)(x_0)\lesssim \mathcal{M}_s(\phi)(x_0)+\mathcal{M}_s(\psi)(x_0).$$

By taking the infimum over all the decompositions of f, we get (15) and the proof is therefore complete. □

Remark 2.15. *In the case where the manifold is of finite measure, the "Fefferman-Stein" inequality (14) has to be replaced by the following one:*

$$\|.\|_{L^q}\simeq \left\|M_s^\sharp(.)\right\|_{L^q}+\|.\|_{L^1}. \tag{17}$$

However when M is of finite measure, we have: $\|L^{-1/2}(f)\|_{L^1}\lesssim\|L^{-1/2}(f)\|_{L^s}$. Then using the $(nhR_{s'})$ property, we deduce that

$$\|L^{-1/2}(f)\|_{L^1}\lesssim\|f\|_{W^{-1,s}}.$$

The reverse inequality of Proposition 2.7 gives us

$$\|L^{-1/2}(f)\|_{L^1}\lesssim\|f\|_{W^{-1,s}}\lesssim\|M_{S,*,s}(f)\|_{L^s}$$

which implies the desired inequality

$$\|L^{-1/2}(f)\|_{L^1}\lesssim\|M_{S,*,s}(f)\|_{L^{r'}}$$

when $s\leq r'$.

We recall criterions that insure our previous assumptions. We refer the reader to the work of P. Auscher, T. Coulhon, X. Duong and S. Hofmann [6] and [5] for more details about all these notions and how they are related.

Theorem 2.16. *Let M be a complete doubling Riemannian manifold.*

- *The inequalities (nhR_2) and $(nhRR_2)$ are always satisfied.*
- *([15]) Assume that the heat kernel p_t of the semigroup $e^{-t\Delta}$ satisfies the following pointwise estimate:*

$$p_t(x,x)\lesssim\frac{1}{\mu(B(x,t^{1/2}))}. \tag{DUE}$$

Then for all $p\in(1,2]$, (nhR_p) and $(nhRR_{p'})$ hold.

- *([16]) Under (D), (DUE) self-improves into the following Gaussian upper-bound estimate of p_t*

$$(UE) \qquad p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c\frac{d^2(x,y)}{t}}.$$

That implies $(L^1 - L^\infty)$ "off-diagonal" decays for $(e^{-t\Delta})_{t>0}$.

- *Under (UE), the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfy "$(L^2 - L^2)$ off-diagonal decays".*

Theorem 2.17 ([20, 21])**.** *The conjunction of (D) and Poincaré inequality (P_2) on M is equivalent to the following Li-Yau inequality*

$$(LY) \qquad \frac{1}{\mu(B(y,t^{1/2}))} e^{-c_1\frac{d^2(x,y)}{t}} \lesssim p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c_2\frac{d^2(x,y)}{t}},$$

with some constants $c_1, c_2 > 0$.

Theorem 2.18 ([6])**.** *Under (D) and Poincaré inequality (P_2), the property (nhR_p) for all $p \in (2, p_0)$ is equivalent to the boundedness*

$$(G_{p_0}) \qquad \left\| \nabla e^{-t\Delta} \right\|_{L^{p_0} \to L^{p_0}} \lesssim \frac{1}{\sqrt{t}}.$$

Moreover under (G_{p_0}), the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfies some $(L^p - L^p)$ "off-diagonal" decays for every $p \in [2, p_0)$.

Theorem 2.19 ([5])**.** *Under Poincaré inequality (P_{p_0}) for $p_0 \in (1, 2]$, $(nhRR_p)$ holds for all $p \in (p_0, 2]$.*

Remark 2.20. *All these results are proved in their homogeneous version, with homogeneous properties (R_p) and (RR_p). It is essentially based on the well-known Calderón-Zygmund decomposition for Sobolev functions. This tool was extended for non-homogeneous Sobolev spaces (see [10]). Thus by exactly the same proof, we can obtain an analogous non-homogeneous version and then prove all these results.*

From Theorems 2.14, 2.16 and 2.19, Remark 2.6 and the self-improvement of Poincaré inequality (proved in [19]) we get:

Corollary 2.21. *Let M be a non-compact Riemannian manifold satisfying the doubling property. If Poincaré inequality (P_{p_0}) holds for some $p_0 \in (1, 2)$, then (H_{s_0,s_1}) is verified for all s_0, s_1 satisfying*

$$s_0 \geq 2 \qquad \text{and} \qquad s_1 \leq p_0'.$$

Corollary 2.22. *In the Euclidean case $M = \mathbb{R}^n$, for all $s_0, s_1 \in (1, \infty)$, the assumption (H_{s_0,s_1}) holds. More generally, on any Riemannian manifold satisfying (D) and (P_1), (H_{s_0,s_1}) holds for all $s_0, s_1 \in (1, \infty)$.*

We begin to understand the link between Sobolev norms and the Lebesgue norms of our maximal operators. This technical result will be useful in Section 3 to develop new results for the interpolation of Sobolev spaces.

3. Interpolation of Sobolev spaces.

In this section, we look for a real interpolation result for the scale of Sobolev spaces $(W^{1,p})_{p\in(1,\infty)}$. We refer the reader to the work of N. Badr (see [10, 9]) for first results. This work is based on a well-known Calderón-Zygmund decomposition for Sobolev functions, initialy explained by P. Auscher in [2]. We refer the reader to [2] for the first use of this one. Many applications follow from this decomposition and there are many versions (for example there are several improvements with weights in [7] and [9]). This very useful tool works under the assumption of Poincaré inequality.

This section is devoted to the description of interpolation results for Sobolev spaces using the results of Section 2.

We recall the important assumption:

Assumption 3.1. *Take two exponents $1 \leq s_0 \leq s_1 < \infty$. We call (H_{s_0,s_1}) the following assumption:*

$$(H_{s_0,s_1}) \qquad \|f\|_{W^{-1,s_1}} \lesssim \|M_{S,*,s_0}(f)\|_{L^{s_1}}.$$

Definition 3.2. *For M a Riemannian manifold, we denote by $\mathcal{I}_M$ the following set:*

$$\mathcal{I}_M := \left\{(s_0,s_1) \in (1,\infty)^2,\ s_0 \leq s_1,\ (H_{s_0,s_1}) \text{ holds}\right\}.$$

Here is the main result of this subsection:

Theorem 3.3. *Let M be a Riemannian manifold satisfying the doubling property (D). Then the scale $(W^{1,p})_{p\in(1,\infty]}$ is an interpolation scale for the real interpolation related to $\mathcal{I}_M$. That is for all $p_0, p_1 \in (1,\infty]$ (with $p_0 \leq p_1$) and $\theta \in (0,1)$ such that*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and satisfying $(p_1', p_\theta') \in \mathcal{I}_M$, we have

$$\left(W^{1,p_0}, W^{1,p_1}\right)_{\theta,p_\theta} = W^{1,p_\theta}.$$

Proof. We set $E := \left(W^{1,p_0}, W^{1,p_1}\right)_{\theta,p_\theta}$. We have to prove the equivalence of norms:

$$\| \ \|_E \simeq \| \ \|_{W^{1,p_\theta}}. \tag{18}$$

From the interpolation theory on Lebesgue spaces, it is obvious that

$$E \hookrightarrow W^{1,p_\theta}.$$

We just have to prove the reverse embedding. We will use the maximal operator $M_{S,*,p_1'}$. Let $q \in [1,p_1]$. We claim that there is a constant $c = c_q$ such that

$$\left\|M_{S,*,p_1'}(h)\right\|_{L^{q',\infty}} \leq c_q \|h\|_{(W^{1,q})^*}. \tag{19}$$

This fact comes from several properties: $(W^{1,q})^* = W^{-1,q'}$ by definition, from (3) and finally from $M_{S,*,p_1'} \leq M_{S,*,q'}$.
We refer the reader to [18] for the proof of

$$E^* = \left[(W^{1,p_0}, W^{1,p_1})_{\theta,p_\theta}\right]^* = (W^{-1,p_0'}, W^{-1,p_1'})_{\theta,p_\theta'}$$

with the concept of “doolittle couple”.
Then by real interpolation on the weak Lebesgue spaces (applied to the sublinear operator $M_{S,*,p_1'}$), we obtain:

$$\left\|M_{S,*,p_1'}(h)\right\|_{L^{p_\theta'}} \lesssim \|h\|_{E^*},$$

which according to the assumption (H_{s_0,s_1}) (for $s_0 = p_1'$ and $s_1 = p_\theta'$) yields

$$\|h\|_{W^{-1,p_\theta'}} \lesssim \|h\|_{E^*}.$$

Thus $E \hookrightarrow W^{1,p_\theta}$ and $E^* \hookrightarrow W^{-1,p_\theta'} = (W^{1,p_\theta})^*$. Using Hahn-Banach Theorem, we deduce that $E = W^{1,p_\theta}$ with equivalent norms. □

To regain results of the same kind as in [10], where the author assumes Poincaré inequality, we describe the following corollary:

Corollary 3.4 (Theorem 0.2)**.** *Assume that M satisfies (D) and admits a Poincaré inequality (P_r) (or the weaker assumption $(nhRR_r)$) for an $r \in (1,2)$. Then for all $p_0 \in (1,2)$ and $\theta \in (0,1)$ such that*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{2} < \frac{1}{r},$$

we have

$$\left(W^{1,p_0}, W^{1,2}\right)_{\theta,p_\theta} = W^{1,p_\theta}.$$

Proof. We set $s_1 = p_\theta'$ and $s_0 = 2$. Thanks to Theorem 3.3, we just have to check that $(s_0,s_1) \in \mathcal{I}_M$. This is a direct consequence of Corollary 2.21. □

Remark 3.5. *In the previous corollary, Poincaré inequality (P_r) could be replaced by the weaker non-homogeneous variant*

$$(\tilde{P}_r) \qquad \left(\fint_Q \left|f - \fint_Q f\right|^r d\mu\right)^{1/r} \leq Cr_Q \left(\fint_Q (|f|^r + |\nabla f|^r d\mu)\right)^{1/r}.$$

As $(\tilde{P}_r)$ is sufficient to obtain the $(nhRR_r)$ property. Moreover Assumption $(nhRR_r)$ is sufficient for the previous corollary, which corresponds to the statement of Theorem 0.2.

Remark 3.6. *In Corollary 3.4, we can chose $p_1 \leq 2$ (and not necessary equal to 2). Then under (DUE) and $(nhRR_r)$, we get the corresponding interpolation result.*

Let us compare these results with [10]. Note first that the results – even the proofs – of [10] in the non-homogeneous case still hold with this variant of Poincaré inequality $\tilde{P}_r$. In [10], the author just requires the condition $p_\theta > r$ to obtain the interpolation result under *local* doubling property and *local* Poincaré inequalities. The main tool (the “well-known” Calderón-Zygmund decomposition for Sobolev functions) of [10] permits to interpolate any Sobolev spaces (not only with $W^{1,2}$ or W^{1,p_1} with $p_1 \leq 2$) under Poincaré inequality (P_r).
The use of the exponent 2 is the most important in the litterature and that is why we mainly deal with it. In the case $p_1 \leq 2$, our assumption $(nhRR_r)$ is weaker than the corresponding Poincaré inequality (P_r). Consequently we regain the results of N. Badr ([10]).

However in the case where $p_1 > 2$, we can not recover her results as we require an extra assumption: the Riesz inequality (nhR_{p_1}). Our assumptions and the ones of [10] are not comparable when $p_1 > 2$. Which is interesting is that even in this case, we success to interpolate Sobolev spaces without assuming Poincaré inequalities.
An interesting question still stays open: we have weakened the assumption of Poincaré inequality, however we do not know which assumptions should be sufficient and necessary to prove an interpolation result. In the case $p_0, p_1 \leq 2$ our assumption $(nhRR_r)$ seems to be the well-adapted assumption ... We emphasize that an assumption has to be done since a recent counterexample of P. Auscher and N. Badr [4].

To finish, we refer the reader to an other work (joined with N. Badr, see [11]), where we develop a new theory for abstract Hardy-Sobolev spaces. Using these maximal inequalities, we prove some results for interpolation between Hardy-Sobolev spaces and Sobolev spaces. In this application, the arguments based on the well-known Calderòn-Zygmund decomposition do not work and these new maximal inequalities play a crucial role.

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Département de Mathématiques, Université Paris-Sud, 91405 Orsay Cedex (France)
E-mail address: frederic.bernicot@math.u-psud.fr
URL: http://www.math.u-psud.fr/∼bernicot