Math. Res. Lett. **16** (2009), *no.* 4, 663–681

ON THE UNIFORMITY OF THE IITAKA FIBRATION

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Abstract. We study pluricanonical systems on smooth projective varieties of positive Kodaira dimension, following the approach of Hacon-McKernan, Takayama and Tsuji succesfully used in the case of varieties of general type. We prove a uniformity result for the Iitaka fibration $X \dashrightarrow \mathrm{Iitaka}(X)$ of smooth projective varieties of positive Kodaira dimension, provided that $\mathrm{Iitaka}(X)$ is not uniruled, the variation of the fibration is maximal, and the general fiber has a good minimal model.

1. Introduction

Following Tsuji [Ts1] and [Ts2], Hacon and McKernan [HM], and Takayama [T] have independently given an algebro-geometric proof of the following beautiful result:

Theorem 1.1 (Hacon-McKernan, Takayama, Tsuji)**.** *For any positive integer n, there exists an integer m_n such that for any smooth complex projective variety X of general type of dimension n, the pluricanonical map*

$$\varphi_{mK_X} : X \dashrightarrow \mathbf{P}H^0(X, \mathscr{O}_X(mK_X))^*$$

is birational onto its image, for all $m \geq m_n$.

The purpose of this paper is to show that the methods used to prove Theorem 1.1 allow to obtain a similar uniformity result concerning the pluricanonical maps of algebraic varieties of arbitrary (positive) Kodaira dimension.

Before stating the result we need to recall some facts. Thanks to the work of Iitaka, it is well-known that, if $\kappa(X) > 0$, for large m such that $h^0(X, mK_X) \neq 0$ the images of the rational maps φ_{mK_X} stabilize i.e. they become birationally equivalent to a fiber space

$$\varphi_\infty : X_\infty \longrightarrow \mathrm{Iitaka}(X),$$

such that the restriction of K_X to a very general fiber F of φ_∞ has Kodaira dimension 0 and $\dim(\mathrm{Iitaka}(X)) = \kappa(X)$. This fibration is called the Iitaka fibration of X (see §2.3 for more details). It is natural to ask (cf. [HM, Conjecture 1.7]) whether the Iitaka fibration of X enjoys a uniformity property as in the case of varieties of general type. When $\kappa(X) = 1$ such a result has been proved in [FM, Theorem 6.1] with a dependence on the smallest integer b such that $h^0(F, bK_F) = 1$, and on the Betti number $B_{\dim(E')}$ of a non-singular model E' of the cover $E \to F$ of the general fiber F associated to the unique element of $|bK_F|$ (when X is a 3-fold with $\kappa(X) = 1$ this extra dependence may be dropped, see [FM, Corollary 6.2]). Here we generalize the Fujino-Mori result to arbitrary Kodaira dimension, under extra hypotheses.

Received by the editors August 13, 2007. Revision received September 11, 2008.

Theorem 1.2. *For any positive integers n, b, k, there exists an integer $m(n,b,k) > 0$ such that, for any algebraic fiber space $f : X \to Y$, with X and Y smooth projective varieties, $\dim(X) = n$, $\dim(Y) = \kappa(X)$, with general fiber F of f of zero Kodaira dimension, and such that:*

(i) *Y is not uniruled;*
(ii) *f has maximal variation;*
(iii) *the general fiber F of f has a good minimal model;*
(iv) *b is the smallest integer such that $h^0(F, bK_F) \neq 0$, and $\mathrm{Betti}_{\dim(E')}(E') \leq k$, where E' is a non-singular model of the cover $E \to F$ of the general fiber F associated to the unique element of $|bK_F|$;*

then the pluricanonical map

$$\varphi_{mK_X} : X \dashrightarrow \mathbf{P}H^0(X, \mathscr{O}_X(mK_X))^*$$

is birationally equivalent to f, for any $m \geq m(n,b,k)$ such that $h^0(X, mK_X) \neq 0$.

Recall that when F is a surface, up to a birational transformation, we may assume that the 12th plurigenus is non-zero and the 2nd Betti number is bounded by 22. Therefore, when $\kappa(X) = n-2$, the integer $m(n,b,k)$ only depends on n (notice that, because of the hypothesis on the maximality of the variation, if $\kappa(X) = \dim(X) - 1$ Theorem 1.2 does not apply as soon as $\dim(X) \geq 3$). The existence of good minimal models is known up to dimension 3 (cf. [Ko+]). On the other hand, condition (iii) is automatically satisfied for interesting classes of fibrations, e.g. those for which $c_1(F)$ is zero (or torsion).

The idea to prove Theorem 1.2 is quite natural. By the important result proved in [BDPP], the hypothesis (i) in Theorem 1.2 is equivalent to the pseudo-effectivity of the canonical divisor of Y. Then, thanks to the canonical bundle formula established in [FM], a positivity result due to Kawamata (cf. [Ka2, Theorem 1.1], where the hypotheses (ii) and (iii) of Theorem 1.2 appear), for the (semistable part of the) direct image of the relative pluricanonical sheaf allows to reduce the problem to the study of effective birationality for multiples of adjoint big divisors $K_Y + M$, where M is a big and nef $\mathbf{Q}$-Cartier divisor such that νM is integral. The hypothesis (iv) of Theorem 1.2 is needed to have an effective bound on the denominator of the $\mathbf{Q}$-divisor M. Then Theorem 1.2 is a consequence of the following result, which we prove using the techniques of [HM], [T], and [Ts1], [Ts2].

Theorem 1.3. *For any positive integers n and ν, there exists an integer $m_{n,\nu}$ such that for any smooth complex projective variety Y of dimension n with pseudo-effective canonical divisor, and any big and nef $\mathbf{Q}$-Cartier divisor M on Y such that νM is integral, the pluriadjoint map*

$$\varphi_{m(K_Y+M)} : Y \dashrightarrow \mathbf{P}H^0(Y, \mathscr{O}_Y(m(K_Y + M)))^*$$

is birational onto its image, for all $m \geq m_{n,\nu}$ divisible by ν.

As for Theorem 1.1, the methods do not lead to an effective constant $m_{n,\nu}$.

During the preparation of this article E. Viehweg kindly informed me that he and D.-Q. Zhang were also working on a generalization of the Fujino-Mori result. In their interesting preprint [VZ] they study the Iitaka fibration for varieties of Kodaira dimension 2, and obtain in this case the same uniformity result without the hypotheses

(i),(ii) and (iii) appearing in Theorem 1.2 (and with an effectively computable constant). Their same result, in the case of three-folds, has been obtained independently by Ringler [R].

2. Preliminaries

We recall a number of basic definitions and results that will be freely used in the paper.

2.1. Notation and conventions. We work over the field of complex numbers. Unless otherwise specified, a divisor will be integral and Cartier. If D and D' are $\mathbf{Q}$-divisors on a smooth variety X we write $D \sim_{\mathbf{Q}} D'$, and say that D and D' are $\mathbf{Q}$-linearly equivalent, if an integral multiple of $D - D'$ is linearly equivalent to zero. We write $D \equiv D'$ when they are numerically equivalent, that is when they have the same degree on every curve. The notation $D \leq D'$ means that $D' - D$ is effective. If $D = \sum a_i D_i$, we denote by $[D]$ the integral divisor $\sum [a_i] D_i$, where, as usual, $[a_i]$ is the largest integer which is less than or equal to a_i. We denote by $\{D\}$ the difference $D - [D]$. A *log-resolution* of a divisor $D \subset X$ is a proper birational morphism of smooth varieties $\mu : X' \to X$, such that the support of $\mathrm{Exc}(\mu) + \mu^* D$ has simple normal crossings. The existence of log-resolutions is insured by Hironaka's theorem. Given a surjective morphism $f : X \longrightarrow Y$ of smooth algebraic varieties the *relative dualizing sheaf* is the invertible sheaf associated to the divisor $K_{X/Y} := K_X - f^* K_Y$. An *algebraic fiber space* is a surjective morphism $f : X \longrightarrow Y$ between smooth projective varieties with connected fibers.

2.2. Volumes, big divisors and base loci. Recall that the volume of a line bundle (see [L1, §2.2.C] for a detailed account on the properties of this invariant) is the number

$$\mathrm{vol}_X(D) := \limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^n / n!}$$

It is actually a limit, and we have $\mathrm{vol}(mD) = m^{\dim(X)} \mathrm{vol}(D)$. Therefore one can define the volume of a $\mathbf{Q}$-divisor D as $\mathrm{vol}(D) := m^{-\dim(X)} \mathrm{vol}(mD)$, where m is an integer such that mD is integral. The volume is invariant by pull-back via a birational morphism. Moreover we have that $\mathrm{vol}(D) > 0$ if and only if D is big, and $\mathrm{vol}(D) = D^{\dim(X)}$ for nef divisors. For a singular variety Y, we denote by $\mathrm{vol}(K_Y)$ the volume of the canonical divisor of a desingularization $Y' \to Y$ (which does not depend on the choice of Y').

If V is a subvariety of X, following [ELMNP2] one defines the restricted volume as :

$$\mathrm{vol}_{X|V}(A) := \limsup_{m \to +\infty} \frac{h^0(X|V, mA)}{m^d / d!}$$

where

$$h^0(X|V, mA) := \dim \mathrm{Im}(H^0(X, mA) \to H^0(V, mA_{|V})).$$

Again, it is a limit, and we have that $\mathrm{vol}_{X|V}(mD) = m^{\dim(V)} \mathrm{vol}_{X|V}(D)$ (see [ELMNP2, Cor. 2.15 and Lemma 2.2]).

We will constantly use Kodaira's lemma : a $\mathbf{Q}$-divisor D is big if and only if $D \sim_{\mathbf{Q}} A + E$, where A is a $\mathbf{Q}$-ample divisor and E a $\mathbf{Q}$-effective one.

If $|T|$ is a linear system on X, its base locus is given by the set-theoretic intersection

$$\mathrm{Base}(|T|) := \bigcap_{L\in|T|} L.$$

Recall that given a Cartier divisor L on a variety X, its stable base locus (see [L1, pp. 127–128]) is

$$\mathbf{B}(L) := \bigcap_{m\geq 1} \mathrm{Base}(|mL|)$$

and its augmented base locus, which has been defined in [ELMNP1], is

$$\mathbf{B}_+(L) := \mathbf{B}(mL - H)$$

for $m \gg 0$ and H ample on X (the latter definition is independent of the choice of m and H). One checks that L is ample if, and only if, $\mathbf{B}_+(L) = \emptyset$, and L is big if, and only if, $\mathbf{B}_+(L) \neq X$. In the latter case, $X \setminus \mathbf{B}_+(L)$ is the largest open set on which L is ample. Equivalently one can define $\mathbf{B}_+(L) = \cap \mathrm{Supp}(E)$, where the intersection is taken over all possible decompositions $L \sim_{\mathbf{Q}} A+E$, where A is ample and E effective.

2.3. Iitaka fibration. We follow [L1, 2.1.A and 2.1.C]. Let L be a line bundle on a projective variety X. The semigroup $\mathbf{N}(L)$ of L is

$$\mathbf{N}(L) := \{m \geq 0 : h^0(X, mL) \neq 0\}.$$

If $\mathbf{N}(L)$ is not zero, then there exists a natural number $e(L)$, called the exponent of L, such that all sufficiently large elements in $\mathbf{N}(L)$ are multiples of $e(L)$. If $\kappa(X, L) = \kappa \geq 0$, then $\dim(\varphi_{m,L}(X)) = \kappa$ for all sufficiently large $m \in \mathbf{N}(L)$. Iitaka's result is the following.

Theorem 2.1 (Iitaka fibrations, see [L1], Theorem 2.1.33, or [Mo])**.** *Let X be a normal projective variety and L a line bundle on X such that $\kappa(X, L) > 0$. Then for all sufficiently large $k \in \mathbf{N}(L)$ there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\quad u_\infty \quad} & X_{\infty,L} \\ \downarrow \varphi_{k,L} & & \downarrow \varphi_{\infty,L} \\ \mathrm{Im}(\varphi_{k,L}) & \xleftarrow{\quad v_{k,L} \quad} & \mathrm{Iitaka}(X, L) \end{array} \tag{2.1}$$

where the horizontal maps are birational. One has $\dim(\mathrm{Iitaka}(X, L)) = \kappa(X, L)$. *Moreover if we set $L_\infty = u_\infty^* L$ and F is the very general fiber of $\varphi_{\infty,L}$, we have $\kappa(F, L_\infty|_F) = 0$.*

We will deal only with the case $L = K_X$, and simply write $\mathrm{Iitaka}(X) := \mathrm{Iitaka}(X, K_X)$. Since the Iitaka fibration is determined only up to birational equivalence, and the questions we are interested in are of birational nature, we will often tacitly assume that $\mathrm{Iitaka}(X)$ is smooth, and that we have an algebraic fiber space $X \longrightarrow \mathrm{Iitaka}(X)$. Notice that as a consequence of the finite generation of the canonical ring proved in [BCHM] we have that, for large m, the images of the pluricanonical maps φ_{mK_X} are all isomorphic to $\mathrm{Proj}(\bigoplus_{m\geq 0} H^0(X, mK_X))$.

2.4. Multiplier ideals. If D is an effective $\mathbf{Q}$-divisor on X one defines the multiplier ideal as follows :

$$\mathscr{I}(X,D) := \mu_* \mathscr{O}_{X'}(K_{X'/X} - [\mu^* D])$$

where $\mu : X' \to X$ is a log-resolution of (X,D), and the definition is independent of the choice of the log-resolution.

Notice that if D is a $\mathbf{Q}$-divisor with simple normal crossings, then $\mathscr{I}(X,D) = \mathscr{O}_X(-[D])$. If D is integral we simply have

$$\mathscr{I}(X,D) = \mathscr{O}_X(-D). \tag{2.2}$$

Again, we refer the reader to Lazarsfeld's book [L2] for a complete treatment of the topic. We now recall Nadel's vanishing theorem.

Theorem 2.2 (see [L2, Theorem 9.4.8])**.** *Let X be a smooth projective variety. Let D be an effective $\mathbf{Q}$-divisor on X, and L a divisor on X such that $L - D$ is big and nef. Then, for all $i > 0$, we have*

$$H^i(X, \mathscr{I}(X,D) \otimes \mathscr{O}_X(K_X + L)) = 0.$$

2.5. Singularities of pairs and Non-klt loci. Recall that, in the literature, a pair (X,D) is a normal variety together with a $\mathbf{Q}$-Weil divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. In this paper the situation is much simpler : the variety will always be smooth and the divisor will be an effective Cartier divisor. A pair (X,D) is Kawamata log-terminal, klt for short, (respectively non-klt) at a point x, if

$$\mathscr{I}(X,D)_x = \mathscr{O}_{X,x} \quad (\text{respectively } \mathscr{I}(X,D)_x \neq \mathscr{O}_{X,x}).$$

A pair is klt if it is klt at each point $x \in X$. A pair (X,D) is log-canonical, lc for short, at a point x, if

$$\mathscr{I}(X,(1-\varepsilon)D)_x = \mathscr{O}_{X,x} \quad \text{for all rational number } 0 < \varepsilon < 1.$$

A pair is lc if it is lc at each point $x \in X$ (for a survey on singularities of pairs and many related results, see [Ko2]). We set

$$\text{Non-klt}(X,D) := \text{Supp}(\mathscr{O}_X / \mathscr{I}(X,D))$$

with the reduced structure and call it the Non-klt locus of the pair (X,D).

A simple though extremely useful way of producing example of non-klt pairs is to consider divisors having high multiplicity at a given point, since we have [L2, Proposition 9.3.2]

$$\text{mult}_x(D) \geq \dim(X) \Rightarrow \mathscr{I}(X,D)_x \neq \mathscr{O}_{X,x}. \tag{2.3}$$

Notice that if $D = \sum a_i D_i$ is an effective $\mathbf{Q}$-divisor with simple normal crossings, the pair (X,D) is klt (respectively lc) if, and only if, $a_i < 1$ (resp. $a_i \leq 1$). We recall two fundamental results describing the effect of small perturbations of D on its Non-klt locus.

Lemma 2.3. *Let X be a smooth projective variety, x_1 and x_2 two distinct points on X, and D an effective $\mathbf{Q}$-divisor such that (X,D) is lc at x_1 and non-klt at x_2. Let V be an irreducible component of Non-klt(X,D) passing through x_1. Let $B \sim_{\mathbf{Q}} A + E$ be a big divisor on X, with A $\mathbf{Q}$-ample and E $\mathbf{Q}$-effective such that $x_1, x_2 \notin \text{Supp}(E)$. Then there exists an effective divisor $F \sim_{\mathbf{Q}} B$ and, for any arbitrarily small rational $\delta > 0$, there exists a unique rational number $b_\delta > 0$ such that:*

(1) $(X, (1-\delta)D + b_\delta F)$ *is lc at* x_1*;*
(2) $(X, (1-\delta)D + b_\delta F)$ *is non-klt at* x_2*;*
(3) *All the irreducible components of Non-klt*$(X, (1-\delta)D + b_\delta F)$ *containing* x_1 *are contained in* V*.*

Moreover $\liminf_{\delta \longrightarrow 0} b_\delta = 0$.

Proof. See e.g. [Pa, Lemma A.3]. The reader may also look at [L2, Lemma 10.4.8] and [Ko2, Th. 6.9.1]. □

Lemma 2.4. *Let* X *be a smooth projective variety and* D *an effective* $\mathbf{Q}$*-divisor. Let* V *be an irreducible component of Non-klt*(X, D) *of dimension* d*. There exists a dense subset* U *in the smooth locus of* V *and a rational number* $\varepsilon_0 : 0 < \varepsilon_0 < 1$ *such that, for any* $y \in U$*, any effective* $\mathbf{Q}$*-divisor* B *whose support does not contain* V *and such that*

$$\operatorname{mult}_y B|_V > d$$

and any rational number $\varepsilon : 0 < \varepsilon < \varepsilon_0$*, the locus Non-klt*$(X, (1-\varepsilon)D + B)$ *contains* y*. If moreover* (X, D) *is lc at the general poit of* V *and* $\mathscr{I}(X, D+B) = \mathscr{I}(X, D)$ *away from* V*, then Non-klt*$(X, (1-\varepsilon)D + B)$ *is properly contained in* V *in a neighborhood of any* $y \in U$*.*

Proof. See e.g. [Pa, Lemma A.4]. Again, for similar statements, see [L2, Lemma 10.4.10] and [Ko2, Th. 6.8.1]. □

3. Positivity results for direct images

In this section we collect results concerning some positivity properties of the direct image of the relative dualizing sheaf that we will use.

3.1. The semistable part and a canonical bundle formula. We recall results contained in [FM, §2 and 4]. Let $f : X \longrightarrow Y$ an algebraic fiber space, whose general fiber F has Kodaira dimension zero. Let b be the smallest integer such that the b-th plurigenus $h^0(F, bK_F)$ of F is non-zero. Then there exists a divisor $L_{X/Y}$ on Y (which is unique modulo linear equivalence, and which depends only on the birational equivalence class of X over Y) such that, up to birationally modify X, we have

$$H^0(Y, ibK_Y + iL_{X/Y}) = H^0(X, ibK_X) \tag{3.1}$$

for all $i > 0$ (the divisor $L_{X/Y}$ is defined by the double dual $f_*\mathscr{O}_X(ibK_{X/Y})^{**}$). Moreover the divisor $L_{X/Y}$ may be written as

$$L_{X/Y} = L^{ss}_{X/Y} + \Delta \tag{3.2}$$

where $L^{ss}_{X/Y}$ is a $\mathbf{Q}$-Cartier divisor, called the semistable part or the moduli part (which is compatible with base change), and Δ is an effective $\mathbf{Q}$-divisor called the boundary part. The divisor $L_{X/Y}$ coincides with its moduli part when f is semistable in codimension 1, and

$$L^{ss}_{X/Y} \text{ is nef.} \tag{3.3}$$

The previous results (3.1), (3.2) and (3.3) are contained in Proposition 2.2, Corollary 2.5 and Theorem 4.5 (iii) of [FM]. The reader may also look at [Ko3] and [Mo, §4-5].

For our application it is important to bound the denominators of $L^{ss}_{X/Y}$. Let B denote the Betti number $B_{\dim(E')}$ of a non-singular model E' of the cover $E \to F$ of the general fiber F associated to the unique element of $|bK_F|$. By [FM, Theorem 3.1] there exists a positive integer $r = r(B)$ such that

$$r \cdot L^{ss}_{X/Y} \text{ is an integral divisor.} \tag{3.4}$$

3.2. Maximal variation and bigness of the semistable part. Let $f : X \longrightarrow Y$ be an algebraic fiber space. Recall that the *variation* of f is an integer $\mathrm{Var}(f)$ such that there exists a fiber space $f' : X' \longrightarrow Y'$ with $\dim(Y') = \mathrm{Var}(f)$, a variety $\bar{Y}$, a generically surjective morphism $\varrho : \bar{Y} \longrightarrow Y'$ and a generically finite morphism $\pi : \bar{Y} \longrightarrow Y$ such that the two fiber spaces induced by ϱ and by π respectively are birationally equivalent. The fibration f has *maximal variation* if $\mathrm{Var}(f) = \dim(Y)$. Equivalently, $f : X \to Y$ has maximal variation if there exists a non-empty open subset $U \subset Y$ such that for any $y_0 \in U$ the set $\{y \in U : f^{-1}(y) \sim_{\text{birational}} f^{-1}(y_0)\}$ is finite. As proved by Fujino ([Fuj, Theorem 3.8]), we always have

$$\kappa(Y, L^{ss}_{X/Y}) \leq \mathrm{Var}(f). \tag{3.5}$$

On the other hand, by a result due to Kawamata [Ka2, Theorem 1.1], if the general fiber of $f : X \longrightarrow Y$ possesses a good minimal model (i.e. a minimal model whose canonical divisor is semiample), then

$$\kappa(Y, L^{ss}_{X/Y}) \geq \mathrm{Var}(f). \tag{3.6}$$

In particular, we have

Corollary 3.1 (Kawamata)**.** *Let $f : X \longrightarrow Y$ be an algebraic fiber space that has maximal variation and such that the general fiber has a good minimal model. Then $L^{ss}_{X/Y}$ is big.*

Fujino's inequality (3.5) implies that the maximality of the variation is a necessary condition for the bigness of $L^{ss}_{X/Y}$.

3.3. Weak positivity. Viehweg introduced the notion of weak positivity for torsion-free coherent sheaf $\mathscr{E}$ on a projective variety V : if V_0 is the largest open subset on which $\mathscr{E}$ is locally free, the sheaf $\mathscr{E}$ is weakly positive if there exists a open dense subset U of V_0 such that for any ample divisor H on V and any positive integer a, there exists a positive integer b such that the sheaf $(\mathrm{Sym}^{ab}\, \mathscr{E}|_{V_0}) \otimes (bH|_{V_0})$ is generated on U by its global sections on V_0 (see [Vie2] for a detailed discussion of this notion). We will make use of the following positivity result for direct images, due to Campana [Ca, Theorem 4.13], which improves on previous results obtained by Kawamata [Ka1], Kollár [Ko1] and Viehweg [Vie1] (see also [Lu, Proposition 9.8]).

Theorem 3.2 (Campana)**.** *Let $f : V' \to V$ be a morphism with connected fibres between smooth projective varieties. Let Δ be an effective $\mathbf{Q}$-divisor on V' whose restriction to the general fibre is lc and has simple normal crossings. Then, the sheaf*

$$f_* \mathscr{O}_{V'}(m(K_{V'/V} + \Delta))$$

is weakly positive for all positive integer m such that $m\Delta$ is integral.

4. Extension of log-pluricanonical forms

In the course of the proof of Theorem 1.3 we will need to lift (twisted) pluricanonical forms on a smooth hypersurface to the ambient variety.

First, we recall Takayama's extension result [T, Theorem 4.1] (cf. [HM, Corollary 3.17] for the corresponding result, which is a generalization of a former result of Kawamata's [Ka3]).

Theorem 4.1 (Takayama)**.** *Let Y be a smooth projective variety. Let $H \subset Y$ be a smooth irreducible hypersurface. Let $L' \sim_{\mathbf{Q}} A' + E'$ a big divisor on Y with*

- *A' a nef and big $\mathbf{Q}$-divisor such that $H \not\subset \mathbf{B}_+(A')$;*
- *E' an effective $\mathbf{Q}$-divisor whose support does not contain H and such that the pair $(H, E'|_H)$ is klt.*

Then the restriction

$$H^0(Y, m(K_Y + H + L')) \longrightarrow H^0(H, m(K_H + L'|_H))$$

is surjective for all integer $m \geq 0$.

The precise statement we need is the following.

Corollary 4.2. *Let Y be a smooth projective variety, M an effective and nef integral Cartier divisor on Y. Let $H \subset Y$ be a smooth irreducible hypersurface such that $H \not\subset \mathrm{Supp}(M)$. Let $L \sim_{\mathbf{Q}} A + E$ a big divisor on Y with*

- *A a nef and big $\mathbf{Q}$-divisor such that $H \not\subset \mathbf{B}_+(A)$;*
- *E an effective $\mathbf{Q}$-divisor whose support does not contain H and such that the pair $(H, E|_H)$ is klt.*

Then the restriction

$$H^0(X, m(K_Y + M + H + L)) \longrightarrow H^0(H, m(K_H + M|_H + L|_H))$$

is surjective for all integer $m \geq 0$.

For other extension results, all inspired by [S2], the reader may look at [C], [Pă] and [Var].

Proof of Corollary 4.2. We want to apply Theorem 4.1 to $L' = M + L$. We can write

$$M + L = (A + M) + E = (\text{big and nef}) + \text{effective},$$

as M is nef. The only thing to check is that

$$H \not\subset \mathbf{B}_+(A + M),$$

assuming $H \not\subset \mathbf{B}_+(A)$. But this is immediate, since by the nefness of M, any decomposition $A \sim_{\mathbf{Q}} (\text{ample}) + E$, with E effective, gives rise to a decomposition $A + M \sim_{\mathbf{Q}} (\text{ample} + M) + E$. Therefore we have

$$\mathbf{B}_+(A + M) \subset \mathbf{B}_+(A)$$

and we are done. □

5. Bounding the restricted volumes from below

It is well-known to specialists that a positive lower bound to the restricted volumes of a big divisor A on a variety Y allows to construct, along the lines of the Angehrn-Siu proof of the Fujita conjecture, a global section of $K_Y + A$ separating two general points on Y (cf. [T, Proposition 5.3] and [ELMNP1, Theorem 2.20]). Such a lower bound is the object of the following result.

Theorem 5.1. *Let Y be a smooth projective variety, M an effective and nef integral Cartier divisor on Y, and $V \subset Y$ be an irreducible subvariety not contained in the support of M. Let L be a big divisor on Y and $L \sim_{\mathbf{Q}} A + E$ a decomposition such that :*

(i) *A is an ample $\mathbf{Q}$-divisor ;*
(ii) *E is an effective $\mathbf{Q}$-divisor such that V is an irreducible component of Non-klt(Y, E) with (Y, E) lc at the general point of V.*

Then : $$\mathrm{vol}_{Y|V}(K_Y + M + L) \geq \mathrm{vol}(K_V + M|_V).$$

The proof of the theorem is a fairly easy consequence of the extension result 4.2 when $\mathrm{codim}(V) = 1$. In the general case, it also requires, among other things, the use of Campana's weak positivity result 3.2. In the following two subsections we prove Theorem 5.1.

5.1. The case $\mathrm{codim}_Y(V) = 1$. First of all, by [T, Lemma 4.6] we may assume without loss of generality that V is smooth. Hypothesis (ii) simply means that V appears with multiplicity 1 in E. We may therefore take a modification $\mu : Y' \to Y$ such that $\mu^* E = V' + F$ has simple normal crossings and moreover

$$V' \not\subset \mathrm{Supp}(F). \tag{5.1}$$

Take an integer $m_0 > 0$ such that $m_0(\mu^* A + \{F\})$ has integer coefficients. By (5.1) the support of this divisor does not contain V', so we have an inclusion

$$\begin{array}{c} H^0(V', m(K_{V'} + \mu^* M|_{V'})) \\ \big\downarrow \\ H^0(V', m(K_{V'} + \mu^* M|_{V'} + (\mu^* A + \{F\})|_{V'})) \end{array} \tag{5.2}$$

for any integer $m > 0$ divisible by m_0. Since the pair $(Y', \{F\})$ is klt, applying the extension result 4.2 to the divisor $\mu^* A + \{F\}$, and observing that $\mu^* L - [F] \sim_{\mathbf{Q}} V' + \mu^* A + \{F\}$, we get a surjection

$$\begin{array}{c} H^0(Y', m(K_{Y'} + \mu^* M + \mu^* L - [F])) \\ \big\downarrow \\ H^0(V', m(K_{V'} + \mu^*(M)|_{V'} + (\mu^* A + \{F\})|_{V'})). \end{array} \tag{5.3}$$

In conclusion we have

$$\begin{aligned} h^0(V, m(K_V + M|_V)) &= h^0(V', m(K_{V'} + \mu^* M|_{V'})) \\ (\ (5.2) + (5.3)\ ) &\leqslant h^0(Y'|V', m(K_{Y'} + \mu^* M + \mu^* L - [F])) \\ &\leqslant h^0(Y'|V', m(K_{Y'} + \mu^* M + \mu^* L)) \\ &= h^0(Y|V, m(K_Y + M + L)) \end{aligned}$$

so Theorem 5.1 is proved in this case.

5.2. The case $\operatorname{codim}_Y(V) \geq 2$**.** We follow here Debarre's presentation [D, §6.2]. We may assume V smooth (see [T, Lemma 4.6]). Take a log-resolution $\mu = Y' \to Y$ of E, and write

$$\mu^* E - K_{Y'/Y} = \sum_F a_F F.$$

By hypothesis V is an irreducible component of Non-klt(Y, E) such that (Y, E) is lc at the general point of V. This means that

(i) if V is strictly contained in $\mu(F)$, then $a_F < 1$;
(ii) if $V = \mu(F)$, then $a_F \leq 1$, with equality for at least one F.

Thanks to the so-called concentration method due to Kawamata and Shokurov (see [KMM, §3-1], [T, Lemma 4.8] and [T, Remark 4.9]) one can further assume that

(iii) there exists a unique divisor (which will be denoted by V') among the F's such that $\mu(V') = V$;
(iv) if $a_F < 0$ then F is μ-exceptional.

Therefore we have the commutative diagram of smooth varieties :

(5.4)
$$\begin{array}{ccc} V' & \hookrightarrow & Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mu} \\ V & \hookrightarrow & Y \end{array}$$

and if we set $G := \sum_{F \neq V'} a_F F$, and write $[G]$ as a difference of effective divisors without common components $[G] = G_1 - G_2$, by (i)-(iv) above we have that :

- G_2 is μ-exceptional ;
- $V \not\subset \mu(\operatorname{Supp}(G_1))$.

Hence, for any integer $m > 0$, the sheaf $\mu_* \mathscr{O}_{Y'}(-m[G])$ is an ideal sheaf on Y whose cosupport does not contain V, so that

$$\begin{aligned} H^0(Y, m(K_Y + M + L)) &\supset H^0(Y, \mu_* \mathscr{O}_{Y'}(-m[G])(m(K_Y + M + L))) \\ &\cong H^0(Y', m(\mu^*(K_Y + M + A + E) - [G])) \\ &\cong H^0(Y', m(K_{Y'} + \mu^* M + V' + \{G\} + \mu^* A)), \end{aligned} \tag{5.5}$$

as soon as the divisors on the right-hand side are integral. Since the pair $(V', \{G\}|_{V'})$ is klt, we can apply the extension result Corollary 4.2 to the divisor

$$L' := \mu^* L - K_{Y'/Y} - V' - [G] \sim_{\mathbf{Q}} \{G\} + \mu^* A$$

and to the smooth hypersurface $V' \subset Y'$. Hence we get a surjection

$$H^0(Y', m(K_{Y'} + \mu^* M + V' + \{G\} + \mu^* A)) \twoheadrightarrow H^0(V', m(K_{V'} + \mu^*(M)|_{V'} + L'|_{V'})).$$

The last surjection together with the injection (5.5) and, again, the fact that the cosupport of $\mu_*\mathscr{O}_{Y'}(-m[G])$ does not contain V, leads to the inclusion :

$$H^0(V', m(K_{V'} + \mu^*(M)|_{V'} + L'|_{V'})) \subset H^0(Y|V, m(K_Y + M + L)). \tag{5.6}$$

On the other hand, thanks to Campana's theorem 3.2, one can show that for a suitable positive integer m' we have :

$$H^0(V', m'(K_{V'/V} + \{G\}|_{V'} + f^*A|_V)) \neq 0$$

(see [D, pages 17-18]). Hence we obtain by multiplication an injection

$$H^0(V, mm'(K_V + M|_V)) \hookrightarrow H^0(V', mm'(K_{V'} + f^*(M|_V) + L'|_{V'})).$$

The last inclusion, together with (5.6) and the fact that the restricted volumes are limits complete the proof of Theorem 5.1. □

Using the log-concavity property of the restricted volume, established in [ELMNP2], we deduce from Theorem 5.1 the following consequence which will be the key ingredient in the inductive proof of Theorem 6.2.

Corollary 5.2. *Let Y be a smooth projective variety. Let M' be an effective and nef integral Cartier divisor on Y and $V \subset Y$ an irreducible subvariety not contained in the support of M'. Let L be a big divisor on X and $L \sim_{\mathbf{Q}} A + E$ a decomposition such that :*

(i) *A is an ample $\mathbf{Q}-$divisor ;*
(ii) *E is an effective $\mathbf{Q}-$divisor such that V is an irreducible component of Non-klt(Y, E) with (Y, E) lc at the general point of V;*
(iii) *$K_Y + L$ is big and $V \not\subset \mathbf{B}_+(K_Y + L)$.*

Then, for any positive integer ν, we have

$$\mathrm{vol}_{Y|V}(K_Y + \frac{1}{\nu}M' + L) \geq \frac{1}{\nu^{\dim(V)}}\,\mathrm{vol}(K_V + \frac{1}{\nu}M'|_V).$$

Proof. Write

$$K_Y + \frac{1}{\nu}M' + L = \frac{1}{\nu}(K_Y + M' + L) + (1 - \frac{1}{\nu})(K_Y + L).$$

By (iii), thanks to the log-concavity property of the restricted volume proved in [ELMNP2, Theorem A] we have

$$\mathrm{vol}_{Y|V}(K_Y + \frac{1}{\nu}M' + L)^{1/d} \geq \frac{1}{\nu}\,\mathrm{vol}_{Y|V}(K_Y + M' + L)^{1/d} + (1 - \frac{1}{\nu})\,\mathrm{vol}_{Y|V}(K_Y + L)^{1/d},$$

where $d = \dim(V)$. Therefore, by Theorem 5.1, we obtain

$$\mathrm{vol}_{Y|V}(K_Y + \frac{1}{\nu}M' + L)^{1/d} \geq \frac{1}{\nu}\,\mathrm{vol}(K_V + M'|_V)^{1/d} \geq \frac{1}{\nu}\,\mathrm{vol}(K_V + \frac{1}{\nu}M'|_V)^{1/d}$$

and the corollary is proved. □

Remark 5.3. We will apply Corollary 5.2 to the base Y of the fibration $f : X \longrightarrow Y$ and to (multiples of) a divisor $L = K_Y + \alpha L^{ss}_{X/Y}$, where α will be a certain positive rational number. The pseudo-effectivity of K_Y that appears among the hypotheses of Theorem 1.3 is therefore needed here to insure the bigness of the sum $K_Y + L$ that appears in Corollary 5.2, hypothesis (iii).

6. Point separation for big pluriadjoint systems

From Theorem 1.3 it is easy to deduce the existence of a uniform positive lower bound on the volume of big adjoint linear systems $K_Y + M$ with M nef.

Corollary 6.1. *For any positive integers n and ν, any smooth complex projective variety Y of dimension n and any big and nef $\mathbf{Q}$-divisor M with $\nu \cdot M$ integral, and such that $K_Y + M$ is big, we have :*

$$\mathrm{vol}(K_Y + M) \geq \frac{1}{(\nu \cdot m_{n,\nu})^n}$$

where $m_{n,\nu}$ is as in Theorem 1.3.

Proof. Let $m_{n,\nu}$ be as in Theorem 1.3. Let $m = \nu \cdot m_{n,\nu}$. Let $\mu : Y' \to Y$ be the blow-up along the base locus of $|m(K_Y + M)|$. Then we can write

$$\mu^* m(K_Y + M) = |G| + F,$$

where $|G|$ is the base-point-free part, and F is the fixed part. In particular G is nef, so $\mathrm{vol}(G) = G^n$. In conclusion we have :

$$\begin{aligned} \mathrm{vol}(K_Y + M) &= \frac{\mathrm{vol}(\mu^* m(K_Y + M))}{m^n} \geq \frac{1}{m^n}\,\mathrm{vol}(G) \\ &= \frac{1}{m^n} G^n = \frac{1}{m^n}\deg \varphi_{|G|}(Y') \geq \frac{1}{m^n}. \end{aligned} \tag{6.1}$$

□

On the other hand we will see that a sort of converse to Corollary 6.1 is true. Namely, assuming the existence of such a lower bound in dimension $< n$, we will determine an effective multiple of $K_Y + M$ which is birational. The multiple will still depend on its volume but in a very precise way, sufficient to derive Theorem 1.3.

Theorem 6.2. *Let n and ν be positive integers. Suppose there exists a positive constant v such that, for any smooth projective variety V of dimension $< n$ with pseudo-effective canonical divisor, and any big and nef $\mathbf{Q}$-Cartier divisor N on V such that νN is integral, we have $\mathrm{vol}(K_V + N) \geq v$. Then, there exists two positive constants $a := a_{n,\nu}$ and $b := b_{n,\nu}$ such that, for any smooth projective variety Y of dimension n with pseudo-effective canonical divisor, and any big and nef $\mathbf{Q}$-Cartier divisor M on Y such that νM is integral, the rational pluriadjoint map*

$$\varphi_{m(K_Y+M)} : Y \dashrightarrow \mathbf{P}H^0(Y, \mathscr{O}_Y(m(K_Y + M)))^*$$

is birational onto its image, for all

$$m \geq a + \frac{b}{\mathrm{vol}(K_Y + M)^{1/n}}$$

such that mM is integral.

6.1. Proof of Theorem 6.2. The proof follows the approach adopted by Angehrn and Siu [AS] in their study of the Fujita conjecture (see also [Ko2, Theorem 5.9]), with the variations appearing in [HM], [T] and [Ts1],[Ts2] to make it work for big divisors, and it is based on the following application of Nadel's vanishing theorem.

Lemma 6.3. *Let Y be a smooth projective variety. Let M (respectively E) be a big and nef (resp. a pseudo-effective) $\mathbf{Q}$-divisor. Let x_1, x_2 be two points outside the support of E. Suppose there exists a positive rational number t_0 such that the divisor $D_0 \sim t_0(M+E)$ satisfies the following:*

(i) *$x_1, x_2 \in$ Non-klt(Y, D_0);*
(ii) *x_1 is an isolated point in Non-klt(Y, D_0).*

Then, for all integer $m \geq t_0 + 1$ such that $(m-1)E + mM$ is integral, there exists a section $s \in H^0(Y, K_Y + (m-1)E + mM)$ such that $s(x_1) \neq 0$ and $s(x_2) = 0$.

Proof. Take any integer $m \geq t_0 + 1$ such that $(m-1)E + mM$ is integral. Notice that

$$D_0 + (m - t_0 - 1)E$$

is an effective $\mathbf{Q}$-divisor, and that

$$(m-1)E + mM - (D_0 + (m - t_0 - 1)E) = (m - t_0)M$$

is a $\mathbf{Q}$-divisor which is big and nef. Hence by Nadel's vanishing theorem 2.2 we have:

$$H^1(Y, \mathscr{I}(Y, D_0 + (m - t_0 - 1)E) \otimes \mathscr{O}_Y(K_Y + (m-1)E + mM)) = 0.$$

Let V_0 be the subscheme defined by the ideal sheaf $\mathscr{I}(Y, D_0 + (m - t_0 - 1)E)$ and consider the short exact sequence of $V_0 \subset Y$:

$$0 \longrightarrow \mathscr{I}(Y, D_0 + (m - t_0 - 1)E) \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_{V_0} \longrightarrow 0.$$

Tensoring it with $\mathscr{O}_Y(K_Y + (m-1)E + mM)$, and taking cohomology, we thus get a surjection:

$$H^0(Y, K_Y + (m-1)E + mM) \twoheadrightarrow H^0(V_0, (K_Y + (m-1)E + mM)|_{V_0}).$$

Notice that as the points x_1, x_2 lie outside the support of E, around them we have

$$\text{Non-klt}(Y, D_0) = \text{Non-klt}(Y, D_0 + (m - t_0 - 1)E),$$

that is, V_0 still contains x_1, x_2, the former as an isolated point. In particular, there exists a section

$$s \in H^0(Y, K_Y + (m-1)E + mM)$$

such that

$$s(x_1) \neq 0 \text{ and } s(x_2) = 0.$$

□

Using (2.3) it is easy to construct a rational multiple of the big divisor $K_Y + M$ satisfying the condition (i) above. The main problem is that its Non-klt locus may well have positive dimension at x_1. We will then proceed by descending induction and use the lower bound on the restricted volumes proved in Corollary 5.2, in order to cut down the dimension of the Non-klt locus at x_1 and end up with a divisor $D_0 \sim t_0(K_Y + M)$, with $t_0 < a + b/(\text{vol}(K_Y + M))^{1/n}$ and satisfying both hypotheses of Lemma 6.3. In the course of the proof we will invoke the following elementary result.

Lemma 6.4. *Let Y be a smooth projective variety, and M an effective $\mathbf{Q}$-divisor such that $K_Y + M$ is big. Let V be a subvariety passing through a very general point of Y and $\varphi : V' \to V$ a desingularization. Then the $\mathbf{Q}$-divisor $K_{V'} + \varphi^* M$ is big.*

Proof. Thanks to the existence of the Hilbert scheme we may assume there exists a smooth family $\mathscr{V} \to B$ and a finite surjective morphism $\Phi : \mathscr{V} \to Y$ such that its restriction to the general fiber gives $\varphi : V' \to V$. Take an integer $m > 0$ such that mM is integral. Since Φ is finite, and $\Phi^*|m(K_Y + M)| \subset |m(K_{\mathscr{V}} + \Phi^* M)|$, the divisor $K_{\mathscr{V}} + \Phi^* M$ is big, and so is its restriction to the general fiber over B. But the normal bundle of any fiber in the family is trivial, so by adjunction we have $(K_{\mathscr{V}})|_{V'} = K_{V'}$ and we are done. □

Proof of Theorem 6.2. The proof follows the Angehrn-Siu approach, as in the case $M = 0$ that was considered in [HM], [T] and in [Ts1], [Ts2]. We will proceed by descending induction on $d \in \{1, \dots, n\}$ to produce an effective $\mathbf{Q}-$divisor $D_d \sim t_d(K_Y + M)$ such that :

(1) $x_1, x_2 \in \text{Non-klt}(Y, D_d)$;
(2) (Y, D_d) is lc at a non-empty subset of $\{x_1, x_2\}$, say at x_1;
(3) $\text{Non-klt}(Y, D_d)$ has a unique irreducible component V_d passing through x_1 and $\dim V_d \leq d$;
(4) $t_d < t_{d+1} + v_{d+1}$ with $v_{d+1} = \nu(d+1)(2/v')^{1/(d+1)}(t_{d+1} + 2) + \varepsilon$, where v' equals $\text{vol}(K_Y + M)$ (resp. v' equals v) if $d = n - 1$ (resp. if $d < n - 1$), and $\varepsilon > 0$ may be taken arbitrarily small.

(We set $t_n = 0$). Take two very general points x_1 and x_2 on Y. Precisely, they must be outside, the support of the effective divisor E in the Kodaira decomposition of $K_Y + M \sim_{\mathbf{Q}} A + E$, the sub-locus of Y covered by the images of $\mathbf{P}^1$ (since we want our subvarieties V_d to be non-uniruled), and the union of all the log-subvarieties of the pair (Y, M) which are not of log-general type (i.e. subvarieties Z of Y not contained in M and such that $K_{Z'} + \nu^* M$ is not big, where $\nu : Z' \to Z$ is any desingularization on Z). As in the first step of the Angehrn-Siu proof, thanks to the bigness of $K_Y + M$, we can pick an effective $\mathbf{Q}$-divisor $D_{n-1} \sim t_{n-1}(K_Y + M)$ which has multiplicity $> n$ at both points, as soon as $t_{n-1} \leq n 2^{1/n} \,\text{vol}(K_Y + M)^{-1/n} + \varepsilon$ (with $\varepsilon > 0$ arbitrarily small). Indeed having multiplicity $\geq r$ at x_1 and x_2 imposes $2\binom{n+r-1}{n} \sim 2\frac{r^n}{n!}$ conditions. On the other hand, for $m \gg 0$ such that $[mv_n]M$ is integral, the dimension of $H^0(Y, [mv_n](K_Y + M))$ grows as $\text{vol}(K_Y + M)\frac{[mv_n]^n}{n!}$ so we have

$$\text{vol}(K_Y + M)\frac{[mv_n]^n}{n!} > \frac{2m^n n^n}{n!}$$

as soon as $v_n := n\big(2/\,\text{vol}(K_Y + M)\big)^{1/n} + \varepsilon$ (with $\varepsilon > 0$ arbitrarily small). In particular, by (2.3), we get that $\text{Non-klt}(Y, D_{n-1}) \ni x_1, x_2$ and, up to multiplying by a positive rational number ≤ 1, we can assume that (Y, D_{n-1}) is lc at one of the two points, say at x_1. Also, up to performing an arbitrarily small perturbation of D_{n-1}, thanks to Lemma 2.3, we may assume there exists a unique irreducible component of $\text{Non-klt}(Y, D_{n-1})$ through x_1. The base of the induction is therefore completed.

For the inductive step we proceed as follows. Suppose that we have constructed an effective $\mathbf{Q}$-divisor $D_d \sim t_d(K_Y + M)$ satisfying conditions (1), (2), (3) and (4) above.

Suppose for simplicity that D_d is non-klt at x_2 and lc at x_1 (the other two possibilities, which are treated in the same way, but render the discussion more complicated, are discussed in details in [Pa, §A.3], when $M = 0$. The general case can be treated in the same way). Also, we may assume that x_1 is a non-singular point of Non-klt(Y, D_d) (if not, a limiting procedure described in [L2, 10.4.C] allows to conclude). Since the points lie outside the support of E the same is true for $(Y, D_d + tE)$, where $t := [t_d] + 1 - t_d$. Since K_Y is pseudo-effective, adding to it any positive multiple of $K_Y + M$ we still get a big divisor. Therefore we apply Corollary 5.2 to the divisors

$$L := ([t_d] + 1)(K_Y + M) \sim tA + (D_d + tE) \text{ and } M' := \nu M$$

and get

$$\mathrm{vol}_{Y|V_d}(K_Y + M + L) = ([t_d] + 2)^d \,\mathrm{vol}_{Y|V_d}(K_Y + M) \geq \frac{1}{\nu^d} \,\mathrm{vol}(K_{V_d} + M|_{V_d})$$

where V_d is the irreducible component of Non-klt$(Y, D_d + tE)$ through x_1. Since x_1 is general, V_d cannot be contained in M. Moreover, always by generality of x_1 the divisor $K_{V_d} + M|_{V_d}$ is big (see Lemma 6.4), and V_d cannot be uniruled (hence its canonical divisor is pseudo-effective, by [BDPP]). Then, using the hypothesis, we have

$$\mathrm{vol}(K_{V_d} + M|_{V_d}) \geq v.$$

Now, we want to add to D_d an effective $\mathbf{Q}$-divisor of the form $v_d(K_Y + M)$ which has multiplicity $> d$ at x_1, but chosen among those restricting to a non-zero divisor on V_d. Precisely, for small rational $\delta > 0$, we add to $(1 - \delta)D_d$ a divisor G equivalent to

$$(d(2/\,\mathrm{vol}_{Y|V_d}(K_Y + M))^{1/d} + \varepsilon)(K_Y + M)$$

(which is $\leq (\nu d(2/v)^{1/d}(t_d + 2) + \varepsilon)(K_Y + M)$ by Corollary 5.2), Using Lemma 2.4, we get a divisor $D_{d-1} \sim t_{d-1}(K_Y + M)$ with

$$t_{d-1} \leq t_d + \nu d(2/v)^{1/d}(t_d + 2) + \varepsilon$$

and such that its Non-klt locus contains x_1, x_2. Moreover G can be chosen such that the new divisor D_{d-1} is klt around x_1 outside the support of $G|_{V_d}$, i.e.

$$\text{Non-klt}(Y, D_{d-1}) \text{ has dimension at } x_1 \text{ strictly lower than } \dim(V_d).$$

Again, multiplying D_{d-1} by a rational number ≤ 1 and applying Lemma 2.3, also conditions (2) and (3) are satisfied. The inductive step is thus proved.

In conclusion, we have obtained an effective $\mathbf{Q}$-divisor $D_0 \sim t_0(K_Y + M)$ with

$$t_0 < a'_{n,\nu} + b'_{n,\nu} t_{n-1} \leq a'_{n,\nu} + b'_{n,\nu} n(2/\,\mathrm{vol}(K_Y + M))^{1/n}$$

whose Non-klt locus contains x_1 as an isolated point, and x_2. Therefore, by Lemma 6.3, we deduce the existence of a global section

$$s \in H^0(Y, K_Y + (m-1)K_Y + mM) = H^0(Y, m(K_Y + M))$$

separating the two points, for all $m > a'_{n,\nu} + b'_{n,\nu} n(2/\,\mathrm{vol}(K_Y + M))^{1/n}$ divisible by ν. □

6.2. Proof of Theorem 1.3.

Proof of Theorem 1.3. The proof is by induction on the dimension of the varieties. Theorem 1.3 holds for $n = 1$. Suppose it holds for $n-1$. From Corollary 6.1 we deduce the existence of a positive lower bound :

$$\mathrm{vol}(K_V + N) \geq v_{n-1,\nu}$$

for all pairs (V, N) where V is a smooth projective variety of dimension $\leq n-1$, and N is a big and nef $\mathbf{Q}$-divisor on V such that νN is integral. In particular the hypotheses of Theorem 6.2 are fullfilled. Notice that here we use in a crucial way the hypothesis that the denominators of the N's are bounded. Otherwise the righthand side in the inequality (6.1), which is $1/m^n$, with m divisible by ν, would go to zero for $\nu \longrightarrow +\infty$.

Consider the pairs (Y, M), where Y is n-dimensional and M is a big and nef $\mathbf{Q}$-divisor such that νM is integral. For those such that volume of $K_Y + M$ is bounded from below, say $\mathrm{vol}(K_Y + M) \geq 1$, then Theorem 6.2 implies that $|m(K_Y + M)|$ separates points, for all

$$m > a + b \geq a + b/\,\mathrm{vol}(K_Y + M)^{1/n}.$$

such that m is divisible by ν. For those such that $\mathrm{vol}(K_Y + M) < 1$, then *a priori* the quantity $a + b/\,\mathrm{vol}(K_Y + M)^{1/n}$ may still be arbitrarily large. This does not occur : using Theorem 6.2 and projecting down, we have that the variety Y is birational to a subvariety of $\mathbf{P}^{2n+1}$ of degree :

$$\begin{aligned} &\leq \quad \Big(a + \frac{b}{\mathrm{vol}(K_Y + M)^{1/n}}\Big)^n \mathrm{vol}(K_Y + M) \\ &= \quad \Big(a\,\mathrm{vol}(K_Y + M)^{1/n} + b\Big)^n \leq (a+b)^n. \end{aligned}$$

Such varieties are parametrized by an algebraic variety (the Chow variety), so thanks to the Lemma 6.5 below, the volumes of $K_Y + M$ are also bounded from below by a positive constant $c_{n,\nu}$ (which is not effective!). Hence we may take

$$m_{n,\nu} := [1 + a + b/c_{n,\nu}^{1/n}]$$

and we conclude that the pluricanonical system $|m(K_Y + M)|$ separates two general points for all $m \geq m_{n,\nu}$ divisible by ν. □

Lemma 6.5 (see [T], Lemma 6.1)**.** *Consider the Chow variety*

$$\mathit{Chow} := \cup_{d \leq d_n} \mathit{Chow}_{n,d_n}(\mathbf{P}^{2n+1}).$$

Let $T = \cup_i T$ be the Zariski closure of those points in Chow. For any i, we have $\inf\{\mathrm{vol}(K_{Y_t} + M_t) : t \in T_i$ *and* $K_{Y_t} + M$ *is big*$\} > 0$.

Proof. It is of course sufficient to prove the statement for one $T = T_i$. We argue by induction on $\dim(T)$. If the dimension is zero, there is nothing to prove. Suppose $\dim(T) > 0$. We consider the universal family $U \to T$ and $\widetilde{U} \to U$ a desingularization, together with a line bundle whose restriction to a smooth fiber is $K_{Y_t} + M_t$. Let $T^o \subset T$ be the open subset over which the induced map $p : \widetilde{U} \to T$ is smooth. Let $S \subset T$ be the Zariski dense subset whose points correspond to varieties with $K_{Y_t} + M_t$ big. The

$S \cap T^o$ is also dense. By construction, for any $s \in S \cap T^o$, the fiber $\widetilde{Y}_s := p^{-1}(s)$ is a smooth variety with $K_{Y_t} + M_t$ big. By the upper semicontinuity of the h^0, the same is true for *every* fiber over T^o, and moreover we have $\inf\{\mathrm{vol}(K_{Y_t} + M_t) : t \in T^o\} > 0$. On the other hand, as for the complement $S \cap (T \setminus T^o)$, we invoke the inductive hypothesis and are done. □

From Theorem 1.3 we deduce our main result.

Proof of Theorem 1.2. Fix n, b and k. Let $r := r(k) = max\{r(B) : B \leq k\}$, where $r(B)$ is the integer appearing in (3.4) and $\nu := r \cdot b$. Set

$$m(n, b, k) := max\{m_{\dim(Y),\nu} : 1 \leq \dim(Y) \leq n\}$$

where $m_{\dim(Y),\nu}$ is as in Theorem 1.3. Let $f : X \longrightarrow Y$ be an algebraic fiber space verifying the hypotheses of Theorem 1.2. Notice that by (3.1) and (3.2) we have :

$$H^0(Y, ibK_Y + iL^{ss}_{X/Y}) \subset H^0(Y, ibK_Y + iL_{X/Y}) = H^0(X, ibK_X) \tag{6.2}$$

for all $i > 0$ divisible by the integer r. Take $M := \frac{1}{b} L^{ss}_{X/Y}$. By (3.3) the $\mathbf{Q}$-divisor M is nef and by (3.4) the divisor $\nu \cdot M$ is integral. The variety Y is non-uniruled, therefore by [BDPP] its canonical divisor K_Y is pseudo-effective. Notice moreover that, since by (3.1) the divisor M is big, we have that $K_Y + M$ is big. Then, by Theorem 1.3, we get the birationality of the pluriadjoints maps $\varphi_{m(K_Y+M)}$, for all $m \geq m(n, b, k)$ divisible by ν. The inclusion (6.2) yields the desired uniformity result for the Iitaka fibration of X. □

Acknowledgements

I am grateful to S. Boucksom, F. Campana, L. Caporaso, B. Claudon, O. Debarre, A. Lopez, M. Roth, E. Viehweg and D.-Q. Zhang for useful discussions and/or comments. I wish to thank the Dipartimento di Matematica of the Università Roma Tre, where this work was done, for the warm hospitality and the stimulating atmosphere. My stay in Roma was made possible by an "accueil en délégation au CNRS", which is gratefully acknowledged.

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