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A CLASS OF SOLUTIONS OF THE VACUUM EINSTEIN CONSTRAINT EQUATIONS WITH FREELY SPECIFIED MEAN CURVATURE

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Abstract. We give a sufficient condition, with no restrictions on the mean curvature, under which the conformal method can be used to generate solutions of the vacuum Einstein constraint equations on compact manifolds. The condition requires a so-called global supersolution but does not require a global subsolution. As a consequence, we construct a class of solutions of the vacuum Einsten constraint equations with freely specified mean curvature, extending a recent result [16] which constructed similar solutions in the presence of matter. We give a second proof of this result showing that vacuum solutions can be obtained as a limit of [16] non-vacuum solutions. Our principal existence theorem is of independent interest in the near-CMC case, where it simplifies previously known hypotheses required for existence.

1. Introduction

The Cauchy problem of general relativity requires initial data (a metric and a second fundamental form defined on a 3-manifold) that satisfy a system of nonlinear PDEs known as the Einstein constraint equations. The constraint equations admit many solutions (permitting the specification of different initial conditions) and it is important to understand the structure of the set of all possible initial data on a given manifold. Various approaches have been given for constructing solutions including parabolic methods [4] and gluing constructions [14] [13]. From the point of view of classifying the set of all possible solutions, the most fruitful technique has been the conformal method initiated by Lichnerowicz [24] and extended by Choquet-Bruhat and York [12].

In the conformal method, one specifies the conformal class of the initial metric, a piece of the second fundamental form corresponding to part of the time derivative of the conformal class, and the trace of the second fundamental form (i.e. the mean curvature). One then seeks a solution of the constraint equations matching this data. For constant mean curvature (CMC) data, this approach has lead to a complete classification of solutions on compact [12] [18], asymptotically Euclidean [6] [7] (with a correction in [26]), and asymptotically hyperbolic [2] [1] manifolds. On the other hand, we have very few results concerning non-CMC solutions, and most of these are perturbative. Near-CMC solutions have been constructed on compact [10] [20] [19] [16] and asymptotically Euclidean [8] [11] manifolds, and we have a near-CMC non-existence theorem for certain data [21]. It is remarkable, however, that despite the success of the conformal method in the CMC case, very little is known about the construction of solutions in the absence of restrictions on the mean curvature.

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An important recent result of Holst, Nagy, and Tsogtgerel [16] (see also the summary in [17]) gives the first construction, using the conformal method, of a class of initial data without a near-CMC hypothesis. The authors of that paper show that solutions of the constraint equations on compact manifolds can be constructed using the conformal method when global sub- and supersolutions (defined in Section 1.1) can be found. For Yamabe-positive metrics, for non-vanishing matter fields, and under a certain smallness condition not involving the mean curvature, [16] provides a global subsolution/supersolution pair that does not have a near-CMC hypothesis and hence yields the existence of certain far-from-CMC solutions.

It is natural to ask if this far-from-CMC construction can be extended to the vacuum case. The global supersolution of [16] (hereafter the HNT global supersolution) is also applicable in vacuum, and indeed requires that the matter fields, if present, be weak. The corresponding HNT global subsolution, however, requires the presence of matter. It is not unusual for the conformal method to require non-vanishing conditions on parts of the specified data, so it was conceivable that the non-vacuum hypothesis was necessary.

In this paper we show that this is not the case, and that the conformal method can be used to construct a corresponding set of vacuum solutions. We give two proofs of this fact. First, we prove that solutions exist, under certain mild technical conditions, whenever a global supersolution can be found (Theorem 1). The proof relies on an a-priori estimate (Proposition 10) that replaces the need for a global subsolution. Hence the HNT global supersolution alone is sufficient to deduce the existence of solutions via the conformal method, and we obtain vacuum far-from-CMC solutions. The second proof considers a sequence of HNT non-vacuum solutions where the matter fields are converging to zero. Again, a lower bound (Proposition 16) is found for the sequence and is used to obtain a corresponding subsequence converging to a vacuum far-from-CMC solutions. The key steps in both proofs rely on a technique from [25] for constructing subsolutions.

1.1. The conformal method. On a given smooth 3-manifold M, the Einstein constraint equations for a metric $\bar{g}$ and a symmetric $(0,2)$-tensor $\bar{K}$ are

$$\begin{aligned} R_{\bar{g}} - \left|\bar{K}\right|_{\bar{g}}^2 + \operatorname{tr}_{\bar{g}} \bar{K}^2 &= 2\bar{\rho} \\ \operatorname{div}_{\bar{g}} \bar{K} - d \operatorname{tr}_{\bar{g}} \bar{K} &= \bar{J}, \end{aligned} \tag{1}$$

where $R_{\bar{g}}$ is the scalar curvature of $\bar{g}$, $\bar{\rho}$ is the matter density, and $\bar{J}$ is the momentum density. We are primarily interested in the vacuum case where $\bar{\rho} \equiv 0$ and $\bar{J} \equiv 0$.

Data for the vacuum conformal method on a compact smooth manifold M consists of a Riemannian metric g specifying a conformal class, a transverse traceless (i.e symmetric, trace-free and divergence-free) $(0,2)$-tensor σ specifying part of the time derivative of the conformal class, and a scalar function τ specifying the mean curvature. We seek a solution $(\bar{g}, \bar{K})$ of the constraint equations of the form

$$\begin{aligned} \bar{g} &= \phi^4 g \\ \bar{K} &= \phi^{-2} \left(\sigma + \mathbf{L}\, W\right) + \frac{\tau}{3}\tilde{g}. \end{aligned} \tag{2}$$

In equations (2) the unknowns are a positive function ϕ and a vector field W, while $\mathbf{L}$ is the conformal Killing operator defined by

$$\mathbf{L}\,W_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{3}\operatorname{div} W g_{ab}. \tag{3}$$

If matter is present, it can be specified by scaled sources ρ and J which are conformally related to $\bar{\rho}$ and $\bar{J}$ by $\bar{\rho} = \phi^{-8}\rho$ and $\bar{J} = \phi^{-6}J$.

It follows that $\bar{g}$ and $\bar{K}$ solve the vacuum constraint equations so long as

$$-8\,\Delta\,\phi + R_g\phi = -\frac{2}{3}\tau^2\phi^5 + \left|\sigma + \mathbf{L}\,W\right|^2\phi^{-7} \tag{4}$$

$$\operatorname{div}\mathbf{L}\,W = \frac{2}{3}\phi^6\,d\tau. \tag{5}$$

If matter is present we must add the terms $2\rho\phi^{-3}$ and J to the right-hand sides of (4) and (5) respectively. The operator $\operatorname{div}\mathbf{L}$ is the vector Laplacian, and hence these equations are a coupled nonlinear elliptic system for ϕ and W.

If τ is constant then equation (5) has a trivial solution and the problem reduces to an analysis of the Lichnerowicz equation (4). One technique for finding solutions of the Lichnerowicz equation is via the method of sub- and supersolutions, which was used previously in the work of Kazden and Warner [22] on the prescribed scalar curvature problem. Isenberg [18] used this method to complete the classification of CMC solutions on compact manifolds. A generalization of the method applies in the non-CMC setting as well, and we review the terminology now.

Consider the equation

$$-8\,\Delta\,\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\beta|^2\phi^{-7} \tag{6}$$

where β is a symmetric $(0,2)$-tensor. We say ϕ_+ is a **supersolution** of (6) if

$$-8\,\Delta\,\phi_+ + R\phi_+ \geq -\frac{2}{3}\tau^2\phi_+^5 + |\beta|^2\phi_+^{-7}. \tag{7}$$

A **subsolution** is defined similarly with the inequality reversed.

For the coupled system, we follow [16] and define global subsolutions and global supersolutions as follows. We say ϕ_+ is a **global supersolution** if whenever $0 < \phi \leq \phi_+$, then

$$-8\,\Delta\,\phi_+ + R\phi_+ \geq -\frac{2}{3}\tau^2\phi_+^5 + \left|\sigma + \mathbf{L}\,W_\phi\right|^2\phi_+^{-7}, \tag{8}$$

where W_ϕ is the solution of (5) obtained from ϕ. We say $\phi_- > 0$ is a **global subsolution** if whenever $\phi \geq \phi_-$, then

$$-8\,\Delta\,\phi_- + R\phi_- \leq -\frac{2}{3}\tau^2\phi_-^5 + \left|\sigma + \mathbf{L}\,W_\phi\right|^2\phi_-^{-7}. \tag{9}$$

The existence result of [16] states that if $\phi_- \leq \phi_+$ are global sub- and supersolutions, then there exists a solution (ϕ, W) of system (4)–(5) such that $\phi_- \leq \phi \leq \phi_+$. The authors of that paper also present a number of global sub- and supersolution pairs, including one that is used to construct far-from-CMC solutions.

1.2. Summary of results. Our primary result concerning the solution of system (4)–(5) has three cases depending on the Yamabe invariant $\mathcal{Y}_g$ of the metric. Recall that

$$\mathcal{Y}_g = \inf_{\substack{f\in C^\infty(M)\\ f\not\equiv 0}} \frac{\int_M 8\,|\nabla f|_g^2 + R_g f^2 \; dV_g}{||f||_{L^6}^2}. \tag{10}$$

(Our notation for L^p spaces and Sobolev spaces $W^{k,p}$ follows that of [23] with the additional convention that subspaces of positive functions are indicated by a subscript +.)

Theorem 1. *Let $g \in W^{2,p}$ with $p > 3$ be a metric on a smooth, compact 3-manifold. Suppose g has no conformal Killing fields and that one of the following conditions holds for a transverse traceless tensor $\sigma \in W^{1,p}$ and a function $\tau \in W^{1,p}$.*

(1) $\mathcal{Y}_g > 0$, $\sigma \not\equiv 0$,
(2) $\mathcal{Y}_g = 0$, $\sigma \not\equiv 0$, $\tau \not\equiv 0$
(3) $\mathcal{Y}_g < 0$ *and there exists $\hat{g}$ in the conformal class of g such that* $R_{\hat{g}} = -\frac{2}{3}\tau^2$.

If $\phi_+ \in W^{2,p}_+$ is a global supersolution for (g, σ, τ), then there exists a solution $(\phi, W) \in W^{2,p}_+ \times W^{2,p}$ of system (4)–(5) such that $\phi \leq \phi_+$.

The new results of Theorem 1 are Cases 1 and 2; Case 3 can be deduced from the existence of a global subsolution found in [16]. Note that Theorem 1 is only an existence theorem. It is not known if the solutions provided by Theorem 1 are unique.

If τ is constant then the conditions of the three cases reduce to precisely the same conditions under which CMC solutions of the constraints can be found (aside from one additional singular case $\mathcal{Y}_g = 0$, $\sigma \equiv 0$, $\tau \equiv 0$). The hypothesis on τ in Case 3 is necessary for Yamabe-negative metrics since it is needed for solutions of the Lichneowicz equation to exist [25]. It is not known if the condition $\sigma \not\equiv 0$ in Cases 1–2 is necessary. However, it was proved in [21] that if $\mathcal{Y}_g \geq 0$ and if $\sigma \equiv 0$, then there do not exist near-CMC solutions of (4)–(5) unless $\mathcal{Y}_g = 0$ and $\tau \equiv 0$ also. Hence some condition involving σ (and possibly also τ) must be required. The hypothesis in Case 2 that $\tau \not\equiv 0$ (if $\sigma \not\equiv 0$) can be shown to be necessary – otherwise the metric would be be Yamabe-positive.

Our first application of Theorem 1 is to the HNT supersolution, which exists under the following hypotheses.

Proposition 1 ([16]). *Suppose $g \in W^{2,p}$ with $p > 3$, and that $\mathcal{Y}_g > 0$, $\tau \in W^{1,p}$, and $\sigma \in W^{1,p}$. If $||\sigma||_\infty$ is sufficiently small, then there exists a global supersolution of (4)–(5).*

A proof of Proposition 1 can be found in Section 4.2. From Theorem 1 and Proposition 1 we immediately obtain the following result, which is the primary aim of this paper.

Corollary 1. *Let $g \in W^{2,p}$ with $p > 3$ be a Yamabe-positive metric on a smooth compact 3-manifold. Suppose g has no conformal Killing fields, $\sigma \in W^{1,p}$ is a transverse traceless tensor, and $\tau \in W^{1,p}$. If $\sigma \not\equiv 0$ and if $||\sigma||_\infty$ is sufficiently small, then there exists a solution $(\phi, W) \in W^{2,p}_+ \times W^{2,p}$ of system (4)–(5).*

We also provide a second proof of Corollary 1 in Section 5 that is independent of Theorem 1 but instead uses a sequence of HNT non-vacuum solutions.

Theorem 1 permits a strengthening of the current existence theory for near-CMC data inasmuch as it removes conditions in current theorems required to find subsolutions. In [19] the authors present an existence and uniqueness theorem for Yamabe non-negative metrics. They present constant global supersolutions so long as

$$\frac{\max(|d\tau|)}{\min(|\tau|)} \text{ is sufficiently small,} \tag{11}$$

and also assume non-scale invariant conditions on the size of $|d\tau|$ to obtain subsolutions and to obtain uniqueness. Previously [20] gave a similar proof for Yamabe-negative metrics, again presenting a global supersolution under condition (11) and making an additional assumption about the absolute size of $|d\tau|$. [16] provides a global supersolution under a near-CMC condition similar to (11), but also requires in the Yamabe-nonnegative case either a non-vacuum hypothesis or that $\min|\sigma|$ is sufficiently large to obtain a subsolution. Using Theorem 1 we have the following simplified existence result.

Corollary 2. *Let the conditions of one of the cases of Theorem 1 hold. If*

$$\frac{\max(|d\tau|)}{\min(|\tau|)} \tag{12}$$

is sufficiently small, then there exists a solution (ϕ, W) *of system* (4)–(5).

The utility of Theorem 1 is limited to cases where a global supersolution can be found. It is not known if the converse of Theorem 1 is true; in particular, given a solution of system (4)–(5) it is not known if there also exists a corresponding global supersolution.[1] Nevertheless, Theorem 1 makes clear that any future advances in the existence theory of non-CMC initial data using the method of sub- and supersolutions need only focus on supersolutions.

In the following, Sections 2 and 3 provide a summary of the basic results we require in the analysis of equations (4) and (5) respectively. Section 4 is devoted to the proof of Theorem 1, which is obtained using the Schauder fixed point theorem in an approach similar to one outlined in [16]. The key step in Section 4 is Proposition 10 which eliminates the need for a subsolution. Section 5 provides the alternative proof of Corollary 1 using a sequence of non-vacuum solutions.

2. The Lichnerowicz operator

In this section we consider properties of the map taking $(0, 2)$-tensors β and scalar functions τ to a solution ϕ of the Lichnerowicz equation

$$-8\,\Delta\,\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\beta|^2\,\phi^{-7}. \tag{13}$$

The solvability of this equation has been considered in several works under various hypotheses on the zeros of β and τ as well as the Yamabe class of g. Building on previous work in [18] and [9], a complete description of solvability of this equation

[1] Given a solution (ϕ, W) of system (4)–(5), it is not clear if ϕ itself is a global supersolution. Although ϕ is, for one particular W, a solution and hence a supersolution of (4), it is not apparent that it is a supersolution for the whole class of vector fields W required for it to be a global supersolution.

on compact manifolds, including the Yamabe negative case, appeared in [25]. In the context of the function spaces used in the current paper we have the following classification.

Proposition 2. *Suppose $\beta, \tau \in L^{2p}$ and $g \in W^{2,p}$ where $p > 3$. Then there exists a positive solution $\phi \in W^{2,p}_{+}$ of (13) if and only if one of the following is true.*

(1) *$\mathcal{Y}_g > 0$ and $\beta \not\equiv 0$,*
(2) *$\mathcal{Y}_g = 0$ and $\beta \not\equiv 0$, $\tau \not\equiv 0$,*
(3) *$\mathcal{Y}_g < 0$ and there exists $\hat{g}$ in the conformal class of g such that $R_{\hat{g}} = -\frac{2}{3}\tau^2$,*
(4) *$\mathcal{Y}_g = 0$, $\beta \equiv 0$, $\tau \equiv 0$.*

In Cases 1–3 the solution is unique. In Case 4 any two solutions are related by scaling by a constant multiple.

In [25], Proposition 2 was proved under low regularity assumptions on the conformal data. In this paper we work for convenience with metrics in $W^{2,p}$ with $p > 3$. This level of regularity ensures that the metric is $C^{1,\alpha}$. The corresponding hypothesis in Proposition 2 that $\beta, \tau \in L^{2p}$ arises to ensure that the solution $\phi \in W^{2,p}$ and is related to the fact that $-\Delta + V : W^{2,p} \to L^p$ is an isomorphism if $V \in L^p$, $V \geq 0$, and $V \not\equiv 0$ (see, e.g, [9]). We will later make the stronger assumption that $\sigma, \tau \in W^{1,p}$ when working with the coupled system.

We are primarily interested in the map that, for fixed τ, takes β to a solution of equation (13). We say that g and τ are **Lichnerowicz compatible** if they satisfy one of the conditions of Cases 1–3 and we say that β is **admissible** if it further satisfies the same condition. We will not need to consider the singular Case 4, which has no bearing on the construction of non-CMC solutions.

If g and τ are Lichnerowicz compatible, we define the Lichnerowicz operator $\mathcal{L}_\tau$ to be the map taking β to the unique solution of (13). Proposition 2 effectively describes the domain of $\mathcal{L}_\tau$ as an open subset $\mathcal{D}_\tau$ of L^{2p}; $\mathcal{D}_\tau = L^{2p} \setminus \{0\}$ if $\mathcal{Y}_g \geq 0$ and $\mathcal{D}_\tau = L^{2p}$ if $\mathcal{Y}_g < 0$.

2.1. Sub- and supersolutions. The existence of solutions in Cases 1–3 of Proposition 2 follows from the method of sub- and supersolutions In the context of the function spaces used in this paper we have the following propositions, which can be deduced, e.g., from the results for less regular metrics in [25].

Proposition 3. *If $g \in W^{2,p}$ and $\tau \in L^{2p}$ are Lichnerowicz compatible and if $\beta \in L^{2p}$ is admissible, then there exist a subsolution ϕ_- and a supersolution ϕ_+ of (13) such that $\phi_- \leq \phi_+$.*

Proposition 4. *Suppose $g \in W^{2,p}$ and $\beta, \tau \in L^{2p}$ for some $p > n$. If ϕ_-, $\phi_+ \in W^{2,p}$ are a subsolution and a supersolution respectively of (13) such that $\phi_- \leq \phi_+$, then there exists a solution $\phi \in W^{2,p}(M)$ of (13) such that $\phi_- \leq \phi \leq \phi_+$.*

An important technical tool used in the proof of Proposition 2 is the well-known conformal covariance of (13), which allows us to pick a convenient conformal representative for g. This covariance can be expressed in terms of sub- and supersolutions.

Lemma 1. *Suppose $g \in W^{2,p}$ and $\beta, \tau \in L^{2p}$ for some $p > 3$. Suppose also that $\psi \in W^{2,p}_+$. Define*

$$\begin{aligned} \hat{g} &= \psi^4 g \\ \hat{\beta} &= \psi^{-2}\beta \\ \hat{\tau} &= \tau. \end{aligned} \tag{14}$$

Then ϕ is a supersolution (resp. subsolution) of (13) *if and only if $\hat{\phi} = \psi^{-1}\phi$ is a supersolution (resp. subsolution) of the conformally transformed equation*

$$-8\,\Delta_{\hat{g}}\,\hat{\phi} + R_{\hat{g}}\hat{\phi} = -\frac{2}{3}\hat{\tau}^2\hat{\phi}^5 + \left|\hat{\beta}\right|^2_{\hat{g}}\hat{\phi}^{-7}. \tag{15}$$

In particular, ϕ is a solution of (13) *if and only if $\psi^{-1}\phi$ is a solution of* (15)*.*

Proof. Let ϕ be a subsolution or supersolution of (13). Let $g' = \phi^4 g$, and let $R_{g'}$ be its scalar curvature. Then it is well known that

$$R_{g'} = \phi^{-5}(-8\,\Delta_g\,\phi + R_g\phi). \tag{16}$$

But $g' = (\psi^{-1}\phi)^4\hat{g}$, so

$$R_{g'} = \psi^5\phi^{-5}(-8\,\Delta_{\hat{g}}(\psi^{-1}\phi) + R_{\hat{g}}\psi^{-1}\phi). \tag{17}$$

Hence

$$\begin{aligned} &-8\,\Delta_{\hat{g}}\,\hat{\phi} + R_{\hat{g}}\hat{\phi} + \frac{2}{3}\hat{\tau}^2\hat{\phi}^5 - \left|\hat{\beta}\right|^2_{\hat{g}}\hat{\phi}^{-7} = \\ &\qquad = -8\,\Delta_{\hat{g}}(\psi^{-1}\phi) + R_{\hat{g}}\psi^{-1}\phi + \frac{2}{3}\tau^2(\psi^{-1}\phi)^5 - \left|\hat{\beta}\right|^2_{\hat{g}}(\psi^{-1}\phi)^{-7} \\ &\qquad = \psi^{-5}(-8\,\Delta_g\,\phi + R_g\phi) + \psi^{-5}\frac{2}{3}\tau^2\phi^5 - \psi^{-5}\left|\beta\right|^2_g\phi^{-7} \\ &\qquad = \left[-8\,\Delta_g\,\phi + R_g\phi + \frac{2}{3}\tau^2\phi^5 - \left|\beta\right|^2_g\phi^{-7}\right]\psi^{-5}. \end{aligned} \tag{18}$$

The result now follows noting that $\psi^{-5} > 0$ everywhere. □

Proposition 4 requires that $\phi_- \le \phi_+$. This never poses a problem in practice, however, since we can always rescale sub- and supersolutions of (13) to obtain this inequality.

Lemma 2. *If ϕ_+ is a supersolution of* (13)*, then for any $\alpha \ge 1$, $\alpha\phi_+$ is also a supersolution. If ϕ_- is a subsolution of* (13)*, then for any $\alpha \le 1$, $\alpha\phi_-$ is also a subsolution.*

Proof. We employ the monotonicity of the terms on the right-hand side of (13). Note that for $\alpha \geq 1$,

(19)
$$\begin{aligned}-8\,\Delta\,\alpha\phi_+ + R\alpha\phi_+ + \frac{2}{3}\tau^2(\alpha\phi_+)^5 - |\beta|^2(\alpha\phi_+)^{-7} \geq \\ \geq \alpha\left[-\frac{2}{3}\tau^2\phi_+^5 + |\beta|^2\phi_+^{-7}\right] + \frac{2}{3}\tau^2(\alpha\phi_+)^5 - |\beta|^2(\alpha\phi_+)^{-7} \\ = (\alpha^5-\alpha)\frac{2}{3}\tau^2\phi_+^5 + (\alpha - \alpha^{-7})|\beta|^2\phi_+^{-7} \\ \geq 0.\end{aligned}$$

Hence $\alpha\phi_+$ is a supersolution. The argument for subsolutions is similar. □

An immediate application of Lemma 2 (implying uniqueness of solutions of (13) for Lichnerowicz compatible data) is the fact that any supersolution at all of (13) provides an upper bound for solutions.

Lemma 3. *Suppose $g \in W^{2,p}$ and $\tau \in L^{2p}$ are Lichnerowicz compatible and $\beta \in L^{2p}$ is admissible. If $\phi_+ \in W^{2,p}_+$ is a positive supersolution of* (13)*, then $\mathcal{L}_\tau(\beta) \leq \phi_+$. An analagous result holds for subsolutions.*

Proof. Suppose ϕ_+ is a given supersolution and let ϕ_- be the subsolution from Proposition 3. Pick $\alpha \leq 1$ such that $\alpha\phi_- \leq \phi_+$ everywhere. For example we can take

$$\alpha = \min\left(1, \min(\phi_+)/\max(\phi_-)\right). \tag{20}$$

Then Proposition 4 implies there exists a solution $\hat{\phi}$ of (13) satisfying $\phi_- \leq \hat{\phi} \leq \phi_+$. Since solutions of (13) for Lichnerowicz compatible data are unique, we conclude that $\hat{\phi} = \phi$ and therefore $\phi \leq \phi_+$. □

3. The vector Laplacian

The vector Laplacian $\operatorname{div}\mathbf{L}$ is well known to be elliptic and its kernel consists of the conformal Killing fields of g. Hence the equation

$$\operatorname{div}\mathbf{L}\,W = X \tag{21}$$

is solvable if and only if $\int_M \langle X, Z\rangle\ dV = 0$ for every conformal killing field Z. In the context of the function spaces used in this paper, we have the following standard existence result (see, e.g., [9]).

Proposition 5. *Suppose $g \in W^{2,p}$ with $p > 3$ has no conformal Killing fields. Given $X \in L^p$ there exists a unique solution W of*

$$\operatorname{div}\mathbf{L}\,W = X. \tag{22}$$

Moreover, there is a constant c independent of X such that

$$||W||_{W^{2,p}} \leq c||X||_{L^p}. \tag{23}$$

The hypothesis that (M, g) has no conformal Killing fields is superfluous. For smooth metrics, [21] proved that a similar existence theorem and estimate follows even in the presence of conformal Killing fields, so long as we take X to be L^2 orthogonal to the subspace of conformal Killing fields. Our construction of non-CMC solutions in this paper requires solvability of equation (21) in general, however, and we must therefore assume that g has no conformal Killing fields. It is a curious fact that all current non-CMC existence theorems require the hypothesis that g does not have any conformal Killing fields (or that τ is constant on the integral curves of all conformal Killing fields) but there are no proofs that these conditions are necessary.

For a scalar field τ in $W^{1,p}$ with $p > n$, define $\mathcal{W}_\tau : L^\infty \to W^{2,p}$ by

$$\mathcal{W}_\tau(\phi) = W \tag{24}$$

where W is the solution of

$$\operatorname{div} \mathbf{L}\, W = \frac{2}{3}\phi^6 d\tau. \tag{25}$$

We have the following standard estimate from [20]; a stronger version that applies even in the case where g has conformal Killing fields can be found in [21].

Proposition 6. *Let $\tau \in W^{1,p}$ with $p > 3$. Then there exists a constant K_τ such that*

$$||\mathbf{L}\, \mathcal{W}_\tau(\phi)||_\infty \le K_\tau ||\phi||_\infty^6 \tag{26}$$

for every $\phi \in L^\infty$.

Proof. From the Sobolev embedding $W^{1,p} \hookrightarrow L^\infty$ and inequality (23) we have for various constants c_k independent of ϕ and τ,

$$\begin{aligned} ||\mathbf{L}\, \mathcal{W}_\tau(\phi)||_\infty &\le c_1 ||\, \mathcal{W}_\tau(\phi)||_{W^{1,\infty}} \\ &\le c_2 ||\, \mathcal{W}_\tau(\phi)||_{W^{2,p}} \le c_3 ||\phi^6 d\tau||_{L^p} \le c_4 ||\tau||_{W^{1,p}} ||\phi||_\infty^6. \end{aligned} \tag{27}$$

Taking $K_\tau = c_4 ||\tau||_{W^{1,p}}$ completes the proof. □

4. Existence of solutions of the coupled system

The standard approach to finding solutions of the coupled system (4)–(5) is via a fixed point argument. In [20] and [19] the authors use the contraction mapping principle to find a (unique) fixed point. Topological methods have also been used to find fixed points, e.g. Leray-Schauder theory in [10] and the Schauder fixed point theorem in [16]. These methods require weaker hypotheses but do not ensure uniqueness. Our existence theorem uses the Schauder fixed point theorem and is closely related to the approach of [16] (although the specific map for which we find a fixed point is different). In particular, we also do not obtain a proof of uniqueness.

In this section we assume that $g \in W^{2,p}$ and $\tau \in W^{1,p}$ (with $p > 3$) are Lichnerowicz compatible and that $\sigma \in W^{1,p}$ is admissible (i.e. $\sigma \not\equiv 0$ if $\mathcal{Y}_g \ge 0$). This is exactly the hypothesis that g, τ, and σ satisfy one of Cases 1–3 of Theorem 1.

Define $\mathcal{N}_{\sigma,\tau} : L_+^\infty \to W_+^{2,p}$ by

$$\mathcal{N}_{\sigma,\tau}(\phi) = \mathcal{L}_\tau(\sigma + \mathbf{L}\, \mathcal{W}_\tau(\phi)). \tag{28}$$

To ensure $\mathcal{N}_{\sigma,\tau}$ is well defined, we assume that g has no conformal Killing fields (so that the domain of $\mathcal{W}_\tau$ is all of L^∞). We must must also verify that $\sigma + \mathbf{L}\, \mathcal{W}_\tau(\phi)$ belongs to the domain of $\mathcal{L}_\tau$ for any choice of $\phi \in L_+^\infty$. It suffices to show that if $\mathcal{Y}_g \ge 0$, then $\sigma + \mathbf{L}\, \mathcal{W}_\tau(\phi) \not\equiv 0$. Since σ is divergence free, it is L^2 orthogonal to the image of $\mathbf{L}$. Hence

$$\int_M |\sigma + \mathbf{L}\, \mathcal{W}_\tau(\phi)|^2 \; dV = \int_M |\sigma|^2 + |\mathbf{L}\, \mathcal{W}_\tau(\phi)|^2 \; dV \ge \int_M |\sigma|^2 \; dV \ne 0, \tag{29}$$

since $\sigma \not\equiv 0$ if $\mathcal{Y}_g \ge 0$.

The solutions of system (4)–(5) for conformal data σ and τ are in one-to-one correspondence with the fixed points of $\mathcal{N}_{\sigma,\tau}$. We will find fixed points of $\mathcal{N}_{\sigma,\tau}$ via an application of the Schauder fixed point theorem, which states that if $f : U \to U$ is a continuous map from a closed convex subset U of a normed space to itself, and if $\overline{f(U)}$ is compact, then f has a fixed point [5].

In Section 4.1 we show that if ϕ_+ is a global supersolution, then there is a constant $K_0 > 0$ such that the set $U = \{\phi \in L^\infty : K_0 \leq \phi \leq \phi_+\}$ is invariant under $\mathcal{N}_{\sigma,\tau}$. Clearly U is closed and convex in L^∞. In Section 4.2 we show that $\mathcal{N}_{\sigma,\tau}(U)$ is precompact in L^∞ and that $\mathcal{N}_{\sigma,\tau}$ is continuous, which establishes Theorem 1.

4.1. An invariant set for $\mathcal{N}_{\sigma,\tau}$. Let $\phi_+ \in W^{2,p}_+$ be a global supersolution. We seek an invariant set of the form $\{\phi \in L^\infty : K_0 \leq \phi \leq \phi_+$ where $K_0 > 0$ is a constant. To begin, it is easy to show that $\{\phi \in L^\infty : 0 < \phi \leq \phi_+\}$ is invariant under $\mathcal{N}_{\sigma,\tau}$.

Proposition 7. *If $\phi \in L^\infty_+$ satisfies $\phi \leq \phi_+$, then*

$$\mathcal{N}_{\sigma,\tau}(\phi) \leq \phi_+. \tag{30}$$

Proof. Let $\psi = \mathcal{N}_{\sigma,\tau}(\phi)$, so ψ is a solution of

$$-8\,\Delta\,\psi + R\psi = -\frac{2}{3}\tau^2\psi^5 + \left|\sigma + \mathbf{L}\,W\right|^2\psi^{-7} \tag{31}$$

where $W = \mathcal{W}_\tau(\phi)$. Since ϕ_+ is a global supersolution and since $0 \leq \phi \leq \phi_+$, we conclude that ϕ_+ is a supersolution of (31). Lemma 3 then implies $\psi \leq \phi_+$. □

To find the lower bound K_0 for the invariant set we consider the cases $\mathcal{Y}_g \geq 0$ and $\mathcal{Y}_g < 0$ separately. For the case $\mathcal{Y}_g \geq 0$ an estimate for a lower bound for $\mathcal{N}_{\sigma,\tau}(\phi)$ can be obtained from a lower bound for the Green's function of a certain elliptic PDE.

Proposition 8. *Let $V \in L^p$ with $p > 3$ and suppose $V \geq 0$, $V \not\equiv 0$. Then Green's function $G(x, y)$ of the operator $-\Delta + V$ exists and satisfies*

$$G(x, y) \geq m_G \tag{32}$$

for some constant $m_G > 0$.

Proof. Let $H(x, y)$ be a positive Green's function for the Laplacian on M, so

$$-\,\Delta_y\,H(x, y) = \delta_x - \frac{1}{\mathrm{Vol}(M)}. \tag{33}$$

The existence of this Green's function and its properties are established in [3] in the case of smooth metrics; the same techniques apply to $C^{1,\alpha}$ metrics and hence $W^{2,p}$ metrics if $p > 3$. In particular,

$$H(x, y) = \frac{1}{4\pi}\left|x - y\right|^{-1} + h(x, y) \tag{34}$$

where, since $\dim(M) = 3$, $h(x, y)$ is continuous on $M \times M$.

For fixed x, $H(x, \cdot) \in L^{3-\epsilon}$ for any $\epsilon > 0$. Since $V \in L^p$ for some $p > 3$, we conclude that $H(x, \cdot)V(\cdot) \in L^r$ where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{3 - \epsilon} < \frac{2}{3} \tag{35}$$

for ϵ sufficiently small. That is, $H(x, \cdot)V(\cdot) \in L^r$ with $r > 3/2$.

Let $\Psi(x,y)$ be the solution of

$$-\Delta_y \Psi(x,y) + V(y)\Psi(x,y) = \frac{1}{\mathrm{Vol}(M)} - V(y)H(x,y). \tag{36}$$

The solution exists and belongs to $W^{2,r}$ since $H(x,\cdot)V(\cdot) \in L^r$ with $r > 3/2$ [9]. In particular, for fixed x, Sobolev embedding implies $\Psi(x,y)$ is continuous in y. Moreover, the map taking x to $H(x,\cdot)$ is easily seen to be continuous as a map from M to $L^{3-\epsilon}$ and hence the map taking x to $V(\cdot)H(x,\cdot)$ is continuous from M to L^r. It follows that $\Psi(x,y)$ is continuous in both x and y.

Define $G(x,y) = H(x,y) + \Psi(x,y)$. Clearly $G(x,y)$ is the Green's function for $-\Delta + V$. We now show that G is uniformly bounded below by a positive number. Note that the asymptotic structure of G implies that $G(x,y) \geq 1$ in in a neighborhood U of the diagonal in $M \times M$. Since G is continuous on $M \times M \setminus U$, it follows that it achieves a minimum on $M \times M \setminus U$ at a point (x_0, y_0). Take ϵ so small that $G(x_0, y) \geq 1$ on $B_\epsilon(x_0)$. Then on $M \setminus B_\epsilon(x_0)$ we have

$$-\Delta_y G(x_0,y) + V(y)G(x_0,y) = 0 \tag{37}$$

and $G(x_0,y) \geq 1$ on $\partial B_\epsilon(x_0)$. The strong maximum principle of [27] (or [15] Theorem 8.19 if $V \in L^\infty$) then applies and $G(x_0,y_0) > 0$. Setting $m_G = \min(1, G(x_0,y_0))$ completes the proof. □

The estimate for the lower bound of $G(x,y)$ implies an estimate for the lower bound of the solution of $-\Delta \phi + V\phi = f$ whenever f is non-negative.

Proposition 9. *Let $V \in L^p$ with $p > 3$ and suppose $V \geq 0$, $V \not\equiv 0$. There exist positive constants c_1 and c_2 such that for every $f \in L^p$ with $f \geq 0$ the solution ϕ of*

$$-\Delta \phi + V\phi = f \tag{38}$$

satisfies

$$\max(\phi) \leq c_1 ||f||_{L^p} \tag{39}$$

and

$$\min(\phi) \geq c_2 ||f||_{L^1}. \tag{40}$$

Proof. Since $V \geq 0$, $V \not\equiv 0$ we have $-\Delta + V : W^{2,p} \to L^p$ is an isomorphism and $||\phi||_{W^{2,p}} \leq c||f||_{L^p}$ for some constant c independent of f. By Sobolev embedding, $W^{2,p}$ embeds continuously in L^∞ which establishes inequality (39).

Let $G(x,y)$ be the Green's function for $-\Delta + V$. Then, since $f \geq 0$,

$$\phi(x) = \int_M f(y)G(x,y)\, dV(y) \geq m_G \int_M f(y)\, dV(y) = m_G ||f||_{L_1} \tag{41}$$

where m_G is the lower bound for $G(x,y)$ found in Proposition 8. This implies inequality (40) with $c_2 = m_G$. □

We can now establish the desired lower bound (in the Yamabe non-negative case) for $\mathcal{N}_{\sigma,\tau}(\phi)$ when $\phi \leq \phi_+$.

Proposition 10. *Suppose $\phi_+ \in W^{2,p}_+$ is a global supersolution, $\mathcal{Y}_g \geq 0$, and $\sigma \not\equiv 0$. Then there exists a constant $K_0 > 0$ such that whenever $0 < \phi \leq \phi_+$,*

$$K_0 \leq \mathcal{N}_{\sigma,\tau}(\phi). \tag{42}$$

Proof. Suppose $0 < \phi_0 \le \phi_+$ and let $W = \mathcal{W}_\tau(\phi_0)$. We will construct a subsolution ϕ_- of the equation

$$-8\,\Delta\,\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + \mathbf{L}\,W|\,\phi^{-7} \tag{43}$$

and determine a lower bound K_0 for ϕ_- that is independent of the choice of ϕ_0. Estimate (42) then follows from Lemma 3.

The construction of the subsolution follows a procedure found in [25]. Pick $\psi \in W^{2,p}_+$ such that $\hat{g} = \psi^4 g$ has continuous positive or zero scalar curvature depending on the sign of $\mathcal{Y}_g$. Define $\hat{\beta} = \psi^{-2}(\sigma + \mathbf{L}\,W)$ and let η be the solution of

$$-8\,\Delta_{\hat{g}}\,\eta + \left(R_{\hat{g}} + \frac{2}{3}\tau^2\right)\eta = \left|\hat{\beta}\right|^2_{\hat{g}}. \tag{44}$$

Since $R_{\hat{g}} + \frac{2}{3}\tau^2 \ge 0$ and is not identically zero, it follows that the solution η exists and is positive.

We now claim that $\alpha\eta$ is a subsolution of

$$-8\,\Delta_{\hat{g}}\,\phi + R_{\hat{g}}\phi = -\frac{2}{3}\tau^2\phi^5 + \left|\hat{\beta}\right|^2_{\hat{g}}\phi^{-7} \tag{45}$$

if α is taken small enough. To see this, note that

$$-8\,\Delta_{\hat{g}}\,\alpha\eta + R_{\hat{g}}\alpha\eta + \frac{2}{3}\tau^2(\alpha\eta)^5 - \left|\hat{\beta}\right|^2_{\hat{g}}(\alpha\eta)^{-7} = \frac{2}{3}\left[(\alpha\eta)^5 - \alpha\eta\right]\tau^2 + \left[\alpha - (\alpha\eta)^{-7}\right]\left|\hat{\beta}\right|^2. \tag{46}$$

Hence $\alpha\eta$ is a subsolution if $\alpha^8 \le \eta^{-7}$ and $\alpha \le \eta^{-1}$; we take $\alpha = \min(1, \max(\eta)^{-1})$. By Lemma 1 it follows that $\psi^{-1}\alpha\eta$ is a subsolution of (43). If we can determine a uniform lower bound m' for $\alpha\eta$, then setting $K_0 = \min(\psi^{-1})m'$ completes the proof.

To find a uniform lower bound for $\alpha\eta = \min(1, \max(\eta)^{-1})\eta$, it suffices to find uniform upper and lower bounds for η. From Proposition 9 applied to $-\Delta_{\hat{g}} + \frac{1}{8}(R_{\hat{g}} + \frac{2}{3}\tau^2)$ we have constants c_1 and c_2 such that

$$\max(\eta) \le c_1 \left\| \left|\hat{\beta}\right|^2_{\hat{g}} \right\|_{L^p} \tag{47}$$

and

$$\min(\eta) \ge c_2 \left\| \left|\hat{\beta}\right|^2_{\hat{g}} \right\|_{L^1}. \tag{48}$$

Now

$$\int_M \left|\hat{\beta}\right|^{2p}_{\hat{g}}\, d\hat{V} = \int_M \psi^{-12p+6}\,|\beta|^2_g\, dV \le \max(\psi^{12p-6})\int_M |\beta|^{2p}_g\, dV \tag{49}$$

and

$$\int_M \left|\hat{\beta}\right|^2_{\hat{g}}\, d\hat{V} = \int_M \psi^{-6}\,|\beta|^2_g\, dV \ge \min(\psi^{-6})\int_M |\beta|^2_g\, dV. \tag{50}$$

Since ψ is a fixed conformal factor and does not depend on ϕ, it suffices to estimate

$$\int_M |\beta|^{2p}_g\, dV = \int_M |\sigma + \mathbf{L}\,W|^{2p}_g\, dV \text{ from above} \tag{51}$$

and

$$\int_M |\sigma + \mathbf{L}\,W|^2 \text{ from below.} \tag{52}$$

Following the argument at the start of Section 4 we have

$$\int_M |\sigma + \mathbf{L}\,W|^2 \; dV = \int_M |\sigma|^2 + |\mathbf{L}\,W|^2 \; dV \geq \int_M |\sigma|^2 \; dV. \tag{53}$$

Since $\sigma \not\equiv 0$ we have obtained the desired lower bound.

On the other hand,

$$\int_M |\sigma + \mathbf{L}\,W|_g^{2p} \; dV \leq 2^{2p-1} \int_M |\sigma|^{2p} + |\mathbf{L}\,W|^{2p} \; dV. \tag{54}$$

Moreover, from Proposition 6

$$|\mathbf{L}\,W|^{2p} \leq \mathrm{Vol}(M) || \mathbf{L}\,W||_{L^\infty}^{2p} \leq \mathrm{Vol}(M) \left[K_\tau \max(\phi_+)^6\right]^{2p} \tag{55}$$

which establishes the desired upper bound. □

The proof of the lower bound in the Yamabe negative case is much easier. In [20] a global subsolution was found under the hypothesis that τ has no zeros. This was extended by [16] to any compatible τ using a technique from [25]. The proof is short, and we reproduce it here.

Proposition 11. *Suppose $\mathcal{Y}_g < 0$ and that τ is Lichnerowicz compatible. Then there exists a constant $K_0 > 0$ such for any $\phi \in L^\infty_+$,*

$$K_0 \leq \mathcal{N}_{\sigma,\tau}(\phi). \tag{56}$$

Proof. Pick $\eta \in W^{2,p}_+$ such that $\hat{g} = \eta^4 g$ has scalar curvature $-\frac{2}{3}\tau^2$; such a conformal factor exists since τ is Lichnerowicz compatible. Then

$$-8\,\Delta_g\,\eta + R_g\eta + \frac{2}{3}\tau^2\eta^5 - |\beta|^2\,\eta^{-7} = -\frac{2}{3}\tau^2\eta^5 + \frac{2}{3}\tau^2\eta^5 - |\beta|^2\,\eta^{-7} = -\,|\beta|^2\,\eta^{-7} \leq 0. \tag{57}$$

Hence η is a subsolution. Lemma 3 then implies that $\phi \geq \eta$ and hence $K_0 = \min \eta$ is a lower bound. □

4.2. Mapping properties of $\mathcal{N}_{\sigma,\tau}$. Suppose that $\phi_+ \in W^{2,p}$ is a global supersolution. Let K_0 be the constant from Proposition 10 or 11 depending on the sign of $\mathcal{Y}_g$, and define $U = \{\phi \in L^\infty : K_0 \leq \phi \leq \phi_+\}$. We know from Section 4.1 that U is invariant under $\mathcal{N}_{\sigma,\tau}$, and we now complete the proof using the Schauder fixed point theorem that $\mathcal{N}_{\sigma,\tau}$ has a fixed point in U. As mentioned earlier, it suffices to show that $\mathcal{N}_{\sigma,\tau}$ is continuous and $\mathcal{N}_{\sigma,\tau}(U)$ is precompact.

Proposition 12. *There exists a constant M such that for any $\phi \in U$,*

$$||\mathcal{N}_{\sigma,\tau}(\phi)||_{W^{2,p}} \leq M. \tag{58}$$

Proof. Let $W = \mathcal{W}_\tau(\phi)$, and let $\psi = \mathcal{N}_{\sigma,\tau}(\phi)$. We have the elliptic regularity estimate

$$||\psi||_{W^{2,p}} \leq c\left[||\,\Delta\,\psi||_{L^p} + ||\phi||_{L^p}\right]. \tag{59}$$

Since $0 < \psi \leq \phi_+$ we have $||\phi||_{L^p} \leq \mathrm{Vol}(M)^{1/p} \max(\phi_+)$. Also, ψ solves

$$-8\,\Delta\,\psi = -R\psi - \frac{2}{3}\tau^2\psi^5 + |\sigma + \mathbf{L}\,W|^2\,\psi^{-7}. \tag{60}$$

Since $R \in L^p$, $\sigma \in L^{2p}$, $0 < K_0 \leq \psi \leq \phi_+$, and since Proposition 6 implies

$$||\,\mathbf{L}\,W||_{L^\infty} \leq K_\tau \max(\phi_+)^6, \tag{61}$$

it follows that the right-hand side of (60) is bounded in L^p independent of ϕ. Hence inequality (58) holds. □

Corollary 3. *The set $\mathcal{N}_{\sigma,\tau}(U)$ is precompact.*

Proof. From Proposition 12, it follows that $\mathcal{N}_{\sigma,\tau}(U)$ is contained in a ball in $W^{2,p}$ and hence in a ball in $C^{1,\alpha}$. By the compact embedding of $C^{1,\alpha}$ in L^∞, we conclude that $\overline{\mathcal{N}_{\sigma,\tau}(U)}$ is compact. □

To show $\mathcal{N}_{\sigma,\tau}$ is continuous, it is enough to show that $\mathcal{W}_\tau$ and $\mathcal{L}_\tau$ are continuous. That $\mathcal{W}_\tau$ is continuous is obvious, but there is something to show for $\mathcal{L}_\tau$. The continuity in this case follows from the implicit function theorem.

Proposition 13. *If $g \in W^{2,p}$ and $\tau \in L^{2p}$ are Lichnerowicz compatible, then the map $\mathcal{L}_\tau : \mathcal{D}_\tau \to W^{2,p}$ is C^1.*

Proof. Let $\beta_0 \in D_\tau$ and let $\psi_0 = \mathcal{L}_\tau(\beta_0) = \mathcal{L}(\beta_0)$. Define $\hat{g} = \psi_0^4 g$ and let $\hat{\mathcal{L}}$ be the corresponding Lichnerowicz operator. That is, $\hat{\mathcal{L}}(\beta)$ is the solution of

$$-8\,\Delta_{\hat{g}}\,\phi + R_{\hat{g}}\phi = -\frac{2}{3}\tau^2\phi^5 + |\beta|^2\,\phi^{-7}. \tag{62}$$

By conformal covariance we have

$$\mathcal{L}_\tau(\beta) = \psi_0\hat{\mathcal{L}}_\tau(\psi_0^{-2}\beta) \tag{63}$$

and hence to show that $\mathcal{L}$ is C^1 near β_0 it suffices to show that $\hat{\mathcal{L}}$ is C^1 near $\hat{\beta}_0 = \psi_0^{-2}\beta_0$. Noting that $\hat{\mathcal{L}}(\hat{\beta}_0) \equiv 1$, we may drop the hat notation and it suffices to show that $\mathcal{L}$ is C^1 near any point β_0 such that $\mathcal{L}(\beta_0) \equiv 1$.

Define $F : W^{2,p}_+ \times \mathcal{D}_\tau \to L^{2p}$ by

$$F(\phi,\beta) = -8\,\Delta\,\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - |\beta|^2\,\phi^{-7}; \tag{64}$$

the Lichnerowicz operator satisfies $F(\mathcal{L}_\tau(\beta),\beta) = 0$. A standard computation shows that the Gâteaux derivative of F is given by

$$DF_{\phi,\beta}(h,k) = -8\,\Delta\,h + Rh + \frac{10}{3}\tau^2\phi^4 h + 7\,|\beta|^2\,\phi^{-8}h - 2\phi^{-7}\,\langle\beta,k\rangle\,. \tag{65}$$

It is easily seen that the operator DF is continuous in ϕ and β.

Now

$$DF_{1,\beta_0}(h,0) = -8\,\Delta\,h + Rh + \frac{10}{3}\tau^2 h + 7\,|\beta_0|^2\,h. \tag{66}$$

But since $\mathcal{L}(\beta_0) \equiv 1$,

$$R = -\frac{2}{3}\tau^2 + |\beta_0|^2 \tag{67}$$

and hence

$$DF_{1,\beta_0}(h,0) = -8\,\Delta\,h + \left[\frac{8}{3}\tau^2 + 8\,|\beta_0|^2\right] h. \tag{68}$$

Since the potential $(8/3)\tau^2 + 8\,|\beta_0|^2$ is non-negative and does not vanish identically (since g and τ are Lichnerowicz compatible and β_0 is admissible), we conclude that $DF_{1,\beta_0} : W^{2,p} \to L^p$ is an isomorphism. The implicit function theorem then implies that $\mathcal{L}$ is a C^1 function in a neighborhood of β_0. □

This completes the proof of Theorem 1. Our result of primary interest, Corollary 1, relies crucially on the HNK supersolution. For completeness, we give a proof here of its existence.

Proposition 14 ([16]). *Suppose $g \in W^{2,p}$ with $p > 3$, $\mathcal{Y}_g > 0$, $\tau \in W^{1,p}$, and $\sigma \in W^{1,p}$. If $||\sigma||_\infty$ is sufficiently small, then there exists a global supersolution of (4)–(5).*

Proof. Pick $\psi \in W^{2,p}_+$ such that the scalar curvature $\hat{R}$ of $\hat{g} = \psi^4 g$ is strictly positive. We claim that if ϵ is sufficiently small, and if $||\sigma||_{L^\infty}$ is additionally sufficiently small, then $\epsilon\psi$ is a global supersolution.

Suppose $0 < \phi \le \epsilon\psi$, and let W be the corresponding solution of (5). Note that

$$-8\,\Delta(\epsilon\psi) + R(\epsilon\psi) + \tau^2(\epsilon\psi)^5 - |\sigma + \mathbf{L}\,W|^2(\epsilon\psi)^{-7} = \tag{69}$$
$$= \epsilon\hat{R}\psi^5 + \tau^2(\epsilon\psi)^5 - |\sigma + \mathbf{L}\,W|^2\,(\epsilon\psi)^{-7}$$
$$\ge \epsilon\hat{R}\psi^5 - 2\,|\mathbf{L}\,W|^2\,(\epsilon\psi)^{-7} - 2\,|\sigma|^2\,(\epsilon\psi)^{-7}.$$

By Proposition 6 there exists a constant K_τ such that

$$||\,\mathbf{L}\,W||_\infty \le K_\tau ||\phi||_\infty^6 \le K_\tau \epsilon^6 \max(\psi)^6. \tag{70}$$

Hence

$$\epsilon\hat{R}\psi^5 - 2\,|\mathbf{L}\,W|^2\,(\epsilon\psi)^{-7} - 2\,|\sigma|^2\,(\epsilon\psi)^{-7} \ge$$
$$\ge \epsilon\min(\hat{R})\min(\psi)^5 - 2K_\tau^2\epsilon^5\max(\psi)^{12}\min(\psi)^{-7} - 2\,|\sigma|^2\,(\epsilon\psi)^{-7}$$
$$= \epsilon 2K_\tau^2 \frac{\max(\psi)^{12}}{\min(\psi)^7}\left[\frac{\min(\hat{R})}{2K_\tau^2}\left(\frac{\min(\psi)}{\max(\psi)}\right)^{12} - \epsilon^4\right] - 2\,|\sigma|^2\,(\epsilon\psi)^{-7}. \tag{71}$$

Now pick ϵ so small that

$$\frac{\min(\hat{R})}{2K_\tau^2}\left(\frac{\min(\psi)}{\max(\psi)}\right)^{12} - \epsilon^4 \tag{72}$$

is positive. It then follows that $\epsilon\psi$ is a global supersolution so long as $||\sigma||_{L^\infty}$ is so small that the right hand side of (71) remains positive. □

5. Vacuum solutions as the limit of non-vacuum solutions

In this section we give an alternative proof of Corollary 1 using sequences of non-vacuum solutions. We start with the following theorem which is an immediate consequence of the results of [16].

Proposition 15 ([16]). *Let $g \in W^{2,p}$ with $p > 3$ be a metric on a smooth, compact 3-manifold. Suppose that g has no conformal Killing fields, g is Yamabe positive, and that $\sigma \in W^{1,p}$ is a transverse traceless tensor and $\tau \in W^{1,p}$. If $||\sigma||_{L^\infty}$ is sufficiently small, then for each $\rho_n = \frac{1}{n}$ there exists a solution $(\phi_n, W_n) \in W^{2,p}_+ \times W^{2,p}$ of*

$$-8\,\Delta\,\phi_n + R_g\phi_n = -\frac{2}{3}\tau^2\phi_n^5 + 2\rho_n\phi_n^{-3} + |\sigma + \mathbf{L}\,W_n|^2\,\phi_n^{-7} \tag{73}$$

$$\operatorname{div}\mathbf{L}\,W_n = \frac{2}{3}\phi_n^{-6}\,d\tau. \tag{74}$$

Moreover, there exists a constant $N_+ > 0$ independent of n such that $0 < \phi_n \le N_+$ for every n.

We now consider what happens to the sequence (ϕ_n, W_n) and show a subsequence of it converges to a solution (ϕ, W) of the vacuum equations.

Lemma 4. *There is a subsequence of* $\{W_n\}$ *that converges in* $W^{1,p}$ *and weakly in* $W^{2,p}$ *to a limit* W. *Moreover,* $\{\mathbf{L}\, W_n\}$ *converges uniformly to* $\mathbf{L}\, W$.

Proof. From Proposition 6 we have

$$||\,\mathbf{L}\, W_n||_{W^{2,p}} \leq c||d\tau\phi_n^6||_{L^p} \leq c||d\tau||_{L^p}||\phi_n||_{L^\infty}^6 \leq c||d\tau||_{L^p}N_+^6. \tag{75}$$

So the sequence $\{W_n\}$ is bounded in $W^{2,p}$ and has a subsequence that converges weakly in $W^{2,p}$ and strongly in $W^{1,p}$ to a limit W.

Reducing to this subsequence, we know that $\{\mathbf{L}\, W_n\}$ is bounded in C^α since $\{W_n\}$ is bounded in $W^{2,p}$ and therefore in $C^{1,\alpha}$ for some $\alpha > 0$. But then by the Arzelà-Ascoli theorem, a subsequence converges in C^0. Since $\mathbf{L}\, W_n \to \mathbf{L}\, W$ in L^p, we conclude that $\mathbf{L}\, W_n \to \mathbf{L}\, W$ in C^0. □

We henceforth reduce to this subsequence.

Lemma 5. *Suppose* $\sigma \not\equiv 0$. *Then* $\sigma + \mathbf{L}\, W \not\equiv 0$.

Proof. If $\sigma + \mathbf{L}\, W \equiv 0$ then $\operatorname{div} \mathbf{L}\, W = 0$ weakly, and hence W is in the kernel of the vector Laplacian. In particular $\mathbf{L}\, W = 0$, so $\sigma \equiv 0$, a contradiction. □

We henceforth also assume that $\sigma \not\equiv 0$, which is necessary to establish the following lower bound for the sequence.

Proposition 16. *If* $\sigma \not\equiv 0$, *then there is a constant* N_- *such that*

$$0 < N_- \leq \phi_n \tag{76}$$

for every n.

Proof. Pick $\hat{\psi} \in W_+^{2,p}$ such that $\hat{g} = \hat{\psi}^4 g$ has positive scalar curvature $R_{\hat{g}}$; this is possible since $\mathcal{Y}_g > 0$. Let $\beta_n = \sigma + \mathbf{L}\, W_n$, $\beta = \sigma + \mathbf{L}\, W$, $\hat{\beta}_n = \hat{\psi}^{-2}\beta_n$, $\hat{\beta} = \hat{\psi}^{-2}\beta$, and $\hat{\rho}_n = \psi^{-8}\rho_n$.

Following [25] we seek non-constant sub- and supersolutions of

$$-8\Delta_{\hat{g}}\phi + R_{\hat{g}}\phi = -\frac{2}{3}\tau^2\phi^5 + \left|\hat{\beta}\right|_{\hat{g}}^2 \phi^{-7} + 2\hat{\rho}_n\phi^{-3}. \tag{77}$$

We will find a positive lower bound for the sub-solutions and use this lower bound to obtain a positive lower bound for the functions ϕ_n.

For each n, let ψ_n be the solution of

$$-8\Delta_{\hat{g}}\psi_n + \left[R_{\hat{g}} + \frac{2}{3}\tau^2\right]\psi_n = |\hat{\beta}_n|_{\hat{g}}^2 + 2\hat{\rho}_n, \tag{78}$$

which exists since $R_{\hat{g}} + \frac{2}{3}\tau^2 > 0$. Since $\hat{\beta}_n$ and $\hat{\rho}_n$ converge uniformly to $\hat{\beta}$ and 0, it follows that ψ_n converges in $W^{2,p}$ to the solution ψ of

$$-8\,\Delta_{\hat{g}}\,\psi + R_{\hat{g}}\psi + \frac{2}{3}\tau^2\psi = |\hat{\beta}|_{\hat{g}}^2. \tag{79}$$

In particular, from Sobolev embedding, this convergence is in C^0. Note that since β (i.e. $\sigma + \mathbf{L}\, W$) is not identically zero, ψ is not identically zero. From the weak and strong maxiumum principles ([15] Theorems 8.1 and 8.19) it follows that each ψ_n and also ψ is a positive function. Since the convergence is uniform on a compact manifold, there are constants m and M such that $0 < m \leq \psi_n \leq M$ for every n.

Consider the function $\alpha\psi_n$. Then

$$-8\Delta_{\hat{g}}\alpha\psi_n + R_{\hat{g}}\alpha\psi_n + \frac{2}{3}\tau^2(\alpha\psi_n)^5 - |\hat{\beta}|^2_{\hat{g}}(\alpha\psi_n)^{-7} - 2\hat{\rho}_n(\alpha\psi_n)^{-3} = \frac{2}{3}\tau^2\left[(\alpha\psi_n)^5 - \alpha\psi_n\right] + \left|\hat{\beta}\right|^2_{\hat{g}}\left[\alpha - (\alpha\psi_n)^{-7}\right] + \rho\left[\alpha - (\alpha\psi_n)^{-3}\right]. \tag{80}$$

One readily verifies that if $\alpha \geq \max(1, \min(\psi_n)^{-1})$ then each term on the right-hand side of (80) is non-negative and $\alpha\psi_n$ is a supersolution. We define $\alpha_+ = \max(1, m^{-1})$.

Similarly, if $\alpha \leq \min(1, \max(\psi_n)^{-1})$ then each term on the right-hand side of (80) is non-positive and $\alpha\psi_n$ is a subsolution. We define $\alpha_- = \max(1, M^{-1})$.

Since $\alpha_-\psi_n$ and $\alpha_+\psi_n$ are sub- and supersolutions of (77) it follows from Lemma 1 that $\alpha_-\psi^{-1}\psi_n$ and $\alpha_+\psi^{-1}\phi_n$ are sub- and supersolutions of (73). Lemma 3 then implies

$$\alpha_-\psi^{-1}\psi_n \leq \phi_n \leq \alpha_+\psi^{-1}\psi_n \tag{81}$$

for each n. Letting $N_- = \alpha_- \max(\psi)^{-1}m$ completes the proof. □

Proposition 17. *A subsequence of $\{\phi_n\}$ converges uniformly and in $W^{1,p}$ to a function $\phi \in W^{2,p}_+$ that is a solution of*

$$-8\Delta\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + \mathbf{L}\,W|^2\phi^{-7}. \tag{82}$$

Proof. The functions ϕ_n solve

$$-8\Delta\phi_n = -R\phi_n - \frac{2}{3}\tau^2\phi_n^5 + |\sigma + \mathbf{L}\,W_n|^2\phi_n^{-7} + 2\rho_n\phi_n^{-3}. \tag{83}$$

Since the right-hand side of (83) is bounded in L^p (here we use the fact that $N_- \leq \phi_n \leq N_+$ for every n) we conclude from elliptic regularity estimate

$$||\phi_n||_{W^{2,p}} \leq c_1\left(||\,\Delta\,\phi_n||_{L^p} + ||\phi_n||_{L^p}\right) \leq c_2\left(||\,\Delta\,\phi_n||_{L^p} + N_+\right) \tag{84}$$

that the sequence $\{\phi_n\}$ is bounded in $W^{2,p}$. Reducing to a subsequence, we conclude that $\{\phi_n\}$ converges weakly in $W^{2,p}$ and strongly in $W^{1,p}$ and also in C^0 to a limit $\phi \in W^{2,p}$ and $\phi \geq N_- > 0$. A standard convergence argument shows that ϕ is a weak solution of

$$-8\Delta\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + \mathbf{L}\,W|^2\phi^{-7}. \tag{85}$$

Since ϕ is a weak solution and $\phi \in W^{2,p}$ we conclude that ϕ is a strong solution. □

Proposition 18. *The vector field W is a solution of*

$$\operatorname{div}\mathbf{L}\,W = \frac{2}{3}\phi^6 d\tau. \tag{86}$$

Proof. Since $W_n \to W$ in $W^{1,p}$ and $\phi_n^6 d\tau \to \phi^6 d\tau$ in L^p, and since

$$\operatorname{div}\mathbf{L}\,W_n = \frac{2}{3}\phi_n^6 d\tau, \tag{87}$$

a standard argument shows that W weakly solves

$$\operatorname{div}\mathbf{L}\,W = \frac{2}{3}\phi^6 d\tau. \tag{88}$$

Since $W \in W^{2,p}$, W is a strong solution. □

This completes the second proof of Corollary 1.

6. Conclusion

The conformal method of solving the Einstein constraint equations is remarkably effective when the mean curvature is constant, and is remarkably recalcitrant when it is not. In this paper we have made progress towards our understanding of the non-CMC case. We have proved that there exist solutions of the vacuum constraint equations whenever a global supersolution can be found. Using a well-known near-CMC global supersolution, we have simplified the hypotheses required for existence in the near-CMC case. And as a consequence of the HNT supersolution, we have shown that for Yamabe-positive metrics, and for small enough transverse traceless tensors, there exist vacuum solutions of the constraint equations for any choice of mean curvature.

Our existence theorem shows that any potential failure of the conformal method must arise from a loss of control from above of the conformal factor. Currently known global supersolutions impose this control by making strong smallness assumptions, either on the mean curvature, or on the transverse traceless tensor. Presumably one can interpolate between these smallness conditions, but the question of existence for generic large data remains open. There also remain numerous other open questions, including the existence of far-from-CMC solutions for Yamabe-null or Yamabe-negative metrics, uniqueness for far-from-CMC data, and existence for metrics admitting conformal Killing fields. As a consequence, the applicability of the conformal method for general mean curvatures remains largely unknown. Nevertheless, the results of [16] and the current paper are a step towards answering this question.

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