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FIRST CONIVEAU NOTCH OF THE DWORK FAMILY AND ITS MIRROR

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Abstract. If X_λ is a smooth member of the Dwork family over a perfect field k, and Y_λ is its mirror variety, then the motives of X_λ and Y_λ are equal up to effective motives that are in coniveau ≥ 1. If k is a finite field, this provides a motivic explanation for Wan's congruence between the zeta functions of X_λ and Y_λ.

Introduction

Let k be a field. We consider the Dwork family of hypersurfaces X_λ in $\mathbb{P}^n$ defined by the equation

$$\sum_{i=0}^{n} X_i^{n+1} + \lambda X_0 \dots X_n = 0$$

with the parameter $\lambda \in k$. The variety X_λ is a Calabi-Yau manifold when X_λ is smooth. On each member X_λ there is a group action by the kernel G of the character $\mu_{n+1}^{n+1} \to \mu_{n+1}, (\zeta_i) \mapsto \prod_i \zeta_i$, given by

$$G \times X_\lambda \to X_\lambda, \quad (\zeta_0, \dots, \zeta_n) \cdot (x_0 : \dots : x_n) = (\zeta_0 x_0, \dots, \zeta_n x_n).$$

The quotient X_λ/G is a hypersurface with trivial canonical bundle in a toric Fano variety and a singular mirror of X_λ [B]. If Y_λ is a crepant resolution of X_λ/G then (X_λ, Y_λ) provides an example of a mirror pair. Since the birational geometry of Y_λ is independent of the choice of the resolution a natural question arises: to compare the birational motives of X_λ and Y_λ. For a finite field $k = \mathbb{F}_q$ the number of $\mathbb{F}_{q^m}$-rational points modulo q^m is a birational invariant and D. Wan asked to compare the number of rational points of a mirror pair [W]. In the case of the Dwork family he proved a mirror congruence formula [W, Theorem 1.1]:

$$\#X_\lambda(\mathbb{F}_{q^m}) = \#Y_\lambda(\mathbb{F}_{q^m}) \mod q^m$$

for every positive integer m. Fu and Wan studied more general mirror pairs which come from quotient constructions and obtained under certain assumptions on the action of G (see Theorem 3.7) a congruence formula [FW]:

$$\#X(\mathbb{F}_{q^m}) = \#(X/G)(\mathbb{F}_{q^m}) \mod q^m. \tag{0.0.1}$$

The same formula is proved in [BBE, Corollary 6.12] with different assumptions.

The purpose of this paper is twofold. The first theorem compares the motives of X_λ and Y_λ when X_λ is a member of the Dwork family, and provides Wan's congruence formula as a consequence. We also explain what can be expected for general quotient

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constructions in §3. In the second theorem we prove a congruence formula for a quotient singularity X/G and a resolution of singularities $Y \to X/G$:

$$\#(X/G)(\mathbb{F}_{q^m}) = \#Y(\mathbb{F}_{q^m}) \mod q^m.$$

Thus 0.0.1 is sufficient in order to get $\#X(\mathbb{F}_{q^m}) = \#Y(\mathbb{F}_{q^m})$ modulo q^m.

We state now our theorems and several consequences. By a motive we understand a pair (X, P) with X a smooth projective variety and $P \in \mathrm{CH}^{\dim X}(X \times X) \otimes \mathbb{Q}$ a projector. The morphisms are correspondences in rational coefficients. Note that we work with *effective* motives only. The Lefschetz motive is denoted by $\mathbb{Q}(-1) := (\mathbb{P}^1, \mathbb{P}^1 \times p)$ with $p \in \mathbb{P}^1(k)$. For X_λ the cycle $P = 1/|G| \sum_{g \in G} \Gamma(g)$, where Γ denotes the graph, is a projector.

Theorem. *Let k be a perfect field, and $n \geq 2$. We assume that* $\mathrm{char}(k) \nmid n+1$ *if the characteristic of k is positive. Let X_λ be a smooth member of the Dwork family. Then there are effective motives N, N' such that*

$$(X_\lambda, id) \cong (X_\lambda, P) \oplus N \otimes \mathbb{Q}(-1) \quad \textit{and} \quad (Y_\lambda, id) \cong (X_\lambda, P) \oplus N' \otimes \mathbb{Q}(-1).$$

For a finite field $k = \mathbb{F}_q$ the eigenvalues of the geometric Frobenius acting on $H^*_{\text{ét}}(N \otimes \mathbb{Q}(-1)) = H^*_{\text{ét}}(N) \otimes \mathbb{Q}_l(-1)$ lie in $q \cdot \bar{\mathbb{Z}}$, and by using Grothendieck's trace formula this implies Wan's theorem [W, Theorem 1.1]. For $k = \mathbb{C}$ the theorem of Arapura-Kang on the functoriality of the coniveau filtration N^* allows us to conclude that

$$\mathrm{gr}^0_{N^*}(H^*(X_\lambda, \mathbb{Q})) \cong \mathrm{gr}^0_{N^*}(H^*(Y_\lambda, \mathbb{Q}))$$

as Hodge structures (see Corollary 2.4).

We now describe our method. We use birational motives in order to reduce to a statement for zero cycles over $\mathbb{C}$: $\mathrm{CH}_0(X_\lambda) = P \circ \mathrm{CH}_0(X_\lambda)$, i.e. P acts as identity. To prove this we consider, additionally to G, the action of the symmetric group S_{n+1} acting via permutation of the homogeneous coordinates. The transpositions act as -1 on $H^0(X_\lambda, \omega_{X_\lambda})$ and the quotients X_λ/H for suitable subgroups H of $G \rtimes S_{n+1}$ can be shown to be $\mathbb{Q}$-Fano varieties. By the theorem of Zhang [Z] these are rationally chain connected, which yields sufficiently many relations for the zero cycles on X_λ to prove the claim.

Theorem. *Let X be a smooth projective $\mathbb{F}_q$-variety with an action of a finite group G. Let $\pi : X \to X/G$ be the quotient, and $f : Y \to X/G$ be a birational map, where Y is a smooth projective variety. Then*

$$\#Y(\mathbb{F}_q) = \#(X/G)(\mathbb{F}_q) \mod q.$$

For the proof we use the action of the geometric Frobenius F on étale cohomology. Suppose that $Z \subset X/G$ is the set where f is not an isomorphism, then F acts on the cohomology with support in Z with eigenvalues in $q\bar{\mathbb{Z}}$. This is proved by reduction to the case $\pi^{-1}(Z) \subset X$ via a trace map argument. Counting points with Grothendieck's trace formula yields the result.

1. Zero cycles and the first notch of the coniveau

1.1. Notation. Let k be a field. By a motive we understand a pair (X, P) with X a smooth projective variety over k and $P \in \mathrm{Hom}(X, X)$ a projector in the algebra of correspondences. The correspondences are defined to be

$$\mathrm{Hom}(X, Y) = \oplus_i \mathrm{CH}^{\dim X_i}(X_i, Y),$$

where X_i are the connected components of X. Here and in the following we use Chow groups with $\mathbb{Q}$ coefficients. Note that we work with effective motives only.

We simply write $X = (X, id_X)$ for the motive associated with X. The motives form a category $\mathcal{M}_k$ with morphism groups

$$\mathrm{Hom}((X, P), (Y, Q)) = Q \circ \mathrm{Hom}(X, Y) \circ P \subset \mathrm{Hom}(X, Y).$$

The sum and the product in $\mathcal{M}$ are defined by disjoint union and product:

$$(X, P) \oplus (Y, Q) = (X \cup Y, P + Q)$$
$$(X, P) \otimes (Y, Q) = (X \times Y, P \times Q)$$

We denote by $\mathbb{Q}(-1)$ the Lefschetz motive, i.e. $\mathbb{P}^1 = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$. We set $\mathbb{Q}(a) := \mathbb{Q}(-1)^{\otimes -a}$ for $a < 0$ and $\mathbb{Q}(0) := \mathrm{Spec}(k)$. If X is connected then

$$\mathrm{Hom}((X, P) \otimes \mathbb{Q}(a), (Y, Q) \otimes \mathbb{Q}(b)) = P \circ \mathrm{CH}^{\dim X - a + b}(X \times Y) \circ Q.$$

If M is a motive, we define

$$\mathrm{CH}^i(M) := \mathrm{Hom}(\mathbb{Q}(-i), M), \quad \mathrm{CH}_i(M) := \mathrm{Hom}(M, \mathbb{Q}(-i))$$

for $i \geq 0$ and $\mathrm{CH}^i(M) = 0 = \mathrm{CH}_i(M)$ for $i < 0$. We have

$$\mathrm{CH}^i(M \otimes \mathbb{Q}(a)) = \mathrm{CH}^{i+a}(M), \quad \mathrm{CH}_i(M \otimes \mathbb{Q}(a)) = \mathrm{CH}_{i+a}(M) \tag{1.1.1}$$

for all $i \geq 0$ and $a \leq 0$. Note that for a motive $M = (X, P)$ with X connected of dimension n the equality $\mathrm{CH}_i(M) = \mathrm{CH}^{n-i}(M)$ in general doesn't hold.

If $k \subset L$ is an extension of fields then $(X, P) \mapsto (X \times_k L, P \times_k L)$ defines a functor

$$\times_k L : \mathcal{M}_k \to \mathcal{M}_L. \tag{1.1.2}$$

The following Proposition is a consequence of the theory of birational motives [KS] due to B. Kahn and R. Sujatha. We include the proof for the convenience of the reader.

Proposition 1.2. *Let k be a perfect field and X be connected.*

(i) *A motive $M = (X, P)$ can be written as $M \cong N \otimes \mathbb{Q}(-1)$ with some motive N if and only if $\mathrm{CH}_0(M \times_k L) = 0$ for some field extension L of the function field $k(X)$ of X.*

(ii) *There exists an isomorphism $M \cong N \otimes \mathbb{Q}(a)$ with some motive N and $a < 0$ if and only if $\mathrm{CH}_i(M \times_k L) = 0$ for all $i < -a$ and all field extensions $k \subset L$.*

Proof. (i) If $M \cong N \otimes \mathbb{Q}(-1)$ then $M \times_k L \cong (N \times_k L) \otimes \mathbb{Q}(-1)$ and therefore $\mathrm{CH}_0(M \times_k L) = 0$ by 1.1.1.

Suppose now that $\mathrm{CH}_0(M \times_k L) = 0$. By the same arguments as in [BS, Proposition 1] we have

$$P \in \mathrm{image}\left(\mathrm{CH}^{\dim D}(X \times D) \xrightarrow{(id \times \imath)_*} \mathrm{CH}^{\dim X}(X \times X)\right) \tag{1.2.1}$$

for some effective (not necessarily irreducible) Divisor $\imath : D \to X$. For the convenience of the reader we recall the proof. It is well-known that

$$\mathrm{CH}_0(X \times_k k(X)) \to \mathrm{CH}_0(X \times_k L)$$

is injective, and therefore $\mathrm{CH}_0(M \times_k L) = 0$ implies $\mathrm{CH}_0(M \times_k k(X)) = 0$. Let τ be the composite

$$\tau : \mathrm{CH}^{\dim X}(X \times X) \to \varinjlim_{U \subset X} \mathrm{CH}^{\dim X}(X \times U) = \mathrm{CH}^{\dim X}(X \times k(X)),$$

where the limit is over all open subsets $U \subset X$. It is easy to see that the equality $0 = (P \times_k k(X)) \circ \tau(\Delta_X) = \tau(P)$ holds, which shows 1.2.1.

Let $Y \to D$ be an alteration such that Y is regular (and thus smooth), and denote by $f : Y \to D \xrightarrow{\imath} X$ the composite. We have $P = (id_X \times f)_*(Z)$ for a suitable cycle $Z \in \mathrm{CH}^{\dim Y}(X \times Y)$. Define $Q \in \mathrm{End}(Y)$ by $Q = Z \circ P \circ \Gamma(f)^t$ where $\Gamma(f)^t \in \mathrm{CH}^{\dim X}(Y \times X)$ is the graph of f. The equality $\Gamma(f)^t \circ Z = P$ implies $Q^2 = Q$. It is easy to check that

$$(Y, Q) \otimes \mathbb{Q}(-1) \xrightarrow{P \circ \Gamma(f)^t} (X, P) \quad (X, P) \xrightarrow{Z \circ P} (Y, Q) \otimes \mathbb{Q}(-1)$$

are inverse to each other, so that $(Y, Q) \otimes \mathbb{Q}(-1) \cong (X, P)$ as claimed.

(ii) By induction on a and using (i). □

1.3. Motives associated with morphism. Let $\pi : X \to Z$ be a finite surjective morphism of degree d, where X is connected, smooth and projective, but Z may be singular. The cycle $X \times_Z X \subset X \times X$ gives a projector $P = 1/d \cdot [X \times_Z X] \in \mathrm{End}(X)$ and we write $(X, \pi) := (X, P)$ for the corresponding motive.

If $\pi : X \to Y$ is a surjective morphism between connected, smooth and projective varieties of the same dimension, then the graph $\Gamma(\pi)$ of π gives morphisms $\Gamma(\pi) \in \mathrm{Hom}(Y, X)$ and $\Gamma(\pi)^t \in \mathrm{Hom}(X, Y)$. Let d be the degree of π, since $\Gamma(\pi)^t \circ \Gamma(\pi) = d \cdot id_Y$ we get an isomorphism $Y \cong (X, 1/d \cdot \Gamma(\pi) \circ \Gamma(\pi)^t)$. Thus

$$(Y, Q) \cong (X, 1/d \cdot \Gamma(\pi) \circ Q \circ \Gamma(\pi)^t)$$

for every projector Q.

Proposition 1.4. *Let k be a perfect field. In the diagram*

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \pi} & & \\ Z & \xleftarrow[f]{} & Y \end{array}$$

we assume that X, Y are smooth, connected and projective varieties of the same dimension, the morphism π is finite and surjective, and f is birational. The following holds:

(i) *The motive (X, π) is a direct summand in Y.*

(ii) *If $X = (X, \pi) \oplus N' \otimes \mathbb{Q}(-1)$ for some motive N', then*

$$Y \cong (X, \pi) \oplus N \otimes \mathbb{Q}(-1)$$

for some motive N.

Proof. (i) We write S for the unique irreducible component of $X \times_Z Y$ of dimension $\dim X$. Choose an alteration $g : W \to S$ with W regular, W is smooth since k is perfect.

Via $g_1 := pr_1 \circ g$ (resp. $g_2 := pr_2 \circ g$) the motives X, (X, π) (resp. Y) are direct summands of W, we write $P_X, P_{(X,\pi)}, P_Y$ for the corresponding projectors. The inclusion (X, π) factors through Y if and only if $P_{(X,\pi)} \circ P_Y = P_Y \circ P_{(X,\pi)} = P_{(X,\pi)}$ in $\mathrm{End}(W)$. We have

$$\begin{aligned} \deg(g)^2 \deg(\pi)^2 \cdot P_Y \circ P_{(X,\pi)} &= \Gamma(g_2) \circ \Gamma(g_2)^t \circ \Gamma(g_1) \circ [X \times_Z X] \circ \Gamma(g_1)^t \\ &= \deg(g) \cdot \Gamma(g_2) \circ [S] \circ [X \times_Z X] \circ \Gamma(g_1)^t \\ &= \deg(g) \cdot [W \times_Z X] \circ [X \times_Z X] \circ \Gamma(g_1)^t \\ &= \deg(g) \deg(\pi) \cdot \Gamma(g_1) \circ [X \times_Z X] \circ [X \times_Z X] \circ \Gamma(g_1)^t \\ &= \deg(g)^2 \deg(\pi)^2 \cdot P_{(X,\pi)} \end{aligned}$$

That $P_{(X,\pi)} \circ P_Y = P_{(X,\pi)}$ can be proved in the same way. Note that

$$(X, \pi) \xrightarrow{\Gamma(g_1)} W \xrightarrow{\Gamma(g_2)^t} Y$$

does not depend on the choice of W, i.e. (X, π) is in a natural way a direct summand in Y. Indeed, if $h : W' \to W$ then

$$\Gamma(g_2 \circ h)^t \circ \Gamma(g_1 \circ h) = \Gamma(g_2)^t \circ \Gamma(h)^t \circ \Gamma(h) \circ \Gamma(g_1) = \Gamma(g_2)^t \circ \Gamma(g_1),$$

and for another choice W'' we may find W' dominating W and W''.

(ii) Write $Y \cong (X, \pi) \oplus M$. Let $L \supset k$ be a field extension, we have $Y \times_k L \cong (X \times_k L, \pi \times_k L) \oplus M \times_k L$. The map $S \times_k L \to X \times_k L$ is birational and X is smooth, thus

$$\mathrm{CH}_0(S \times_k L) \cong \mathrm{CH}_0(X \times_k L) \cong \mathrm{CH}_0(X \times_k L, \pi \times_k L).$$

The pushforward $\mathrm{CH}_0(S \times_k L) \to \mathrm{CH}_0(Y \times_k L)$ is surjective, and therefore

$$\mathrm{CH}_0(Y \times_k L) = \mathrm{CH}_0(X \times_k L, \pi \times_k L)$$

and $\mathrm{CH}_0(M \times_k L) = 0$. According to Proposition 1.2 this shows $M \cong N \otimes \mathbb{Q}(-1)$. □

1.5. Coniveau filtration. Let $k = \mathbb{C}$, we work with the singular cohomology in rational coefficients $H^i(X) := H^i(X, \mathbb{Q})$ for $i \geq 0$. The coniveau filtration $N^*H^i(X)$ is defined to be

$$N^p H^i(X) := \bigcup_S \ker\left(H^i(X) \to H^i(X - S)\right),$$

where S runs through all algebraic subsets (maybe reducible) of codimension $\geq p$. The coniveau filtration is a filtration of Hodge structures and therefore the graduated pieces $\mathrm{Gr}^p_N := N^p H^i(X)/N^{p+1} H^i(X)$ inherit a Hodge structure.

By the work of Arapura and Kang [AK, Theorem 1.1] the coniveau filtration is preserved (up to shift) by pushforwards, exterior products and pullbacks. Using resolution of singularities it follows that

(1.5.1) $$\mathrm{Gr}^p_N : (X, P) \mapsto \mathrm{image}(P : \oplus_i \mathrm{Gr}^p_N H^i(X) \to \oplus_i \mathrm{Gr}^p_N H^i(X))$$

is a functor from motives to Hodge structures (for all $p \geq 0$). Note, however, that there is no Kuenneth formula for Gr^p_N; even for $p = 0$ the surjection

$$\bigoplus_{s+t=i} \mathrm{Gr}^0_N H^s(X) \otimes \mathrm{Gr}^0_N H^t(Y) \to \mathrm{Gr}^0_N H^i(X \times Y)$$

is not injective in general. For the fiber product with $\mathbb{P}^1$ we have

$$N^p H^i(X \times \mathbb{P}^1) = N^p H^i(X) \oplus N^{p-1} H^{i-2}(X)(-1)$$

and therefore

$$\begin{aligned} &\mathrm{Gr}^p_N(M \otimes \mathbb{Q}(-1)) = \mathrm{Gr}^{p-1}_N(M)(-1) \quad \text{if } p > 0 \\ &\mathrm{Gr}^0_N(M \otimes \mathbb{Q}(-1)) = 0 \end{aligned} \tag{1.5.2}$$

for all motives M.

2. Application: the Dwork family and its mirror

2.1. Let k be a field. We consider the hypersurfaces X_λ in $\mathbb{P}^n_k$ defined by the equation

$$\sum_{i=0}^{n} X_i^{n+1} + \lambda \cdot X_0 \cdots X_n = 0 \tag{2.1.1}$$

with $\lambda \in k$, and we assume that $n+1$ is prime to the characteristic of k.

Let $G \subset (\mu_{n+1})^{n+1}/\Delta(\mu_{n+1})$ ($\Delta(\mu_n) \cong \mu_{n+1}$ diagonally embedded) be the kernel of the character $(\zeta_0, \ldots, \zeta_{n+1}) \mapsto \zeta_0 \cdot \cdots \cdot \zeta_{n+1}$, then G acts on X_λ in the obvious way. We denote by $\pi : X_\lambda \to X_\lambda/G$ the quotient map.

As explained in section 1.3 we get a natural map

$$\mathrm{CH}_0(X_\lambda) \to \mathrm{CH}_0(X_\lambda, \pi).$$

Recall that we use the notation from section 1.1. In particular all Chow groups have coefficients in $\mathbb{Q}$.

Lemma 2.2. *Let k be a field. We assume that* $\mathrm{char}(k) \nmid n+1$ *if* $\mathrm{char}(k) > 0$. *If $n \geq 2$ and X_λ is smooth, then the map*

$$\mathrm{CH}_0(X_\lambda) \to \mathrm{CH}_0(X_\lambda, \pi)$$

is an isomorphism.

Proof. The projector for $(X_\lambda, \pi) \subset X_\lambda$ is $\frac{1}{|G|}\sum_{g \in G} \Gamma(g)$. Therefore the statement is equivalent to

$$\sum_{g \in G} g_*(a) = |G| \cdot a$$

for every $a \in \mathrm{CH}_0(X_\lambda)$.

1. case: $k = \mathbb{C}$. For $n = 2$ the quotient map $\pi : X_\lambda \to X_\lambda/G$ is an isogeny of elliptic curves, and therefore the statement is true.

Consider $\mu_{n+1} \cong H \subset G$ with $\zeta \mapsto (\zeta, \zeta^{-1}, 1, \ldots, 1)$, and $\tau \in \mathrm{Aut}(X_\lambda)$ defined by $\tau^*(X_0) = X_1, \tau^*(X_1) = X_0$, and $\tau^*(X_i) = X_i$ otherwise. We have $H \rtimes \mathbb{Z}/2 \cdot \tau \subset$

$\mathrm{Aut}(X_\lambda)$ and claim that $X_\lambda/(H \rtimes \mathbb{Z}/2 \cdot \tau)$ is rational. Indeed, for the open set $U_\lambda = \{X_n \neq 0\} \subset X_\lambda$ we compute

$$U_\lambda/(H \rtimes \mathbb{Z}/2\tau) \cong \mathrm{Spec}(k[\sigma_1, x_2, \ldots, x_{n-1}, v]/I) \cong \mathrm{Spec}(k[x_2, \ldots, x_{n-1}, v]),$$

with $I = (\sigma_1 + x_2^{n+1} + \cdots + x_{n-1}^{n+1} + \lambda \cdot v \cdot x_2 \ldots x_{n-1})$. Here, the coordinates are defined to be $x_i := X_i/X_n$, $v = x_0 \cdot x_1$, and $\sigma_1 = x_0^{n+1} + x_1^{n+1}$. Since rational varieties are rationally chain connected we conclude that

$$\sum_{g \in H \rtimes \mathbb{Z}/2\tau} \Gamma(g) \cdot a = 2(n+1) \cdot \deg(a) \cdot [p] \tag{2.2.1}$$

for every $a \in \mathrm{CH}_0(X_\lambda)$ and some closed point $p \in X_\lambda \cap \{X_0 = X_1 = 0\}$ (p exists since $n > 2$).

Next, if $\zeta \in \mu_{n+1} \cong H$ then $(\zeta, \tau) \in H \rtimes \mathbb{Z}/2 \cdot \tau$ has order 2, and we consider the quotient $q : X_\lambda \to X_\lambda/(\zeta, \tau)$ by the action of (ζ, τ). We claim that $X_\lambda/(\zeta, \tau)$ is rationally chain connected.

The fixpoint set F is

$$F = \{X_0 - \zeta X_1 = 0\} \quad \text{if } n \text{ is odd,}$$
$$F = \{X_0 - \zeta X_1 = 0\} \cup \{[1 : -\zeta^{-1} : 0 : \cdots : 0]\} \quad \text{if } n \text{ is even.}$$

Let $H = \{X_0 - \zeta X_1 = 0\} \subset F$ be the hyperplane section. One verifies that H is smooth if and only if X_λ is smooth, and for every point $x \in H$ there are coordinates $y, x_1, \ldots, x_{n-2}$ such that y is a local equation for H with $(\zeta, \tau)^* y = -y$ and the x_i are invariant. Thus $y^2, x_1, \ldots, x_{n-2}$ are local coordinates for the quotient which is therefore smooth in the points $q(H)$. So that $X_\lambda/(\zeta, \tau)$ is smooth if n is odd, and $X_\lambda/(\zeta, \tau)$ has an isolated quotient singularity in $q([1 : -\zeta^{-1} : 0 : \cdots : 0])$ if n is even.

In both cases, $2K_{X_\lambda/(\zeta,\tau)}$ is Cartier and $2K_{X_\lambda/(\zeta,\tau)} \cong \mathcal{O}(-q(H))$ (the isomorphism comes from an invariant form in $H^0(X_\lambda, \omega_{X_\lambda}^{\otimes 2}) = H^0(X_\lambda, \omega_{X_\lambda}^{\otimes 2})^{(\zeta,\tau)}$). We have $q^*(\mathcal{O}(q(H))) = \mathcal{O}(2H)$ and therefore $\mathcal{O}(q(H))$ is ample. If n is odd then the Theorem of Campana, Kollár, Miyaoka, Mori ([C],[KMM]) implies that $X_\lambda/(\zeta, \tau)$ is rationally chain connected. If n is even, then $X_\lambda/(\zeta, \tau)$ is a $\mathbb{Q}$-Fano variety with log terminal singularities and we may use the Theorem of Zhang [Z] to prove the claim.

We conclude that

$$a + \Gamma((\zeta, \tau))(a) = 2\deg(a)[p] \tag{2.2.2}$$

for every $a \in \mathrm{CH}_0(X_\lambda)$ and $p \in X_\lambda \cap \{X_0 = X_1 = 0\}$. Using 2.2.1 and 2.2.2 we see

$$\begin{aligned} \sum_{g \in H} \Gamma(g)(a) &= \sum_{g \in H \rtimes \mathbb{Z}/2\tau} \Gamma(g)(a) - \sum_{\zeta \in \mu_{n+1}} \Gamma((\zeta, \tau))(a) \\ &= 2(n+1)\deg(a)[p] - \sum_{\zeta \in \mu_{n+1}} (2\deg(a)[p] - a) = (n+1)a. \end{aligned} \tag{2.2.3}$$

Of course, for the subgroups $\mu_{n+1} \cong H_i \subset G$ defined by $\zeta \mapsto (1, \ldots, 1, \zeta, \zeta^{-1}, 1, \ldots, 1)$ where ζ is put in the i-th position, the same conclusion 2.2.3 holds. Now, the equality

$$\sum_{g \in G} \Gamma(g) = \left(\sum_{g \in H_0} \Gamma(g) \right) \circ \cdots \circ \left(\sum_{g \in H_{n-2}} \Gamma(g) \right)$$

proves the claim.

2. case: $\mathrm{char}(k) = 0$. It is a well-known fact that if $k_0 \subset k$ is a subfield and $X = X_0 \times_{k_0} k$ then the pullback map

$$\mathrm{CH}_0(X_0) \to \mathrm{CH}_0(X) \tag{2.2.4}$$

is injective (without the assumption on $\mathrm{char}(k)$). The variety X_λ is defined over $\mathbb{Q}(\lambda) \subset k$, and every zero cycle can be defined over a subfield $k_0 \subset k$ which is finitely generated over $\mathbb{Q}(\lambda)$. By fixing an embedding $\sigma : k_0 \to \mathbb{C}$, we reduce to the case $k = \mathbb{C}$.

3. case: $\mathrm{char}(k) = p \neq 0$. Again, since 2.2.4 is injective, we may assume that k is algebraically closed. Let W be the Witt vectors of k; W is a complete discrete valuation ring with residue field k and quotient field K with $\mathrm{char}(K) = 0$. Choose a lift $\tilde{\lambda} \in W$ of λ, and let $X_{\lambda,W} \subset \mathbb{P}^n_W$ be the variety $\sum_{i=0}^n X_i^{n+1} + \tilde{\lambda} X_0 \cdots X_n = 0$. The specialization map

$$sp : \mathrm{CH}_0(X_{\lambda,W} \otimes_W K) \to \mathrm{CH}_0(X_\lambda)$$

from [F, §20.3] is surjective, because W is complete (and therefore $X_{\lambda,W}(W) \to X_\lambda(k)$ is surjective). Since $\mathrm{char}(k) \nmid n+1$ we have

$$\mu_{n+1}(k) \xleftarrow{\cong} \mu_{n+1}(W) \xrightarrow{\cong} \mu_{n+1}(K),$$

and the same statement holds for G. Now the compatibility of sp with pushforwards [F, Proposition 20.3] proves the claim. □

Theorem 2.3. *Let k be a perfect field. We assume $\mathrm{char}(k) \nmid n+1$ if $\mathrm{char}(k) > 0$. Let X_λ be a smooth member of the Dwork family for $n \geq 2$. If $\pi : X_\lambda \to X_\lambda/G$ is the quotient of the G-action (see 2.1) and $Y_\lambda \to X_\lambda/G$ is a resolution of singularities, then*

$$X_\lambda \cong (X_\lambda, \pi) \oplus N'_\lambda \otimes \mathbb{Q}(-1), \quad Y_\lambda \cong (X_\lambda, \pi) \oplus N_\lambda \otimes \mathbb{Q}(-1)$$

for some motives N'_λ and N_λ.

Proof. By construction of (X_λ, π) in 1.3 we have $X_\lambda \cong (X_\lambda, \pi) \oplus M_\lambda$ with some motive M_λ. In view of Lemma 2.2 we know that

$$\mathrm{CH}_0(X_\lambda \times_k L) = \mathrm{CH}_0((X_\lambda \times_k L, \pi \times_k L)) = \mathrm{CH}_0((X_\lambda, \pi) \times_k L)$$

for all field extensions $k \subset L$, and thus $\mathrm{CH}_0(M_\lambda \times_k L) = 0$. Proposition 1.2 implies that $M_\lambda \cong N'_\lambda \otimes \mathbb{Q}(-1)$ for some N'_λ, and Proposition 1.4 proves the claim. □

Corollary 2.4. *Under the assumptions of Theorem 2.3.*

(i) *If $k = \mathbb{C}$ then there is an isomorphism of Hodge structures*

$$\mathrm{Gr}^0_N H^*(X_\lambda, \mathbb{Q}) \cong \mathrm{Gr}^0_N H^*(Y_\lambda, \mathbb{Q}).$$

(ii) *If $k = \mathbb{F}_q$, the finite field with q elements, then for all $m \geq 1$:*

$$\# X_\lambda(\mathbb{F}_{q^m}) = \# Y_\lambda(\mathbb{F}_{q^m}) \quad \textit{modulo } q^m.$$

Proof. (i) By 1.5.2.

(ii) If N is a motive (over $\mathbb{F}_q$) then the eigenvalues of the Frobenius acting on $H^*_{\text{ét}}(N \otimes \mathbb{Q}(-1)) = H^*_{\text{ét}}(N) \otimes \mathbb{Q}_l(-1)$ lie in $q \cdot \bar{\mathbb{Z}}$. Now the claim follows from Grothendieck's trace formula. □

3. Conjectures

3.1. For $k = \mathbb{C}$ the Hodge structure of a variety X can be recovered from the associated motive. For an effective motive N we know $h^{i,0}(N \otimes \mathbb{Q}(-1)) = 0$ for all i, so that if

(3.1.1) $$X \cong X' \oplus N' \otimes \mathbb{Q}(-1), \quad Y \cong X' \oplus N \otimes \mathbb{Q}(-1),$$

then $h^{i,0}(X) = h^{i,0}(Y)$.

3.2. Now consider the setting

$$\begin{array}{ccc} X & & \\ \big\downarrow{\scriptstyle \pi} & & \\ X/G & \longleftarrow & Y \end{array}$$

where X is a smooth projective variety with an action of a finite group G, and Y is a resolution of singularities of the quotient X/G. Since X/G has only quotient singularities we know that $H^i(Y, \mathcal{O}_Y) = H^i(X/G, \mathcal{O}_{X/G})$ for all i. The map $\mathcal{O}_{X/G} \to \pi_* \mathcal{O}_X$ is split by $\frac{1}{|G|}\sum_{g \in G} g^*$ and we obtain

$$H^i(X, \mathcal{O}_X)^G = H^i(X/G, \mathcal{O}_{X/G}) = H^i(Y, \mathcal{O}_Y), \quad \text{for all } i.$$

Therefore 3.1.1 can only be expected if the following holds:

(3.2.1) $$H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{O}_X)^G.$$

3.3. On the other hand, the Bloch conjectures on a filtration of the Chow group of zero cycles which is controlled by the Hodge structure (see [V, §23.2] for a precise statement) predict

$$\pi : \mathrm{CH}_0(X) \xrightarrow{\cong} \mathrm{CH}_0(X/G) = \mathrm{CH}_0(X, \pi)$$

whenever 3.2.1 holds, and thus $X \cong (X, \pi) \oplus N' \otimes \mathbb{Q}(-1)$ (in the notation of 1.3). Now, Proposition 1.4 yields 3.1.1 with $X' = (X, \pi)$. So that the Bloch conjectures imply the following conjecture.

Conjecture 3.4. *Let X be a smooth projective variety over a field k of* $\mathrm{char}(k) = 0$, *and let G be a finite group acting on X with $H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{O}_X)^G$ for all i. If Y is a resolution of singularities of the quotient $\pi : X \to X/G$, then there are (effective) motives N, N' such that*

$$X \cong (X, \pi) \oplus N' \otimes \mathbb{Q}(-1), \quad Y \cong (X, \pi) \oplus N \otimes \mathbb{Q}(-1).$$

Unfortunately little is known concerning the Bloch conjecture.

3.5. Let us consider monomial deformations of the degree d Fermat hypersurface in $\mathbb{P}^n$, and $G \subset \mu_d^{n+1}$. The condition 3.2.1 holds only for $d \le n+1$ (or $G = \{1\}$). Therefore there is no generalisation of Theorem 2.3 to degree $d > n+1$. In the case $d < n+1$, X_λ and Y_λ are rationally connected, and thus $\mathrm{CH}_0(X_\lambda) = \mathbb{Q} = \mathrm{CH}_0(Y_\lambda)$. We obtain

$$X_\lambda \cong \mathbb{Q} \oplus N' \otimes \mathbb{Q}(-1), \quad Y_\lambda \cong \mathbb{Q} \oplus N \otimes \mathbb{Q}(-1).$$

from Proposition 1.2.

3.6. For a finite field $k = \mathbb{F}_q$ we don't know the correct assumptions for Conjecture 3.4. However, the assertion implies a congruence formula for the number of rational points:

$$\#X(\mathbb{F}_q) \equiv \#Y(\mathbb{F}_q) \mod q. \tag{3.6.1}$$

The work of Fu and Wan provides a congruence formula for the number of rational points of X and X/G.

Theorem 3.7 ([FW]). *Let X be a smooth projective variety over the finite field $\mathbb{F}_q$. Suppose X has a smooth projective lifting $\mathcal{X}$ over the Witt ring $W = W(\mathbb{F}_q)$ such that the W-modules $H^r(\mathcal{X}, \Omega^s_{\mathcal{X}/W})$ are free. Let G be a finite group of W-automorphisms acting on X. Suppose G acts trivially on $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for all i. Then for any natural number k, we have the congruence*

$$\#X(\mathbb{F}_{q^k}) \equiv \#(X/G)(\mathbb{F}_{q^k}) \pmod{q^k}.$$

By extending the theory of Witt vector cohomology to singular varieties, Berthelot, Bloch and Esnault were able to prove the following theorem.

Theorem 3.8 ([BBE, Corollary 6.12]). *Let X be a proper scheme over $\mathbb{F}_q$, and G a finite group acting on X so that each orbit is contained in an affine open subset of X. If $|G|$ is prime to p, and if the action of G on $H^i(X, \mathcal{O}_X)$ is trivial for all i, then*

$$\#X(\mathbb{F}_q) \equiv \#(X/G)(\mathbb{F}_q) \mod q.$$

In the next section we prove that

$$\#Y(\mathbb{F}_q) \equiv \#(X/G)(\mathbb{F}_q) \mod q$$

if X is a smooth projective variety and Y is a resolution of singularities of X/G. So that with the assumptions of 3.7 or 3.8 we obtain the congruence formula 3.6.1.

4. Congruence formula

4.1. In this section we fix a finite field $k = \mathbb{F}_q$ of characteristic p, and an algebraic closure $\bar{k}$ of k. For a separated scheme X of finite type over k we work with the étale cohomology groups $H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$ (resp. étale cohomology groups with support $H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ for $Z \subset X$) with ℓ a prime number $\neq p$, and $X_{\bar{k}} = X \times_k \bar{k}$. They are finite dimensional $\mathbb{Q}_\ell$-vector spaces, and the Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ acts continuously on them.

We denote by $F \in G_k$ the geometric Frobenius, F acts on $H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$ with eigenvalues that are algebraic integers. If X is proper then we have Grothendieck's trace formula

$$\#X(\mathbb{F}_q) = \sum_i (-1)^i \mathrm{Tr}(F, H^i(X_{\bar{k}}, \mathbb{Q}_\ell)).$$

4.2. Let V be a finite dimensional $\mathbb{Q}_\ell$-vector space, and $F : V \to V$ a linear map. Fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. The vector space $\bar{V} = V \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ decomposes into the generalised eigenspaces of F:

$$\bar{V} = \bigoplus_{\lambda \in \bar{\mathbb{Q}}_\ell} \bar{V}_\lambda,$$

i.e. $\bar{V}_\lambda$ is the maximal subspace such that F acts with eigenvalue λ. For every $g \in G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ we get $g(\bar{V}_\lambda) = \bar{V}_{g(\lambda)}$, and we obtain a decomposition

$$V = \bigoplus_{\lambda \in G_{\mathbb{Q}_\ell} \backslash \bar{\mathbb{Q}}_\ell} \left(\bigoplus_{\lambda' \in G_{\mathbb{Q}_\ell} \cdot \lambda} \bar{V}_\lambda \right)^{G_{\mathbb{Q}_\ell}},$$

where λ runs through all orbits of $G_{\mathbb{Q}_\ell}$ in $\bar{\mathbb{Q}}_\ell$, and λ' through all conjugates of λ. We write

$$V_\lambda = \left(\bigoplus_{\lambda' \in G_{\mathbb{Q}_\ell} \cdot \lambda} \bar{V}_\lambda \right)^{G_{\mathbb{Q}_\ell}}, \quad V = \bigoplus_{\lambda \in G_{\mathbb{Q}_\ell} \backslash \bar{\mathbb{Q}}_\ell} V_\lambda.$$

Let W be another finite dimensional $\mathbb{Q}_\ell$-vector space with a linear operation $F : W \to W$. If $\phi : V \to W$ is a linear map which commutes with the action of F then

$$\phi(V_\lambda) \subset W_\lambda$$

for every $\lambda \in G_{\mathbb{Q}_\ell} \backslash \bar{\mathbb{Q}}_\ell$.

Now, we fix an integer $q \in \mathbb{Z}$, and we assume that all eigenvalues of F are algebraic integers, i.e. $V_\lambda = 0$ if $\lambda \notin G_{\mathbb{Q}_\ell} \backslash \bar{\mathbb{Z}}$ where $\bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}_\ell$ is the integral closure of $\mathbb{Z}$ in $\bar{\mathbb{Q}}_\ell$. We note that the subset $q\bar{\mathbb{Z}} \subset \bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}_\ell$ has an induced action by $G_{\mathbb{Q}_\ell}$, and we define the slope < 1 resp. slope ≥ 1 part of V to be

$$V^{<1} := \bigoplus_{\lambda \notin G_{\mathbb{Q}_\ell} \backslash q\bar{\mathbb{Z}}} V_\lambda, \quad V^{\geq 1} := \bigoplus_{\lambda \in G_{\mathbb{Q}_\ell} \backslash q\bar{\mathbb{Z}}} V_\lambda.$$

We obtain a decomposition

$$V = V^{<1} \oplus V^{\geq 1}$$

with F action on $V^{<1}$ and $V^{\geq 1}$, and the decomposition is functorial for linear maps that commute with the F-operation.

For étale cohomology and $q = |k|$ we thus get for all i a functorial decomposition

$$H^i(X_{\bar{k}}, \mathbb{Q}_\ell) = H^i(X_{\bar{k}}, \mathbb{Q}_\ell)^{<1} \oplus H^i(X_{\bar{k}}, \mathbb{Q}_\ell)^{\geq 1},$$

$$\text{resp.} \quad H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell) = H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell)^{<1} \oplus H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell)^{\geq 1}.$$

Lemma 4.3. *If X is smooth and $Z \subset X$ is a closed subset of codimension ≥ 1 then*

$$H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell)^{<1} = 0 \quad \textit{for all } i.$$

Proof. [E1, Lemma 2.1],[E2, §2.1]. □

In other words all eigenvalues of the Frobenius on $H^i_{Z_{\bar{k}}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ lie in $q \cdot \bar{\mathbb{Z}}$. It is not difficult to extend this lemma to the case when X has quotient singularities.

Lemma 4.4. *Let X be a smooth and quasi-projective variety, and let G be a finite group acting on X. If $\pi : X \to X/G$ is the quotient and $Z \subset X/G$ is a closed subset of codimension ≥ 1 then*

$$H^i_{Z_{\bar{k}}}((X/G)_{\bar{k}}, \mathbb{Q}_\ell)^{<1} = 0 \quad \textit{for all } i.$$

Proof. We write $Y = \pi^{-1}(Z)$ which is a closed subset of X of codimension ≥ 1. Note that $(X/G)_{\bar{k}} = X_{\bar{k}}/G$, i.e. $\pi_{\bar{k}} : X_{\bar{k}} \to (X/G)_{\bar{k}}$ is the quotient for the G action on $X_{\bar{k}}$.

The composite of $\mathbb{Q}_\ell \subset \pi_{\bar{k}*}\mathbb{Q}_\ell$ with

$$\sum_{g\in G} g^* : \pi_{\bar{k}*}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$$

is multiplication by $|G|$. Since

$$H^i_{Z_{\bar{k}}}((X/G)_{\bar{k}}, \pi_{\bar{k}*}\mathbb{Q}_\ell) = H^i_Y(X_{\bar{k}}, \mathbb{Q}_\ell)$$

we get

$$H^i_{Z_{\bar{k}}}((X/G)_{\bar{k}}, \mathbb{Q}_\ell) \cong H^i_Y(X_{\bar{k}}, \mathbb{Q}_\ell)^G,$$

and this map is compatible with the Frobenius action. Now, Lemma 4.3 implies the statement. □

Theorem 4.5. *Let X be a smooth projective $\mathbb{F}_q$-variety with an action of a finite group G. Let $\pi : X \to X/G$ be the quotient, and $f : Y \to X/G$ be a birational map, where Y is a smooth projective variety. Then*

$$\#Y(\mathbb{F}_q) = \#(X/G)(\mathbb{F}_q) \qquad \text{mod } q.$$

Proof. Let U be an open (dense) subset of X/G such that $f^{-1}(U) \xrightarrow{\cong} U$ is an isomorphism. Write $Z = (X/G)\backslash U$ and $Z' = Y\backslash f^{-1}(U)$. We consider the map of long exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_{Z'_{\bar{k}}}(Y_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^i(Y_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^i(f^{-1}(U)_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow \\
& \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & H^i_{Z_{\bar{k}}}((X/G)_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^i((X/G)_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow & H^i(U_{\bar{k}}, \mathbb{Q}_\ell) & \longrightarrow
\end{array}$$

Here all maps commute with the action of the Frobenius. By using Lemma 4.4 we get

$$\begin{array}{ccc}
H^i(Y_{\bar{k}}, \mathbb{Q}_\ell)^{<1} & \xrightarrow{\cong} & H^i(f^{-1}(U)_{\bar{k}}, \mathbb{Q}_\ell)^{<1} \\
\uparrow & & \uparrow{=} \\
H^i((X/G)_{\bar{k}}, \mathbb{Q}_\ell)^{<1} & \xrightarrow{\cong} & H^i(U_{\bar{k}}, \mathbb{Q}_\ell)^{<1}.
\end{array}$$

This implies

$$H^i(Y_{\bar{k}}, \mathbb{Q}_\ell)^{<1} \cong H^i((X/G)_{\bar{k}}, \mathbb{Q}_\ell)^{<1} \quad \text{for all } i.$$

With Grothendieck's trace formula we obtain

$$\#Y(\mathbb{F}_q) - \#X/G(\mathbb{F}_q) = \sum_i (-1)^i \left(\mathrm{Tr}(F, H^i(Y_{\bar{k}}, \mathbb{Q}_\ell))^{\geq 1} - \mathrm{Tr}(F, H^i((X/G)_{\bar{k}}, \mathbb{Q}_\ell))^{\geq 1}\right).$$

The right-hand side is a number in $\mathbb{Z} \cap q\bar{\mathbb{Z}} = q\mathbb{Z}$, which proves the congruence. □

Remark 4.6. It seems that the fibre $f^{-1}(x)$ of a point $x \in (X/G)(\mathbb{F}_q)$ satisfies the congruence

$$\# f^{-1}(x)(\mathbb{F}_q) = 1 \qquad \text{mod } q.$$

Of course this would imply the statement of Theorem 4.5.

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