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A LOCAL CRITERION FOR THE SAITO-KUROKAWA LIFTING OF CUSPFORMS WITH CHARACTERS

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Abstract. Let f be a holomorphic degree-2 Siegel cuspform of weight κ, level N, and nebentype a primitive Dirichlet character χ. Let f be an eigenfunction of the regular Hecke operators T_p, T_{p^2} at primes $p \nmid N$ and an eigenfunction of the Frobenius operators Π_p and their duals Π_p^* at primes $p|N$. For certain χ, we give conditions on the Satake parameters of f which imply that f is lifted from an elliptic cuspform ϕ of weight $2\kappa-2$, level N, and nebentype χ^2. We also show that for such f and ϕ, the eigenvalues of the Frobenius operators on f are eigenvalues of Hecke operators on ϕ.

1. Introduction

The Saito-Kurokawa lift is a Hecke equivariant map from elliptic cuspforms to degree-2 Siegel cuspforms so that the spin L-function of a Siegel cuspform in the image of the lift decomposes into more elementary L-functions in a precise way. Classically, cuspforms that are in the image of the lift are characterized by conditions on their Fourier coefficients. In particular, let f be a Siegel cuspform of weight κ and level 1. For a totally positive definite matrix $H = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ with $a, b, c \in \mathbb{Z}$, write the H^{th} Fourier coefficient of f by $A_f(H) = A_f(a, b, c)$. Then f is in the image of the Saito-Kurokawa lifting (the Maass space) if and only if the Fourier coefficients of f satisfy the Maass relations [9],

$$A_f(a,b,c) = \sum_{d|\gcd(a,b,c)} d^{\kappa-1} A_f\left(\frac{ac}{d^2}, \frac{b}{d}, 1\right).$$

The existence of the Saito-Kurokawa lift was demonstrated in a series of papers [1], [13], [24] and an exposition of the proof is in [9] and [20]. The lift has been generalized to cuspforms of higher level [14], and to Siegel cuspforms of degree $2n$ [11].

For $N \in \mathbb{Z}_{>0}$ we denote the space of holomorphic degree-2 Siegel cuspforms of weight κ, level N, and character χ by $\mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$. In this paper we consider eigenfunctions of the regular Hecke operators T_p, T_{p^2} for $p \nmid N$ and eigenfunctions of the Frobenius operators Π_p, Π_p^* for $p|N$, as defined by Andrianov in [3]. We define a Saito-Kurokawa lift, that is a lift from holomorphic elliptic cuspforms of weight $2\kappa-2$, level N, and nebentype χ^2 into $\mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$, in Section 2. For $p \nmid N$ we define the Satake parameters $\{\alpha_{0p}, \alpha_{1p}, \alpha_{2p}\}$ of f in Section 2, see [19]. To a Dirichlet character χ we define a character $\widetilde{\chi}$ of $\mathbb{Z}[i]$ by $\widetilde{\chi}(a) = \chi(a\overline{a})$ as in [2].

Theorem 1. *Let $N \in \mathbb{Z}_{>0}$ be odd and squarefree. Let χ be a primitive Dirichlet character of conductor N so that the character $\widetilde{\chi}$ is primitive. Let $\kappa > 2$ and let $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ be a Hecke eigenfunction of T_p and T_{p^2} at primes $p \nmid N$ and an*

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eigenfunction of the Frobenius operators Π_p *and their duals* Π_p^* *at* $p|N$. *If* $A_f(1_2) \neq 0$ *and the Satake parameters of* f *satisfy the condition*

$$\alpha_{1p}\alpha_{2p} = p^{\pm 1} \quad or \quad \alpha_{1p}\alpha_{2p}^{-1} = p^{\pm 1} \quad for \quad p \nmid N,$$

then f *is in the image of the Saito-Kurokawa lift.*

The condition $A_f(1_2) \neq 0$ is a technical condition that allows the results from [2] and [18] to be more easily applied. Note that from Remark 4 in [2] there are examples of primitive characters χ so that $\widetilde{\chi}$ is not a primitive character of $\mathbb{Z}[i]$. Also, note that a local characterization for the Saito-Kurokawa lift was obtained in [23], using [16] and a converse theorem from [22]. Recently another local characterization of the lifting was obtained by Pitale and Schmidt in [17].

Theorem 2. *Let* f *be as in Theorem 1 and in the image of the Saito-Kurokawa lift of an elliptic cuspform* ϕ *of weight* $2\kappa - 2$, *level* N, *and nebentype* χ^2. *For* $p|N$, *let* $\rho_f(p)$ *be the eigenvalue of the Frobenius operator* Π_p *on* f *and let* $\lambda_\phi(p)$ *be the eigenvalue of the* p^{th} *Hecke operator on* ϕ. *Then* $\rho_f(p) = \lambda_\phi(p)$.

In Section 2 we define the cuspforms and the lifting that we study. In Section 3 we prove Theorem 2. We introduce spinor zeta functions twisted by Dirichlet characters as in [18], and prove a result on Gauss sums in Section 4. In the last section we prove Theorem 1, using the results from Section 4 which allow us to apply a converse theorem due to Booker, [5].

2. Automorphic Forms and the Saito-Kurokawa Lifting

Let

$$\mathfrak{H}_2 = \{z \in M_2(\mathbb{C}) \mid z^T = z, \ -i(z - z^*) > 0\}$$

denote the Siegel upper-half space, where $M_2(\mathbb{C})$ denotes the set of 2×2 matrices with entries in $\mathbb{C}$, z^T is matrix transpose, and $z^* = \overline{z}^T$. For $N \in \mathbb{Z}_{>0}$ consider the congruence subgroup

$$\Gamma_0^2(N) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_2(\mathbb{Z}) \ \Big| \ c \equiv 0_2 \pmod{N} \Big\}.$$

Let $\Gamma_0(N)$ denote the analogous congruence subgroup in $SL_2(\mathbb{Z})$, as in [12].

Let $\mu(g, z) = \det(cz + d)^{-1}$ and let χ be a Dirichlet character modulo N. A holomorphic Siegel modular form of weight κ, level N, and character χ is a holomorphic $\mathbb{C}$-valued function f on $\mathfrak{H}_2$ so that

$$f(g(z))\mu(g, z)^\kappa \chi(\det(d))^{-1} = f(z)$$

for all $z \in \mathfrak{H}_2$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^2(N)$, and where $g(z) = (az + b)(cz + d)^{-1}$. Let $\mathcal{M}_\kappa^2(\Gamma_0^2(N), \chi)$ denote the space of such Siegel modular forms and let $\mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ denote the subspace of cuspforms. Similarly, let $\mathcal{S}_\kappa^1(\Gamma_0(N), \chi^2)$ be the space of holomorphic elliptic cuspforms of weight κ, level N, and character χ^2.

The Fourier expansion of $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ is of the form

$$f(z) = \sum_{H \in \mathcal{A}_2^+} A_f(H) e^{\pi i \sigma(Hz)}$$

where

$$\mathcal{A}_2^+ = \{H = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}) \mid a, b, c \in \mathbb{Z},\ H > 0\}$$

and σ is the trace. For $p \nmid N$, we define the Hecke operators T_p, T_{p^2} and for $p|N$ the Frobenius operators Π_p, Π_p^* for the space of such modular forms as in [3]. Note that the Frobenius operator Π_m commutes with the Hecke operators and has a natural action on f by

$$\Pi_m(f)(z) = \sum_{H \in \mathcal{A}_2^+} A_f(mH) e^{\pi i \sigma(Hz)},$$

see [3]. The following is a consequence of Theorem 22 from [3].

Theorem 3 ([3], Theorem 22)**.** *Let N be a product of distinct odd primes and χ a Dirichlet character modulo N so that χ^2 is primitive. Then there exists an orthonormal basis of $\mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ consisting of eigenfunctions of the Hecke operators T_p, T_{p^2} for $p \nmid N$ and Frobenius operators Π_p, Π_p^* for $p|N$. The eigenvalues of Π_p and Π_p^* have absolute value equal to $p^{\kappa-3/2}$.*

In particular, for p an odd prime and χ primitive there is an orthonormal basis for $\mathcal{S}_\kappa^2(\Gamma_0^2(p), \chi)$ consisting of such eigenfunctions.

Let

$$J = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$$

and let $G = GSp_2 = \{g \in GL_4 \mid g^T J g = \nu(g) J\}$ where ν is a multiplicative (similitude) character. Let $\mathbb{A}_\mathbb{Q}$ denote the adeles of $\mathbb{Q}$ and let $G_\mathbb{A}$ be the group G with entries in $\mathbb{A}_\mathbb{Q}$. We can associate to $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ a smooth cuspform on $G_\mathbb{A}$ in the following way. Let $\mathbb{A}_0$ denote the finite adeles of $\mathbb{Q}$. Let $\mathbb{Q}_p$ be the completion of the valuation $|\cdot|_p$ of $\mathbb{Q}$ and let G_p be the local points of $G_\mathbb{A}$. Let $\Gamma_0^2(N)_\mathbb{A}$ denote the congruence subgroup of $G_\mathbb{A}$.

For $p < \infty$ set $K_p = G_{\mathbb{Z}_p}$ and let $K_{\mathbb{A}_0} = \prod_{p<\infty} K_p$. For $p|\infty$ define

$$K_p = \{g \in G_p \mid g(i1_2) = i1_2\} \cong U(2)$$

where $U(2) = \{g \in GL_4(\mathbb{C}) \mid g^* g = 1_4\}$. Then K_p is a maximal compact subgroup of G_p. If $g \in K_\infty = \prod_{p|\infty} K_p$ then let $\rho_\kappa(g) = \det(a+ib)^\kappa$. Let $K_\mathbb{A} = K_\infty K_{\mathbb{A}_0}$. For $g \in G_\mathbb{A}$ and $z \in \mathfrak{H}_2$ define

$$\mu(g, z) = (\mu(g_p, z_p))_{p \in \mathbb{A}_\infty}.$$

For f a function on $\mathfrak{H}_2$ and $g \in G_\mathbb{A}$, define a function on $\mathfrak{H}_2$ by $(f|_\kappa g)(z) = \chi(\det(d))^{-1} \mu(g, z)^\kappa f(g(z))$. A holomorphic automorphic form is therefore a function $f : \mathfrak{H}_2 \to \mathbb{C}^\times$ so that f is holomorphic and $f|_\kappa g = f$ for all $g \in \Gamma_0^2(N)_\mathbb{A}$. Then $\chi(\det(d))^{-1} \mu(1, g(i1_n))^\kappa f(g(i1_n))$ is a function on $G_\mathbb{A}$ which we also label f. By strong approximation on $G_\mathbb{A}$, f is a left $G_\mathbb{Q}$-invariant, right (K_∞, ρ_κ)-equivariant, right $\Gamma_0^2(N)_{\mathbb{A}_0}$-invariant function on $G_\mathbb{A}$, and so f is a holomorphic modular form on $G_\mathbb{A}$ of weight κ, level N and character χ.

We say that f on $G_\mathbb{A}$ as defined above is an eigenfunction of the Hecke operators T_p, T_{p^2} or of the Frobenius operators Π_p, Π_p^* if the corresponding function f on $\mathfrak{H}_2$ is an eigenfunction of T_p, T_{p^2} or Π_p, Π_p^* respectively, in the sense of [3].

For a smooth cuspform f on $G_{\mathbb{A}}$ that is a Hecke eigenfunction at almost all primes, let $\pi_{\mathbb{A}}$ be the automorphic representation of $G_{\mathbb{A}}$ generated by f under the right regular representation. We have the factorization $\pi_{\mathbb{A}} \cong \otimes_p' \pi_p$ [10], which is a completed restricted product and π_p is an irreducible unitarizable automorphic representation of G_p.

Consider the Borel subgroup of G_p,

$$B_p = \left\{ \begin{pmatrix} \lambda a_1 & * & * & * \\ 0 & \lambda a_2 & * & * \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & * & a_2^{-1} \end{pmatrix} \in G_p \;\middle|\; \lambda, a_1, a_2 \in \mathbb{Q}_p^\times \right\}.$$

By [7] and [8] an irreducible admissible spherical representation embeds into an unramified principal series. By dualization, an admissible spherical representation is also an image of an unramified principal series. Thus for $p < \infty$ so that π_p is a spherical representation of G_p, we have a surjection of G_p-representation spaces

$$I_p(\chi_p) = \text{c-Ind}_{B_p}^{G_p} \chi_p \delta_B^{1/2} \to \pi_p$$

where χ_p is an unramified character on B_p [8], δ_B is the modular function of B_p, and c-Ind is compactly-supported smooth induction. Thus π_p is isomorphic to a quotient of $I_p(\chi_p)$.

Let χ_1, χ_2, σ be unramified characters of $\mathbb{Q}_p^\times$ so that for $b \in B_p$ we have $\chi_p(b) = \chi_1(a_1)\chi_2(a_2)\sigma(\lambda)$. For $p \nmid N$ let $\alpha_{0p} = p^{\kappa-3/2}\sigma(p)$, $\alpha_{1p} = \chi_1(p)$, and $\alpha_{2p} = \chi_2(p)$. We consider the 8 element set $\{\alpha_{0p}^{\pm 1}, \alpha_{1p}^{\pm 1}, \alpha_{2p}^{\pm 1}\}$ to be the set of Satake parameters of f, see [19], where this is the orbit of $\{\alpha_{0p}, \alpha_{1p}, \alpha_{2p}\}$ under the action of the Weyl group.

Let $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ be an eigenfunction of the Hecke operators T_p, T_{p^2} for $p \nmid N$ so that

$$T_p f(z) = \lambda_f(p) f(z) \quad \text{and} \quad T_{p^2} f(z) = \lambda_f(p^2) f(z).$$

The (Langlands) spinor zeta function of f on $G_{\mathbb{A}}$ is

$$Z_L(s,f) = \prod_p Z_p(s,f)$$

where the product is over the finite primes and for $p \nmid N$ we have $Z_p(s,f) = Q_{p,f}(p^{-s})^{-1}$ where

$$\begin{aligned} Q_{p,f}(x) = 1 - \lambda_f(p)x + \{p\lambda_f(p^2) + \chi(p^2)p^{2\kappa-5}(p^2+1)\}x^2 \\ - \chi(p^2)p^{2\kappa-3}\lambda_f(p)x^3 + \chi(p^4)p^{4\kappa-6}x^4. \end{aligned}$$

In the sequel, products defining L-functions are understood to be over the finite places.

Note that for $p \nmid N$ we also have

$$Z_p(s,f) = \left[(1-\alpha_{0p}p^{-s})(1-\alpha_{0p}\alpha_{1p}p^{-s})(1-\alpha_{0p}\alpha_{2p}p^{-s})(1-\alpha_{0p}\alpha_{1p}\alpha_{2p}p^{-s})\right]^{-1}.$$

The standard degree-5 L-function attached to f is

$$L_{\text{St}}(s,f) = \prod_p L_p(s,f)$$

where for $p \nmid N$,

$$L_p(s,f) = \left[(1-p^{-s})(1-\alpha_{1p}p^{-s})(1-\alpha_{1p}^{-1}p^{-s})(1-\alpha_{2p}p^{-s})(1-\alpha_{2p}^{-1}p^{-s})\right]^{-1}.$$

It was shown in [4] (for level 1) and [21] that $L_{\mathrm{St}}(s,f)$ is holomorphic except possibly for a finite number of simple poles. The standard degree-2 L-function attached to $\phi \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$ is

$$L(s,\phi) = \prod_{p \text{ prime}} L_p(s,\phi) = \prod_{p\nmid N}(1-\lambda_\phi(p)p^{-s}+\chi(p)p^{2\kappa-3-2s})^{-1}\prod_{p|N}(1-\lambda_\phi(p)p^{-s})^{-1},$$

where $\lambda_\phi(p)$ is the eigenvalue of the p^{th} Hecke operator on ϕ, as in [12].

Let f be an eigenfunction of the Hecke operators at $p \nmid N$. We say that f is in the image of the Saito-Kurokawa lifting of the elliptic cuspform ϕ if and only if the incomplete spinor zeta function attached to f decomposes

$$Z^S(s,f) = \prod_{p\nmid N} Z_p(s,f) = L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L^S(s,\phi) \tag{1}$$

where $S = \{p \text{ prime} \mid p|N\}$ and $L^S(s,\phi) = \prod_{p\nmid N} L_p(s,\phi)$ is the incomplete L-function of ϕ. Let $\mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)^*$ denote the subspace of $\mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)$ generated by such cuspforms in the image of the Saito-Kurokawa lift.

3. Eigenvalues of the Frobenius Operators and the Saito-Kurokawa Lifting

Let $\phi \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$ and let $f \in \mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)$. For $w_{N,1} = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ define $\phi^*(z) = \phi\big|_{w_{N,1}}(z) = N^{\kappa-1}(Nz)^{-2\kappa-2}\phi(w_{N,1}(z))$ (for z in the complex upper-half plane) and for $w_{N,2} = \left(\begin{smallmatrix} 0_2 & -1_2 \\ N\cdot 1_2 & 0_2 \end{smallmatrix}\right)$ define $f^*(z) = f\big|_{w_{N,2}}(z) = N^\kappa(Nz)^{-\kappa}f(w_{N,2}(z))$ (for $z \in \mathfrak{H}_2$). Let $A^*_f(1_2)$ denote the Fourier coefficient of f^* at $H = 1_2$. Then $A_f(1_2) \neq 0$ if and only if $A^*_f(1_2) \neq 0$, from [2].

Lemma 1. *If $\phi \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$ is a Hecke eigenfunction then so is $\phi^* \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\overline{\chi}^2)$.*

If $f \in \mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)$ is a Hecke eigenfunction for $p \nmid N$ and an eigenfunction of the Frobenius operators Π_p, Π^*_p for $p|N$ then so is $f^* \in \mathcal{S}^2_\kappa(\Gamma^2_0(N),\overline{\chi})$.

Proof. The first part is Theorem 4.5.5 on p.137 in [15] and the second is from Theorem 2 in [2]. □

Let f be an eigenfunction of the Hecke operators T_p, T_{p^2} for $p \nmid N$ as in Section 2, and an eigenfunction of the Frobenius operators Π_p, Π^*_p for $p|N$, as in [2] or [3]. Let

$$\Pi_p f(z) = \rho_f(p)f(z) \quad \text{and} \quad \Pi^*_p f(z) = \rho^*_f(p)f(z).$$

Following [2], for $\mathrm{Re}(s) > \kappa$ and f as above, define the (Andrianov) spinor zeta function

$$Z_A(s,f) = \prod_{p\nmid N} Q_{p,f}(p^{-s})^{-1}\prod_{p|N}(1-\rho_f(p)p^{-s})^{-1}.$$

From Theorems 1 and 3 in [2] we have the analytic continuation and a functional equation for $Z_A(s,f)$. Note that the degree-1 factors are from [2], and differ from the factors of the Langlands spinor zeta function at $p|N$.

By Theorem 2 in [2] and Lemma 1, f^* is also an eigenfunction for the Hecke operators and Frobenius operators and we have

$$Z_A(s,f^*)=\prod_{p\nmid N}Q_{p,f}(\overline{\chi}(p^2)p^{-s})^{-1}\prod_{p|N}(1-\rho_{f^*}(p)p^{-s})^{-1}.$$

Lemma 2. *Let $f\in\mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)^*$ be an eigenfunction of the Hecke operators at $p\nmid N$ and in the image of $\phi\in\mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$. Then $f^*\in\mathcal{S}^2_\kappa(\Gamma^2_0(N),\overline{\chi})^*$ and is in the image of $\phi^*\in\mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\overline{\chi}^2)$.*

Proof. As f is in the image of the lift, from (1) we have

$$Z^S(s,f)=L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L^S(s,\phi)$$

where $\phi\in\mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$. From a direct computation, for $p\nmid N$ the decomposition

$$Q_{p,f}(p^{-s})=L_p(s-\kappa+1,\chi)L_p(s-\kappa+2,\chi)L_p(s,\phi)$$

is equivalent to the equations

$$\begin{aligned}\lambda_f(p)&=\lambda_\phi(p)+\chi(p)p^{\kappa-2}(p+1)\\ \lambda_f(p^2)&=\chi(p)^2p^{2\kappa-4}+\chi(p)\lambda_\phi(p)p^{\kappa-3}(p+1)-\chi(p)^2p^{2\kappa-6}.\end{aligned}$$

Let $p_1=\begin{pmatrix}1_2&\\&p\cdot 1_2\end{pmatrix}$ and $p_2=\begin{pmatrix}1&&&\\&p&&\\&&p^2&\\&&&p\end{pmatrix}$. Then $T_pf^*=\lambda_{f^*}(p)f^*$ and $T_{p^2}f^*=\lambda_{f^*}(p^2)f^*$ where

$$\lambda_{f^*}(p)=\overline{\chi}(\nu(p_1)^2)\lambda_f(p)\text{ and }\lambda_{f^*}(p^2)=\overline{\chi}(\nu(p_2)^2)\lambda_f(p^2)$$

from 0.7 and (2) of Theorem 2 in [2]. As ν is the similitude character, then $\nu(p_1)^2=p^2$ and $\nu(p_2)^2=p^4$. From Theorem 6.27 p.113 of [12] we have $\lambda_{\phi^*}(p)=\chi(p)^{-2}\lambda_\phi(p)$ and so

$$\begin{aligned}\lambda_{f^*}(p)&=\overline{\chi}(\nu(p_1)^2)\lambda_f(p)=\overline{\chi}(p)^2(\lambda_\phi(p)+\chi(p)p^{\kappa-2}(p+1))\\ &=\lambda_{\phi^*}(p)+\overline{\chi}(p)p^{\kappa-2}(p+1))\end{aligned}$$

and

$$\begin{aligned}\lambda_{f^*}(p^2)&=\overline{\chi}(\nu(p_2)^2)\lambda_f(p^2)=\overline{\chi}(p)^4\lambda_f(p^2)\\ &=\overline{\chi}(p)^4(\chi(p)^2p^{2\kappa-4}+\chi(p)\lambda_\phi(p)p^{\kappa-3}(p+1)-\chi(p)^2p^{2\kappa-6})\\ &=\overline{\chi}(p)^2p^{2\kappa-4}+\overline{\chi}(p)\lambda_{\phi^*}(p)p^{\kappa-3}(p+1)-\overline{\chi}(p)^2p^{2\kappa-6}.\end{aligned}$$

These equations imply the decomposition

$$\begin{aligned}Z_p(s,f^*)&=Q_{p,f}(\overline{\chi}(p^2)p^{-s})^{-1}=Q_{p,f^*}(p^{-s})^{-1}\\ &=\big(1-\lambda_{f^*}(p)p^{-s}+\{p\lambda_{f^*}(p^2)+\overline{\chi}(p^2)p^{2\kappa-5}(p^2+1)\}p^{-2s}\\ &\qquad\qquad-\overline{\chi}(p^2)\lambda_{f^*}(p)p^{2\kappa-3-3s}+\overline{\chi}(p^4)p^{4\kappa-6-4s}\big)^{-1}\\ &=\big((1-\overline{\chi}(p)p^{\kappa-1-s})(1-\overline{\chi}(p)p^{\kappa-2-s})(1-\lambda_{\phi^*}(p)+\overline{\chi}(p)^2p^{2\kappa-3-2s})\big)^{-1}\\ &=L_p(s-\kappa+1,\overline{\chi})L_p(s-\kappa+2,\overline{\chi})L_p(s,\phi^*).\end{aligned}$$

□

Lemma 3. *Let $f \in \mathcal{S}_k^2(\Gamma_0^2(N), \chi)^*$ be an eigenfunction of the Hecke and Frobenius operators as in Theorem 1 and in the image of the lift of $\phi \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N), \chi^2)$. Then*

$$Z_A(s,f) = L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L(s,\phi)P_{N,1}(s,f,\phi)$$

where

$$P_{N,1}(s,f,\phi) = \prod_{p|N} \frac{1-\lambda_\phi(p)p^{-s}}{1-\rho_f(p)p^{-s}}.$$

Proof. For f and ϕ as in the lemma, we have from (1),

$$\begin{aligned} Z_A(s,f) =& L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)\prod_{p\nmid N} L_p(s,\phi)\prod_{p|N}(1-\rho_f(p)p^{-s})^{-1} \\ =& L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L(s,\phi)P_{N,1}(s,f,\phi), \end{aligned}$$

as $L_p(s,\phi) = (1-\lambda_\phi(p)p^{-s})^{-1}$ for $p|N$. □

Define the Gauss sums

$$\mathfrak{g}(\chi) = \sum_{a \pmod N} \chi(a)e^{2\pi i a/N},$$

$$G(\chi) = \sum_{a_1,a_2 \pmod N} \chi(a_1^2+a_2^2)e^{2\pi i a_1/N}.$$

If χ is primitive then $|\mathfrak{g}(\chi)|^2 = N$ (see [15] for example). Let $K = \mathbb{Q}(i)$ and let $\mathcal{O}_K$ denote the integers of K. Let $\widetilde{\chi}$ be the character on $\mathcal{O}_K$ defined by $\widetilde{\chi}(a_1+ia_2) = \chi(a_1^2+a_2^2)$ (as in Section 1). If $\widetilde{\chi}$ is primitive then $|G(\chi)| = N$, from Lemma 3 on p.17 of [18].

Proposition 1. *Let χ be a primitive Dirichlet character modulo N so that $\widetilde{\chi}$ is a primitive character of $\mathcal{O}_K$. Let $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)^*$ be an eigenfunction of the Hecke operators T_p, T_{p^2} at $p \nmid N$ and Frobenius operators Π_p, Π_p^* at $p|N$. Then*

$$A_f(1_2) = A_f^*(1_2)(-1)^{\kappa+u}\frac{G(\chi)}{\mathfrak{g}(\chi)^2}.$$

Further, if f is in the image of $\phi \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N), \chi^2)$ then $\rho_f(p) = \lambda_\phi(p)$ for $p|N$.

Proof. From the functional equation of $Z_A(s,f)$ in Theorem 3 of [2], we have

$$A_f(1_2)\Psi(s,f) = A_f^*(1_2)\chi(-1)G(\chi)N^{3\kappa-3s-4}\Psi(2\kappa-2-s,f^*)$$

where $\Psi(s,f) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)Z_A(s,f)$. Note that the power of N here is different than that explicitly stated in [2]. This is due to the definition of f^*, where from 0.9 in [2] it is given by $f^*(z) = f\big|_{w_{N,2}}(z) = N^{2\kappa-3}(Nz)^{-\kappa}f(w_{N,2}(z))$. Also note that the coefficient in Theorem 3 of [2] is $A_f(2\cdot 1_2)$. This is the same as $A_f(1_2)$ here, from the definition of the Fourier coefficients of f given in Section 2. Therefore, from Lemma 3 we have

$$\begin{aligned} &A_f(1_2)(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L(s,\phi)P_{N,1}(s,f,\phi) \\ &\quad= A_f^*(1_2)\chi(-1)G(\chi)N^{3\kappa-3s-4}(2\pi)^{-2(2\kappa-2-s)}\Gamma(2\kappa-2-s)\Gamma(\kappa-s) \\ &\qquad \times L(\kappa-1-s,\overline{\chi})L(\kappa-s,\overline{\chi})L(2\kappa-2-s,\phi^*)P_{N,2}(s,f^*,\phi^*) \end{aligned}$$

where

$$P_{N,2}(s,f^*,\phi^*)=\prod_{p|N}\frac{1-\lambda_{\phi^*}(p)p^{s-2\kappa+2}}{1-\overline{\rho}(p)p^{s-2\kappa+2}}.$$

For α a primitive Dirichlet character modulo A and $\alpha(-1)=(-1)^u$ let

$$\Lambda_1(s,\alpha)=(A/\pi)^{s/2}\Gamma(\frac{s+u}{2})L(s,\alpha).$$

As α is primitive, for any $a\in\mathbb{C}$ we can write the functional equation of the Hecke L-function $L(s,\alpha)$ in the form

$$\Lambda_1(s-a,\alpha)=i^{-u}\mathfrak{g}(\alpha)A^{-1/2}\Lambda_1(1-s+a,\overline{\alpha}),\tag{2}$$

see (12.7) p. 204 in [12] for example. Applying (2) to $L(s,\chi)$ with $a=\kappa-1$ and $a=\kappa-2$, the above functional equation for $Z_A(s,f)$ becomes

$$\begin{aligned}&A_f(1_2)(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)L(s,\phi)P_{N,1}(s,f,\phi)\\ &\quad=A_f^*(1_2)\chi(-1)G(\chi)N^{3\kappa-3s-4}(2\pi)^{-2(2\kappa-2-s)}\Gamma(\kappa-s)\Gamma(2\kappa-2-s)\\ &\quad\quad\times L(2\kappa-2-s,\phi^*)\big(N/\pi\big)^{2s-2\kappa+2}\frac{(-1)^uN}{\mathfrak{g}(\chi)^2}P_{N,2}(s,f^*,\phi^*)\\ &\quad\quad\times\Gamma(\frac{s-\kappa+u+1}{2})\Gamma(\frac{s-\kappa+u+2}{2})/\Gamma(\frac{\kappa-s+u}{2})\Gamma(\frac{\kappa-s+u-1}{2}).\end{aligned}$$

For $\Lambda_2(s,\phi)=N^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\phi)$, the functional equation is

$$\Lambda_2(s,\phi)=i^{2\kappa-2}\Lambda_2(2\kappa-2-s,\phi^*).$$

Applying this and simplifying, the previous equation becomes

$$\begin{aligned}&A_f(1_2)(2\pi)^{2\kappa-2-2s}\Gamma(s-\kappa+2)\Gamma(\frac{\kappa-s+u-1}{2})\Gamma(\frac{\kappa-s+u}{2})P_{N,1}(s,f,\phi)\\ &=A_f^*(1_2)\chi(-1)\frac{G(\chi)}{\mathfrak{g}(\chi)^2}\Gamma(\kappa-s)(-1)^{u+\kappa-1}\pi^{2\kappa-2-2s}\Gamma(\frac{s-\kappa+u+1}{2})\Gamma(\frac{s-\kappa+u+2}{2})\\ &\quad\times P_{N,2}(s,f^*,\phi^*).\end{aligned}$$

Applying the identity $2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})=\sqrt{\pi}\Gamma(2s)$ to this and simplifying, we obtain

$$\begin{aligned}A_f(1_2)\Gamma(s-\kappa+2)\Gamma(\kappa-s+u-1)P_{N,1}(s,f,\phi)&=A_f^*(1_2)\frac{\chi(-1)G(\chi)(-1)^{u+\kappa-1}}{\mathfrak{g}(\chi)^2}\\ &\quad\times\Gamma(\kappa-s)\Gamma(s-\kappa+u+1)P_{N,2}(s,f^*,\phi^*).\end{aligned}$$

For $u=1$ we can simplify directly, and for $u=0$ we apply the identity $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$. In both cases we obtain

$$A_f(1_2)P_{N,1}(s,f,\phi)=A_f^*(1_2)(-1)^{\kappa+u}\frac{G(\chi)}{\mathfrak{g}(\chi)^2}P_{N,2}(s,f^*,\phi^*).$$

Since $|\rho_f(p)|=p^{\kappa-3/2}$ from Theorem 3, and $|\lambda_\phi(p)|=p^{\kappa-3/2}$ from Theorem 6.3, p.117 from [12] for example, we must have

$$(1-\lambda_\phi(p)p^{-s})(1-\overline{\rho}_f(p)p^{s-2\kappa+2})=C_p(1-\rho_f(p)p^{-s})(1-\lambda_{\phi^*}(p)p^{s-2\kappa+2})$$

where $C_p = C_p(f, \phi, \psi)$ is nonzero and independent of s. This gives

$$\begin{aligned}1 + \lambda_\phi(p)\overline{\rho}_f(p)p^{-2\kappa+2} &- \lambda_\phi(p)p^{-s} - \overline{\rho}_f(p)p^{s-2\kappa+2} \\ &= C_p(1 + \lambda_{\phi^*}(p)\rho_f(p)p^{-2\kappa+2} - \rho_f(p)p^{-s} - \lambda_{\phi^*}(p)p^{s-2\kappa+2}).\end{aligned}$$

This implies the equations

$$\begin{aligned}\lambda_\phi(p) &= C_p\rho_f(p) \\ \overline{\rho}_f(p) &= C_p\lambda_{\phi^*}(p) \\ 1 + \lambda_\phi(p)\overline{\rho}_f(p)p^{-2\kappa+2} &= C_p(1 + \lambda_{\phi^*}(p)\rho_f(p)p^{-2\kappa+2}).\end{aligned}$$

Combining these, the third equation becomes

$$1 + C_p\rho_f(p)\overline{\rho}_f(p)p^{-2\kappa+2} = C_p + \rho_f(p)\overline{\rho}_f(p)p^{-2\kappa+2}$$

which by Theorem 3 gives $1 + C_p p^{-\kappa+1/2} = C_p + p^{-\kappa+1/2}$. Thus $C_p = 1$ and subsequently $\rho_f(p) = \lambda_\phi(p)$. This gives $P_{N,1}(s, f, \phi) = P_{N,2}(s, f^*, \phi^*) = 1$ and the result. □

Note that for $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)^*$ in the image of $\phi \in \mathcal{S}_{2\kappa-2}^1(\Gamma_0(N), \chi^2)$ and an eigenfunction of the Hecke operators and Frobenius operators as in Proposition 1, the above results imply the decomposition

$$Z_A(s, f) = L(s - \kappa + 1, \chi)L(s - \kappa + 2, \chi)L(s, \phi). \tag{3}$$

4. Twisted Spinor Zeta Functions and Gauss Sums

For ψ a Dirichlet character of conductor M, following [18] we define the twisted spinor zeta function attached to f and ψ to be

$$Z_A(s, f \otimes \psi) = \prod_{p\nmid N} Q_{p,f}(\psi(p)p^{-s})^{-1} \prod_{p|N}(1 - \psi(p)\rho_f(p)p^{-s})^{-1}.$$

Define the Gauss sum

$$S(\psi) = \sum_{H\in\mathcal{A}_2(M)} \psi(\sigma(H^{-1}))e^{-2\pi i\sigma(H)/M},$$

where

$$\mathcal{A}_2(M) = \{H \in M_2(\mathbb{Z}_M) \mid H^T = H, \quad H \text{ is invertible modulo } M\}.$$

For ψ primitive we have $|S(\psi)| = M^{3/2}$ from Lemma 2 on p.14 of [18].

Let

$$\Psi(s, f \otimes \psi) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - \kappa + 2)Z_A(s, f \otimes \psi). \tag{4}$$

In order to apply the converse theorem from [5] (given in Section 5 here), we need to slightly generalize the main result from [18]. In particular, statement *(iv)* from Theorem 5 in Section 5 requires hypotheses on χ, ψ, and M that are slightly more general than the hypotheses explicitly stated in the Haupttheorem in [18]. Let $\psi_1 = \chi\psi$. As in Section 1 we can contruct the characters $\widetilde{\psi}$ and $\widetilde{\psi}_1$ of $\mathcal{O}_K$.

Theorem 4 ([18], Haupttheorem). *Let $\kappa > 2$ and let $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N), \chi)$ with $A_f(1_2) \neq 0$. Let ψ be a primitive Dirichlet character modulo M. Then $\Psi(s, f \otimes \psi)$ is meromorphic in $s \in \mathbb{C}$ with at most simple poles at $s = \kappa$ and $\kappa - 2$.*

If $(M, N) = 1$ and χ, ψ, and $\widetilde{\psi}_1$ are primitive characters then $\Psi(s, f \otimes \psi)$ is holomorphic and satisfies the functional equation

$$A_f(1_2)\Psi(s, f \otimes \psi) = A_f^*(1_2)\frac{\chi(-M^2)\psi(N)G(\psi_1)S(\overline{\psi})}{M^{4s-4\kappa+6}N^{3s-3\kappa+4}\mathfrak{g}(\overline{\psi})}\Psi(2\kappa - 2 - s, f^* \otimes \overline{\psi}). \tag{5}$$

Proof. The functional equation (5) with the hypotheses on M, N and ψ, χ, ψ_1 is the Haupttheorem on p.12 of [18]. Here we show that the hypotheses there can be weakened at the expense of the possibility of poles of $\Psi(s, f \otimes \psi)$, in order to obtain the first part of the theorem.

Following [2] and [18], recall that $\mathcal{O}_K$ denotes the integers of $K = \mathbb{Q}(i)$. Let

$$\mathbb{H} = \{(x + iy, r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$$

be upper-half hyperbolic 3-space. Then $PSL_2(\mathcal{O}_K)$ acts on $\mathbb{H}$ by linear fractional transformations and there is a natural embedding $\imath : \mathbb{H} \hookrightarrow \mathfrak{H}_2$. See [2] for example.

Let $u = (x + iy, r) \in \mathbb{H}$, let $s \in \mathbb{C}$ with sufficiently large real part, and let α be a Dirichlet character modulo A. As in (4.7) of [2], define the Eisenstein series

$$E(u, s, \alpha) = \frac{v^s}{2} \sum_{(\gamma,\delta)\in\mathcal{O}_K - \{(0,0)\}} \frac{\alpha(\delta\overline{\delta})}{(|A\gamma z + \delta|^2 + |A\gamma|^2)^s}.$$

By (5.12) from [2] we have the expression

$$(2\pi)^{-s}\Gamma(s)E(u, s, \alpha) = \Delta(u, s, \alpha) - \frac{\alpha(0)}{sA^s} + \frac{G(\alpha, 0)}{(s-2)A^{s+2}}$$

where

$$\Delta(u, s, \alpha) = A^{-2s}\int_{1/A}^{\infty} t^{1-s}\Bigg(\sum_{\gamma,\delta\in\mathcal{O}_K - \{0,0\}} G(\alpha, \gamma)e^{-\frac{\pi t}{v}(|\gamma\overline{z} - \delta|^2 + |\gamma|^2 v^2)}\Bigg)\, dt$$

$$+ \int_{1/A}^{\infty} t^{s-1}(\Theta(t, u, \alpha) - \alpha(0))\, dt \tag{6}$$

is a holomorphic function of s, $\Theta(t, u, \alpha)$ is a theta-series from (5.1) in [2], and

$$G(\alpha, \gamma) = \sum_{d\in\mathcal{O}_K/(A)} \alpha(d\overline{d})e^{-2\pi i \mathrm{Re}(\gamma d/A)}.$$

Note that $G(\alpha, 1) = G(\alpha)$ from Section 3. It follows that $E(u, s, \alpha)$ has a meromorphic continuation to $s \in \mathbb{C}$, with possible poles only at the points $s = 0, 2$ and these are at most simple poles.

Let

$$\widetilde{\Gamma}_0(N) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathcal{O}_K) \;\Big|\; c \equiv 0 \pmod{N} \Big\}$$

and let $\mathcal{F}_N$ be a fundamental domain for the action of $\widetilde{\Gamma}_0(N)$ on $\mathbb{H}$. Let $M, N \in \mathbb{Z}_{>0}$ (not necessarily relatively prime) and let ℓ be the least common multiple of M and

N. For $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N),\chi)$ then from Lemma 1.1 on p.27 of [18], following (4.6) in Lemma 2 of [2], we have the integral representation of the radial Dirichlet series

$$R(s, f \otimes \psi) = \sum_{m=1}^{\infty} \psi(m) A_f(m \cdot 1_2) m^{-s}$$

given by

$$(4\pi)^{-s}\Gamma(s)R(s, f\otimes\psi) = \prod_{\mathfrak{p}\nmid MN} (1 - \psi_1(N(\mathfrak{p}))N(\mathfrak{p})^{-(s-\kappa+2)}) \int_{\mathcal{F}_N} f(\imath(u))E(u, s-\kappa+2, \psi_1)r^{\kappa-3}\, du,$$

where $(\mathfrak{p}) \subset \mathcal{O}_K$. Therefore, from the above expression for the Eisenstein series we get

$$(4\pi)^{-s}\Gamma(s)R(s, f\otimes\psi) = \prod_{\mathfrak{p}\nmid MN} (1 - \psi_1(N(\mathfrak{p}))N(\mathfrak{p})^{-(s-\kappa+2)})(2\pi)^{-(s-\kappa+2)}\Gamma(s-\kappa+2)$$
$$\times\bigg(\int_{\mathcal{F}_N} f(\imath(u))\Delta(u, s-\kappa+2, \psi_1)r^{\kappa-3}\, du - \frac{\psi_1(0)}{(s-\kappa+2)\ell^{s-\kappa+2}}P_N(f) + \frac{G(\psi_1,0)}{(s-\kappa)\ell^{s-\kappa}}P_N(f)\bigg)$$

where

$$P_N(f) = \int_{\mathcal{F}_N} f(\imath(u))r^{\kappa-3}\, du.$$

Following p.23 of [18], for f as above with the Fourier expansion as in Section 2, and ψ a primitive Dirichlet character modulo M, we define

$$f_\psi(z) = \sum_{H\in\mathcal{A}_2^+} \psi(\sigma(H))A_f(H)e^{2\pi i\sigma(Hz)} \in \mathcal{S}_\kappa^2(\Gamma_0^2(M^2N, M), \psi_1)$$

where

$$\Gamma_0^2(M^2N, M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2 \;\middle|\; c \equiv 0 \pmod{M^2N}\ d \equiv \begin{pmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{pmatrix} \pmod{M} \right\}.$$

Note that $\Gamma_0^2(M^2N, M) < \Gamma_0^2(M^2N)$ and since ℓ is the least common multiple of M and N we have

$$\prod_{\mathfrak{p}\nmid MN} (1 - \psi_1(N(\mathfrak{p}))N(\mathfrak{p})^{-s})^{-1} = \prod_{\mathfrak{p}\nmid \ell} (1 - \psi_1(N(\mathfrak{p}))N(\mathfrak{p})^{-s})^{-1}.$$

Thus by (1) on p.62 of [18] we have $Z_A(s, f\otimes\psi) = Z_A(s, f_\psi)$ and so

$$A_f(1_2)Z_A(s, f\otimes\psi) = \prod_{\mathfrak{p}\nmid\ell} (1 - \psi_1(N(\mathfrak{p}))N(\mathfrak{p})^{-(s-\kappa+2)})^{-1} R(s, f\otimes\psi).$$

Putting these and (4) together, we obtain the expression

$$A_f(1_2)\Psi(s, f\otimes\psi) = 2^s(2\pi)^{-\kappa+2}\bigg(\int_{\mathcal{F}_N} f(\imath(u))\Delta(u, s-\kappa+2, \psi_1)r^{\kappa-3}\, du - \frac{\psi_1(0)}{(s-\kappa+2)\ell^{s-\kappa+2}}P_N(f) + \frac{G(\psi_1,0)}{(s-\kappa)\ell^{s-\kappa}}P_N(f)\bigg). \tag{7}$$

As f is a cuspform, the integral part above is a holomorphic function of s. Thus $\Psi(s, f\otimes\psi)$ has possible poles only at $s=\kappa, \kappa-2$ and these are at most simple. Furthermore, if ψ_1 is primitive then $\psi_1(0)=G(\psi_1,0)=0$ and it follows that $\Psi(s, f\otimes\psi)$ is holomorphic. This gives the first part. The functional equation (5) is the Hauptheorem on p.17 of [18]. □

As a consequence of the theorem, we have following result on Gauss sums.

Proposition 2. *Let $(M,N)=1$. Let χ be a primitive Dirichlet character modulo N and ψ a primitive Dirichlet character modulo M so that the characters $\widetilde{\chi}$ and $\widetilde{\psi}$ are primitive. Then*

$$G(\psi_1)=G(\chi)G(\psi)\chi(M)^2\psi(N)^2$$

and

$$S(\overline{\psi})=\frac{\mathfrak{g}(\psi)^4\mathfrak{g}(\overline{\psi})}{G(\psi)}.$$

Note that the second equation is due to Rombach [18], p.66.

Proof. Let f and ϕ satisfy the hypotheses of Proposition 1. From (3) and (4) we have

$$\Psi(s,f\otimes\psi)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)L(s-\kappa+1,\psi_1)L(s-\kappa+2,\psi_1)L(s,\phi\otimes\psi)$$

and similarly,

$$\begin{aligned}\Psi(2\kappa-2-s,f^*\otimes\overline{\psi}) =&(2\pi)^{-2(2\kappa-2-s)}\Gamma(2\kappa-2-s)\Gamma(\kappa-s)\\ &\times L(\kappa-1-s,\overline{\psi}_1)L(\kappa-s,\overline{\psi}_1)L(2\kappa-2-s,\phi^*\otimes\overline{\psi}).\end{aligned}$$

We put these into equation (5) and then apply (2) with $a=\kappa-1$ and $a=\kappa-2$. This gives

$$\begin{aligned}&A_f(1_2)\Gamma(s)\Gamma(s-\kappa+2)\Gamma(\frac{\kappa-s+u}{2})\Gamma(\frac{\kappa-s-1+u}{2})L(s,\phi\otimes\psi)\\ &\quad=A_f^*(1_2)\frac{1}{M^{2s-2\kappa+3}N^{s-\kappa+1}}\frac{\chi(-M^2)\psi(N)G(\psi_1)S(\overline{\psi})}{\mathfrak{g}(\overline{\psi})\mathfrak{g}(\psi_1)^2(-1)^u}\pi^{2\kappa-2s-2}(2\pi)^{-2(\kappa-s-1)}\\ &\times\Gamma(2\kappa-2-s)\Gamma(\kappa-s)\Gamma(\frac{s-\kappa+1+u}{2})\Gamma(\frac{s-\kappa+2+u}{2})L(2\kappa-2-s,\phi^*\otimes\overline{\psi}).\end{aligned}$$

Applying the identity $2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})=\sqrt{\pi}\Gamma(2s)$ and simplifying, this becomes

$$\begin{aligned}&A_f(1_2)\Gamma(s)\Gamma(s-\kappa+2)\Gamma(\kappa-s-1+u)L(s,\phi\otimes\psi)\\ &\quad=A_f^*(1_2)\frac{1}{M^{2s-2\kappa+3}N^{s-\kappa+1}}\frac{\chi(-M^2)\psi(N)G(\psi_1)S(\overline{\psi})}{\mathfrak{g}(\overline{\psi})\mathfrak{g}(\psi_1)^2(-1)^u}\pi^{2\kappa-2s-2}2^{-2(\kappa-s-1)}\\ &\qquad\times\Gamma(2\kappa-2-s)\Gamma(\kappa-s)2^{-(s-\kappa+u)}\Gamma(s-\kappa+1+u)L(2\kappa-2-s,\phi^*\otimes\overline{\psi})\end{aligned}$$

For $u=0$ we use the identity $\Gamma(s)\Gamma(1-s)=\pi/\sin(s\pi)$ in order to simplify the above expression further, and for $u=1$ the simplification follows directly. In both cases, the above equation becomes

$$\begin{aligned}A_f(1_2)\Gamma(s)L(s,\phi\otimes\psi)=&A_f^*(1_2)\frac{1}{M^{2s-2\kappa+3}N^{s-\kappa+1}}\frac{\chi(-M^2)\psi(N)G(\psi_1)S(\overline{\psi})}{\mathfrak{g}(\overline{\psi})\mathfrak{g}(\psi_1)^2(-1)}\\ &\times(2\pi)^{-2(\kappa-s-1)}\Gamma(2\kappa-2-s)L(2\kappa-2-s,\phi^*\otimes\overline{\psi}).\end{aligned}$$

Let

$$\Lambda_3(s,\phi\otimes\psi) = (\sqrt{M^2N}/2\pi)^s\Gamma(s)L(s,\phi\otimes\psi)$$

where $L(s,\phi\otimes\psi)$ is the standard degree-2 L-function of ϕ twisted by ψ. As $\phi\in\mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$ and ψ is primitive, the functional equation is

$$\Lambda_3(s,\phi\otimes\psi) = i^{2\kappa-2}\chi(M)^2\psi(N)\mathfrak{g}(\psi)^2M^{-1}\Lambda_3(2\kappa-2-s,\phi^*\otimes\overline{\psi}),$$

see Theorem 7.6, p.126 of [12]. Applying this to the previous equation and simplifying, we obtain

$$A_f(1_2) = A_f^*(1_2)(-1)^{\kappa+u}\frac{G(\psi_1)S(\overline{\psi})}{\mathfrak{g}(\overline{\psi})\mathfrak{g}(\psi_1)^2\mathfrak{g}(\psi)^2}.$$

Combine this with Proposition 1 to get

$$S(\overline{\psi}) = \frac{\mathfrak{g}(\psi_1)^2\mathfrak{g}(\psi)^2\mathfrak{g}(\overline{\psi})G(\chi)}{\mathfrak{g}(\chi)^2G(\psi_1)}$$

for any primitive χ,ψ modulo N,M respectively, with $(M,N)=1$. For such χ and ψ we can apply $\mathfrak{g}(\psi_1)=\mathfrak{g}(\chi)\mathfrak{g}(\psi)\chi(M)\psi(N)$ (see Lemma 3.1.2 on p.81 in [15] for example) and get

$$S(\overline{\psi}) = \frac{\mathfrak{g}(\psi)^4\mathfrak{g}(\overline{\psi})G(\chi)\chi(M)^2\psi(N)^2}{G(\psi_1)}.$$

Note the right hand side is independent of N. Setting the arbitrary N and the $N=1$ cases equal we get $G(\psi_1)=G(\chi)G(\psi)\chi(M)^2\psi(N)^2$ and $S(\overline{\psi})=\mathfrak{g}(\psi)^4\mathfrak{g}(\overline{\psi})/G(\psi)$. This gives the result. □

5. Satake Parameters and a Converse Theorem

In this section we prove Theorem 1. Let χ and f satisfy the hypotheses of Theorem 1. Then $A_f(1_2)\neq 0$ and f satisfies the condition

$$\alpha_{1p}\alpha_{2p}=p^{\pm 1}\quad\text{or}\quad \alpha_{1p}\alpha_{2p}^{-1}=p^{\pm1}\quad\text{for}\quad p\nmid N. \tag{8}$$

As $\alpha_{0p}^2\alpha_{1p}\alpha_{2p}=p^{2\kappa-3}$ then $\alpha_{0p}=p^{\kappa-2\mp1}$ and this implies the decomposition

$$Z_A(s,f\otimes\psi) = L(s-\kappa+1,\psi_1)L(s-\kappa+2,\psi_1)D_N(s,f,\psi) \tag{9}$$

where

$$D_N(s,f,\psi) = \prod_{p\nmid N}\big((1-\psi(p)\alpha_{0p}\alpha_{1p}p^{-s})(1-\psi(p)\alpha_{0p}\alpha_{2p}p^{-s})\big)^{-1} \times\prod_{p\mid N}(1-\psi(p)\rho_f(p)p^{-s})^{-1}.$$

Now, we have that f^* also satisfies (8) and

$$Z_A(2\kappa-2-s,f^*\otimes\overline{\psi}) = L(\kappa-1-s,\overline{\psi}_1)L(\kappa-s,\overline{\psi}_1)D_N(2\kappa-2-s,f^*,\overline{\psi}).$$

Lemma 4. *Let $\kappa>2$ and let $(M,N)=1$. Let χ be a primitive Dirichlet character modulo N and ψ a primitive Dirichlet character modulo M so that $\widetilde{\chi}$ and $\widetilde{\psi}$ are primitive. Let $f\in\mathcal{S}^2_\kappa(\Gamma_0^2(N),\chi)$ satisfy (8) and let*

$$\Lambda_4(s,f,\psi) = (\sqrt{M^2N}/2\pi)^s\Gamma(s)D_N(s,f,\psi).$$

Then $\Lambda_4(s,f,\psi)$ *satisfies the functional equation*

$$A_f(1_2)\Lambda_4(s,f,\psi) = A^*_f(1_2)(-1)^{u-1}\frac{G(\chi)}{\mathfrak{g}(\chi)^2}\chi(M)^2\psi(N)\mathfrak{g}(\psi)^2M^{-1}\Lambda_4(2\kappa-2-s,f^*,\overline{\psi}).$$

Proof. For $f \in \mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)$, we apply the functional equation (5) from Theorem 4. This gives

$$A_f(1_2)\Psi(s,f\otimes\psi) = A^*_f(1_2)\frac{\chi(-M^2)\psi(N)G(\psi_1)S(\overline{\psi})}{M^{4s-4\kappa+6}N^{3s-3\kappa+4}\mathfrak{g}(\overline{\psi})}\Psi(2\kappa-2-s,f^*\otimes\overline{\psi}).$$

By decomposition (9) and Proposition 2 we have

$$\begin{aligned}
&A_f(1_2)(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)L(s-\kappa+1,\psi_1)L(s-\kappa+2,\psi_1)D_N(s,f,\psi)\\
&\qquad=A^*_f(1_2)\frac{\chi(-M^4)\psi(N^3)G(\chi)\mathfrak{g}(\psi)^4}{M^{4s-4\kappa+6}N^{3s-3\kappa+4}}(2\pi)^{-2(2\kappa-2-s)}\Gamma(2\kappa-2-s)\Gamma(\kappa-s)\\
&\qquad\quad\times L(\kappa-1-s,\overline{\psi}_1)L(\kappa-s,\overline{\psi}_1)D_N(2\kappa-2-s,f^*,\overline{\psi}).
\end{aligned}$$

As ψ_1 is primitive, we can apply the functional equation (2) of the L-functions $L(s,\psi_1)$ to this, with $a=\kappa-1$ and $a=\kappa-2$. After simplifying we get

$$\begin{aligned}
&A_f(1_2)(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)\Gamma(\frac{\kappa-s+u'}{2})\Gamma(\frac{\kappa-s-1+u'}{2})D_N(s,f,\psi)\\
&\quad=A^*_f(1_2)\frac{\chi(-M^2)\psi(N)G(\chi)\mathfrak{g}(\psi)^2}{M^{2s-2\kappa+3}N^{s-\kappa+1}\mathfrak{g}(\chi)^2}\pi^{-2s+2\kappa-2}(-1)^{u'}(2\pi)^{-2(2\kappa-2-s)}\\
&\quad\times\Gamma(2\kappa-2-s)\Gamma(\kappa-s)\Gamma(\frac{s-\kappa+1+u'}{2})\Gamma(\frac{s-\kappa+2+u'}{2})D_N(2\kappa-2-s,f^*,\overline{\psi})
\end{aligned}$$

where $\psi_1(-1)=(-1)^{u'}$. As before, we apply the identity $2^{2s-1}\Gamma(s)\Gamma(s+1/2)=\sqrt{\pi}\Gamma(2s)$ and similar properties of the gamma function. Simplifying, this gives us

$$\begin{aligned}
&A_f(1_2)(2\pi)^{-s}(\sqrt{M^2N})^s\Gamma(s)D_N(s,f,\psi) = A^*_f(1_2)(-1)^{\kappa+u}\frac{G(\chi)}{\mathfrak{g}(\chi)^2}i^{2\kappa-2}\chi(M)^2\psi(N)\\
&\qquad\times\mathfrak{g}(\psi)^2M^{-1}(2\pi)^{2\kappa-2-s}(\sqrt{M^2N})^{2\kappa-2-s}\Gamma(2\kappa-2-s)D_N(2\kappa-2-s,f^*,\overline{\psi}),
\end{aligned}$$

which is the result. □

The standard degree-5 L-function attached to f and a Dirichlet character ψ is

$$L_{\mathrm{St}}(s,f\otimes\psi)=\prod_p L_p(s,f\otimes\psi)$$

where for $p\nmid N$ we have

$$\begin{aligned}
L_p(s,f\otimes\psi) = \big[&(1-\psi_1(p)p^{-s})(1-\psi(p)\alpha_{1p}p^{-s})(1-\psi(p)\alpha_{1p}^{-1}p^{-s})\\
&\times(1-\psi(p)\alpha_{2p}p^{-s})(1-\psi(p)\alpha_{2p}^{-1}p^{-s})\big]^{-1}.
\end{aligned}$$

The analytic continuation of $L_{\mathrm{St}}(s,f\otimes\psi)$ was studied in [21]. As with $L_{\mathrm{St}}(s,f)$, it is shown that there are at most a finite number of simple poles.

Lemma 5. *Let* $f\in\mathcal{S}^2_\kappa(\Gamma^2_0(N),\chi)$ *satisfy (8). Then the local factor at* $p\nmid N$ *of the standard degree-5 L-function attached to* f *and* ψ *is*

(10) $$L_p(s,f\otimes\psi)=L_p(s,\psi_1)D_p(s+\kappa-1,f,\psi)D_p(s+\kappa-2,f,\psi).$$

Proof. Note that for $p \nmid N$,

$$D_p(s,f,\psi) = \left[(1-\psi(p)\alpha_{0p}\alpha_{1p}p^{-s})(1-\psi(p)\alpha_{0p}\alpha_{2p}p^{-s})\right]^{-1}.$$

where $\alpha_{0p} = p^{\kappa-1}$ if $\alpha_{1p}\alpha_{2p} = p^{-1}$, or $\alpha_{0p} = p^{\kappa-3}$ if $\alpha_{1p}\alpha_{2p} = p$. In the case $\alpha_{1p}\alpha_{2p} = p^{-1}$ and $\alpha_{0p} = p^{\kappa-1}$ we have $\alpha_{1p}^{-1} = \alpha_{2p}p$ and $\alpha_{2p}^{-1} = \alpha_{1p}p$. Thus the local factor of the standard L-function becomes

$$\begin{aligned} L_p(s,f\otimes\psi) =& \big[(1-\psi_1(p)p^{-s})(1-\psi(p)\alpha_{1p}p^{-s})(1-\psi(p)\alpha_{2p}p^{-s}) \\ &\quad \times (1-\psi(p)\alpha_{2p}^{-1}p^{-s})(1-\psi(p)\alpha_{1p}^{-1}p^{-s})\big]^{-1} \\ =& \big[(1-\psi_1(p)p^{-s})(1-\psi(p)\alpha_{1p}p^{-s})(1-\psi(p)\alpha_{2p}p^{-s}) \\ &\quad \times (1-\psi(p)\alpha_{1p}p^{-s+1})(1-\psi(p)\alpha_{2p}p^{-s+1})\big]^{-1} \\ =& \big[(1-\psi_1(p)p^{-s})(1-\psi(p)\alpha_{0p}\alpha_{1p}p^{-s-\kappa+1})(1-\psi(p)\alpha_{0p}\alpha_{2p}p^{-s-\kappa+1}) \\ &\quad \times (1-\psi(p)\alpha_{0p}\alpha_{1p}p^{-s-\kappa+2})(1-\psi(p)\alpha_{0p}\alpha_{2p}p^{-s-\kappa+2})\big]^{-1} \\ =& L_p(s,\psi_1)D_p(s+\kappa-1,f,\psi)D_p(s+\kappa-2,f,\psi). \end{aligned}$$

The case $\alpha_{1p}\alpha_{2p} = p$ and $\alpha_{0p} = p^{\kappa-3}$ is done similarly. □

Lemma 6. *Let $f \in \mathcal{S}_\kappa^2(\Gamma_0^2(N),\chi)$ with χ a primitive Dirichlet character so that $\widetilde{\chi}$ is primitive. Let ψ be a Dirichlet character that is primitive or trivial. Then $D_N(s,f,\psi)$ is meromorphic in $s \in \mathbb{C}$. There are at most a finite number of poles, which are of finite order and lie in the strip* $\mathrm{Re}(s) \in [\kappa-2,\kappa]$.

Proof. From the decomposition (9) and the analyticity of $\Psi(s,f\otimes\psi)$ from Theorem 4 and [18] (and [2] for ψ trivial), we have that $D_N(s,f,\psi)$ has a meromorphic continuation to $\mathbb{C}$. By the Euler product expansion, $D_N(s,f,\psi)$ is holomorphic and nonvanishing for $\mathrm{Re}(s) > \kappa$.

Let $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) < \kappa-2$. Then for any primitive Dirichlet character ψ,

$$\Psi(s,f\otimes\psi) = (2\pi)^{-s}\Gamma(s)\Gamma(s-\kappa+2)L(s-\kappa+1,\psi_1)L(s-\kappa+2,\psi_1)D_N(s,f,\psi)$$

is holomorphic at $s = s_0$, by Theorem 4. Thus, if $D_N(s,f,\psi)$ has a pole at s_0 then one of the two L-functions must vanish at s_0. Now, the only (possible) points where $L(s,\psi_1)$ vanishes for $\mathrm{Re}(s) < 0$ are at the negative integers, and $\Gamma(s)$ has poles at the negative integers of the same order as the zeroes of the L-function. Also, we have that $L(s-\kappa+1,\psi_1) = 0$ if and only if $L(s-\kappa+2,\psi_1) \neq 0$. Thus the zeroes of the L-functions are cancelled by the poles of the Γ-functions, and it follows that $D_N(s,f,\psi)$ cannot have a pole at s_0.

From (10) of Lemma 5 we have the decomposition

(11)
$$L_{\mathrm{St}}(s,f\otimes\psi) = \prod_p L_p(s,f\otimes\psi) = L(s,\psi_1)D_N(s+\kappa-1,f,\psi)D_N(s+\kappa-2,f,\psi)F(s)$$

where

$$F(s) = \prod_{p|N} \frac{(1-\psi(p)\rho_f(p)p^{-(s+\kappa-1)})(1-\psi(p)\rho_f(p)p^{-(s+\kappa-2)})}{L_p(s,f\otimes\psi)}.$$

Note that $F(s)$ only vanishes at a finite number of values of $s \in \mathbb{C}$, and these come from the zeros of the numerator of $F(s)$ and the poles of the denominator. Thus the possible zeros of $F(s)$ are all of finite order (in fact of order at most 2).

Let $s_0 \in \mathbb{C}$ be in the strip $1 < \mathrm{Re}(s) < 2$ so then $\kappa - 1 < \mathrm{Re}(s_0 + \kappa - 2) < \kappa$ and $\mathrm{Re}(s_0+\kappa-1) > \kappa$. Thus $D_N(s, f, \psi)$ is holomorphic and nonvanishing at $s = s_0+\kappa-1$ and $L(s, \psi_1)$ is holomorphic and nonvanishing at $s = s_0$. Now, $L_{\mathrm{St}}(s, f \otimes \psi)$ has at most a finite number of poles and these are all of finite order, and $F(s)$ has a finite number of zeros of finite order. It follows from decomposition (11) therefore, that there are at most a finite number of poles of $D_N(s, f, \psi)$ in the strip $\kappa - 1 < \mathrm{Re}(s) < \kappa$, and these are of finite order.

Let $s_0 \in \mathbb{C}$ be in the strip $1 < \mathrm{Re}(s_0) < 0$ so then $\kappa - 2 < \mathrm{Re}(s_0 + \kappa - 1) < \kappa - 1$ and $\mathrm{Re}(s_0 + \kappa - 2) < \kappa - 2$. Thus $D_N(s, f, \psi)$ is holomorphic at $s = s_0 + \kappa - 2$ and $L(s, \psi_1)$ is holomorphic and nonvanishing at $s = s_0$. Now, $L_{\mathrm{St}}(s, f \otimes \psi)$ has at most a finite number of poles and these are all of finite order, and $F(s)$ has a finite number of zeros of finite order. It follows from decomposition (11) therefore, that there are at most a finite number of poles of $D_N(s, f, \psi)$ in the strip $\kappa - 2 < \mathrm{Re}(s) < \kappa - 1$, and these are of finite order. Similarly, we can see that $D(s, f, \psi)$ has at most a finite number of poles of finite order on the lines $\mathrm{Re}(s) = \kappa, \kappa - 2$.

It follows that for ψ trivial or primitive, $D_N(s, f, \psi)$ has at most a finite number of poles. The poles are at of finite order and lie in the strip $\kappa - 2 \le \mathrm{Re}(s) \le \kappa$.

Note that in the case where $N = 1$, then $\Psi(s, f \otimes \psi)$ has simple poles at $s = \kappa$ and $\kappa - 2$. It follows that $D_N(s, f, \psi)$ is holomorphic. □

We are now able to apply a converse theorem to the functions $A_f(1_2)D_N(s, f, \psi)$ and $A_f^*(1_2)(-1)^{\kappa+u} \frac{G(\chi)}{\mathfrak{g}(\chi)^2} D_N(s, f^*, \overline{\psi})$. We state the theorem, essentially in the form given in [6], as Theorem 5. It was applied to Artin's conjecture in [5] and a detailed proof of this is in [6].

Let $\phi_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n$ with $a_{n,j} \in O(n^\alpha)$ for $j = 0, 1$ and α some positive number. Let $D(s, \phi_j) = \sum_{n=1}^{\infty} a_{n,j} n^{-s}$. For ψ a Dirichlet character of conductor M let

$$D(s, \phi_j \otimes \psi) = \sum_{n=1}^{\infty} a_{n,j} \psi(n) n^{-s}$$

and set

$$\Lambda(s, \phi_j \otimes \psi) = (\sqrt{M^2N}/2\pi)^s \Gamma(s) D(s, \phi_j \otimes \psi).$$

Theorem 5 ([6], Theorem 6.2)**.** *Assume that $D(s, \phi_1)$ satisfies the following:*

i) Has an Euler product expansion so that each factor at a prime p is the reciprocal of a polynomial in p^{-s}.

ii) Can be expressed as ratios of entire functions of finite order.

iii) Has at most finitely many poles.

iv) For all primitive Dirichlet characters ψ, the functions $\Lambda(s, \phi_1 \otimes \psi)$ and $\Lambda(s, \phi_2 \otimes \overline{\psi})$ have meromorphic continuations with no poles outside the strip $\mathrm{Re}(s) \in (0, 2\kappa-2)$.

v) For ψ primitive of conductor 1 or a prime p, the functional equation

$$\Lambda(s, \phi_1 \otimes \psi) = i^{2\kappa-2} \chi'(p) \psi(N) \mathfrak{g}(\psi)^2 p^{-1} \Lambda(2\kappa - 2, \phi_2 \otimes \overline{\psi})$$

is satisfied.

Then $\phi_1 \in \mathcal{S}^1_{2\kappa-2}(\Gamma_0(N), \chi')$ for χ' a Dirichlet character modulo N.

Clearly $A_f(1_2)D_N(s,f)$ can be expressed as a Dirichlet series whose coefficients are $O(n^\alpha)$ for some $\alpha \in \mathbb{R}$. As $A_f(1_2)D_N(s,f)$ has an Euler product expansion, then (i) is satisfied. From (7) in the proof of Theorem 4 and (6.1) in [2], we have

$$A_f(1_2)\Psi(s, f\otimes\psi) = c_1 a_1^s\Big(\int_{\mathcal{F}_N} f(\imath(u))\Delta(u, s-\kappa+2, \psi_1) r^{\kappa-3}\, du + c_2 a_2^s + c_3 a_3^s\Big)$$

where a_i, c_i are constants. We have from (6) (see also (5.14) in [2]),

$$\Delta(u,s,\psi_1) = \ell^{-2s}\int_{1/\ell}^{\infty} t^{1-s}\Bigg(\sum_{\gamma,\delta\in\mathcal{O}_K-\{0,0\}} G(\psi_1,\gamma)e^{-\frac{\pi t}{v}(|\gamma\overline{z}-\delta|^2+|\gamma|^2v^2)}\Bigg)\, dt$$
$$+\int_{1/\ell}^{\infty} t^{s-1}(\Theta(t,u,\psi_1)-\psi_1(0))\, dt$$

where $\Theta(t,u,\psi_1)$ is a theta-series from (5.1) in [2]. It follows that $Z_A(s, f\otimes\psi)$ is of finite order for ψ primitive or trivial. Thus there is a polynomial $Q(x)$ so that $Z_A(s,f)Q(s)$ is entire and of finite order. Similarly, there is a polynomial $P(x)$ so that $L(s,\chi)P(s)$ is entire and of finite order. Thus we have

$$A_f(1_2)D_N(s,f) = \frac{A_f(1_2)Z_A(s,f)Q(s)P(s-\kappa+1)P(s-\kappa+2)}{Q(s)L(s-\kappa+1,\chi)P(s-\kappa+1)L(s-\kappa+2,\chi)P(s-\kappa+2)}$$

which is a ratio of entire functions of finite order, and so (ii) is satisfied. By Lemma 6 we have that $A_f(1_2)D_N(s,f)$ has finitely many poles and the possible poles are in the strip $[\kappa-2,\kappa]\subset(0,2\kappa-2)$ for $\kappa>2$, and so we get (iii). By Theorem 4 and Lemmas 4 and 6 we see that (iv) holds for $\Lambda(s,\phi\otimes\psi)$ and $\Lambda(s,\phi^*\otimes\overline{\psi})$. The functional equation in the statement of the theorem follows from Lemma 4 with $\chi'=\chi^2$, $\phi_1 = A_f(1_2)\phi$, and $\phi_2 = A_f^*(1_2)(-1)^{\kappa+u}\frac{G(\chi)}{\mathfrak{g}(\chi)^2}\phi^*$. This gives (v).

Applying Theorem 5 therefore, we have that $A_f(1_2)D_N(s,f) = \sum_{n=1}^\infty a_n n^{-s} = a_1 L(s,\phi)$ for some $\phi\in\mathcal{S}^1_{2\kappa-2}(\Gamma_0(N),\chi^2)$. Thus $Z_A(s,f)$ decomposes into

$$L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)D_N(s,f) = \frac{a_1}{A_f(1_2)}L(s-\kappa+1,\chi)L(s-\kappa+2,\chi)L(s,\phi)$$

and we have $a_1 = A_f(1_2)$. This gives decomposition (1) and it follows that f is in the image of the Saito-Kurokawa lift. This finishes the proof of Theorem 1.

References

[1] A.N. Andrianov, *Modular descent and the Saito-Kurokawa conjecture*, Invent. Math. **53** (1979), 267-280.

[2] A.N. Andrianov, *On functional equations satisfied by spinor Euler products for Siegel modular forms of genus 2 with characters*, Abh. Math. Sem. Univ. Hamburg **71** (2001), 123-142.

[3] A.N. Andrianov, *On diagonalization of singular Frobenius operators on Siegel modular forms*, Amer. J. Math. **125** (2003), 139-165.

[4] S. Böcherer, *Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe*, J. Reine Angew. Math. **362** (1985), 146-168.

[5] A.R. Booker, *Poles of Artin L-functions and the strong Artin conjecture*, Ann. of Math. **158** (2003), 1089-1098.

[6] A.R. Booker, *Numerical Tests of Modularity*, Ph.D. Thesis, Princeton University, 2004.

[7] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233-259.

[8] W. Casselman, *The unramified principal series of* $\mathfrak{p}$*-adic groups, I: the spherical function*, Comp. Math. **40** no. 2 (1980), 387-406.

[9] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser, Progr. in Math. **55**, Basel, 1985.

[10] D. Flath, *Decomposition of representations into tensor products*, in Automorphic Forms, Representations, and L-Functions, Proc. Symp. Pure Math. Vol **33** Part I (1979), 179-183.

[11] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. **154** (2001), 641-681.

[12] H. Iwaniec, *Topics in Classical Automorphic Forms*, Grad. Studies in Math. **17**, Amer. Math. Soc., Providence, RI, 1997.

[13] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades. I,II,III*, Invent. Math. **52** (1979), 95-104, 249-253, 255-265.

[14] M. Manickam, B. Ramakrishnan, and T.C. Vasudevan, *On Saito-Kurokawa descent for congruence subgroups*, Manuscr. Math. **81** (1993), 161-182.

[15] T. Miyake, *Modular Forms*, Springer-Verlag, Berlin, Heidelberg, 1989.

[16] I.I. Piatetski-Shapiro, *On the Saito-Kurokawa lifting*, Invent. Math. **71** (1983), 309-338.

[17] A. Pitale and R. Schmidt, *Sign changes of Hecke eigenvalues of Siegel cusp forms of degree 2*, preprint.

[18] A. Rombach, *Über Twists in siegelschen Modulformen zweiten Grades und ihue Spinorzetafunktion*, Inauguraldissertation, Universität Mannheim, 2004.

[19] I. Satake, *Theory of spherical functions on reductive algebraic groups over* p*-adic fields*, Publ. Math. I.H.E.S. **18** (1963), 1-69.

[20] R. Schmidt, *On classical Saito-Kurokawa liftings*, J. Reine Angew Math. **604** (2007), 211-236.

[21] G. Shimura, *Arithmeticity in the theory of automorphic forms*, Math. Surveys and Monographs **82**, Amer. Math. Soc., Providence, RI, 2000.

[22] R. Weissauer, *Der Heckesche Umkehusatz*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 83-119.

[23] R. Weissauer, *On certain degenerate eigenvalues of Hecke operators*, Comp. Math. **83** no. 2 (1992), 127-145.

[24] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass)*, Seminar on Number Theory, Paris 1979-80, p. 371-394, Progr. Math. **12**, Birkhäuser, Boston, MA 1981.

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