

A DEGENERATE MONGE–AMPÈRE EQUATION AND THE BOUNDARY CLASSES OF KÄHLER CONES

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1. Introduction

It is well known that a numerically effective (nef) line bundle on a compact complex manifold M^n may not admit any smooth Hermitian metric whose curvature is everywhere nonnegative. The first such example were discovered by Demailly, Peternell, and Schneider in 1994 [3]. They showed that for a non-splitting extension

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$$

on an elliptic curve C , the line bundle L dual to the tautological line bundle of the projectivized bundle $M^2 = \mathbf{P}(\mathcal{E})$ does not admit any smooth Hermitian metric with nonnegative curvature. In fact, they showed that any singular Hermitian metric on L with nonnegative curvature must have logarithmic singularity, so the metric cannot even be continuous. Clearly, L is a nef line bundle on the ruled surface M , since \mathcal{E} is nef on C .

Here we should point out that, dated back to 1973, the second author in fact gave a talk on this example in the AMS conference at Stanford. He thought that this would give a counterexample to the Calabi conjecture. The mistake was found a few months later. The mistake was due to different definitions of semipositivity in this example.

In this article, we are interested in knowing under what conditions, any nef line bundles, or more generally any boundary classes of the Kähler cone, will always be representable by a smooth (or $C^{1,1}$) closed $(1, 1)$ form that is everywhere nonnegative. The existence of such forms often has direct geometric applications in the study of Monge–Ampère foliations or Kähler submersions.

The existence problem can often be interpreted as a degenerate Monge–Ampère equation, which in general would be difficult to solve. Here as an exploration and a test case, we prove the existence of smooth nonnegative $(1, 1)$ forms for any boundary class of the Kähler cone when the manifold satisfies certain curvature conditions.

Theorem 1. *Let (M^n, g) be a compact Kähler manifold satisfying the following curvature condition: for any orthonormal tangent frame $\{e_1, \dots, e_n\}$ at any $x \in M$, and*

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for any real numbers a_1, \dots, a_n :

$$\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0 \quad (*)$$

Then any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed $(1,1)$ form that is everywhere nonnegative.

Note that when $n = 2$, the condition $(*)$ simply means $R_{u\bar{u}v\bar{v}} \geq 0$ for any pair of orthogonal tangent vectors u, v of type $(1,0)$, namely, M^2 has *nonnegative orthogonal bisectonal curvature*. This is certainly a very restrictive condition, and it is possible that any such manifold actually admit Kähler metric with nonnegative bisectonal curvature, thus being one of the spaces in Mok's solution to the generalized Frankel's conjecture [6].

For $n > 2$, the condition $(*)$ becomes much less restrictive than nonnegative orthogonal bisectonal curvature. It would be an interesting question to try to classify all manifolds satisfying the curvature condition $(*)$, as a generalization to Mok's Theorem.

The referee of the article kindly pointed out to us the following fact: The above curvature condition $(*)$ appeared implicitly for the first time in the literature in 1965, in the work [1] by Bishop and Goldberg. In that paper, the authors showed that any compact Kähler manifold M^n with positive bisectonal curvature must have its second Betti number equal to 1.

It would be interesting to know what kind of restriction the second Betti number must obey under the curvature condition $(*)$, say under the strict inequality case (namely, the left hand side is positive whenever a_i are not all equal). Notice that the product of \mathbf{CP}^1 (equipped with a sufficiently positively curved metric) with another curve always satisfies the curvature condition $(*)$ in the strict sense. So we cannot expect the second Betti number to always be 1, even when $n = 2$. On the other hand, as proved in [5] (see the proof in front of Lemma 1), on a compact Kähler manifold M^n satisfying the curvature condition $(*)$, any harmonic $(1,1)$ form must be parallel. So in particular, the Hodge number $h^{1,1}(M)$ is equal to 1 if M is locally irreducible.

Note that as a consequence of Theorem 1, we know that the ruled surface M^2 in the aforementioned example of the second author and Demailly–Peternell–Schneider does not admit any Kähler metric with nonnegative orthogonal bisectonal curvature. On the other hand, M^2 clearly has nef tangent bundle.

2. A degenerate Monge–Ampère Equation

Let (M^n, g_0) be a compact Kähler manifold. Denote by

$$H(M) = H_{\mathbf{R}}^{1,1}(M) = H^{1,1}(M) \cap H^2(M, \mathbf{R})$$

the vector space of real $(1,1)$ classes. Write $\mathcal{K}(M)$ for the Kähler cone in $H(M)$, namely, the convex cone formed by all the cohomology classes that can be represented

by smooth closed $(1, 1)$ forms that are everywhere positive. We are interested in the boundary set $\mathcal{B} = \overline{\mathcal{K}} \setminus \mathcal{K}$ of the Kähler cone. We will call a (non-trivial) cohomology class α in \mathcal{B} a *boundary Kähler class* or simply a *boundary class* of M . We want to know when α can be represented by a closed, smooth $(1, 1)$ form that is everywhere nonnegative.

As one could imagine, the case $\alpha^n > 0$ and the case $\alpha^n = 0$ behaves quite differently. The former means the class α is ‘nef’ and ‘big’ in the algebraic geometry terminology, and is in general easier to deal with. The latter case means α is nef but not big, which is a more degenerate situation, corresponding to the volume collapsing case under differential geometric terms. In this article we will deal with a special situation in this latter case.

Let ω_0 be the Kähler form of g_0 . Consider a path in $H(M)$ from $[\omega_0]$ to α :

$$\alpha_t := (1 - t)[\omega_0] + t\alpha, \quad t \in [0, 1].$$

Note that for $0 \leq t < 1$, α_t lies in the Kähler cone. Then

$$a(t) := \frac{1}{V} \int_M \alpha_t^n$$

is positive for $0 \leq t < 1$, and $a(1) = 0$. Here we denote by

$$V = \int_M \omega_0^n.$$

Let us fix a smooth $(1, 1)$ -form η in the class α (certainly if η happens to be nonnegative then we are done.). Since $a(t)\omega_0^n$ defines a smooth volume form on M , by a theorem of the second author [7], there exists a smooth function u_t on M , unique up to a constant, satisfying the following equations

$$(2.1) \quad \begin{aligned} (\omega_0 + t(\eta - \omega_0) + dd^c u_t)^n &= a(t)\omega_0^n, \\ \omega_0 + t(\eta - \omega_0) + dd^c u_t &> 0, \end{aligned}$$

for all $0 \leq t < 1$. If there is a smooth limit of u_t , say u_* , as $t \rightarrow 1$, then $\eta + dd^c u_*$ will be a desired nonnegative $(1, 1)$ form representing the nef class α .

From now on, let us consider a slightly more general case: Let ω_0 be a Kähler form on M . Let Φ be a d -closed $(1, 1)$ form on M , such that the cohomology class represented by $\omega_0 + t\Phi$ is positive for each $0 \leq t < 1$. In other words, for each $0 \leq t < 1$, there exists a smooth function f_t on M such that

$$\omega_0 + t\Phi + dd^c f_t > 0 \quad \text{on } M.$$

We assume that

$$(2.2) \quad \int_M (\omega_0 + \Phi)^n = 0.$$

Our goal is to find a smooth solution v satisfying equations

$$(2.3) \quad (\omega_0 + \Phi + dd^c v)^n = 0,$$

$$(2.4) \quad \omega_0 + \Phi + dd^c v \geq 0.$$

We remark that (2.2) is exactly the compatibility condition for (2.3).

Theorem 2. *Let ω_0 and Φ be given as above. Suppose that the curvature tensor of ω_0 satisfies the condition (*) in Theorem 1. Then there exists a smooth solution v for (2.3) and (2.4).*

In particular, let $\Phi = \eta - \omega_0$ in (2.1), we obtain a desired nonnegative smooth $(1, 1)$ -form $\eta + dd^c h$ which represents the boundary class α . This will establish Theorem 1 for the case of a $(1, 1)$ -class α on the boundary of the Kähler cone such that $\alpha^n = 0$. But the same proof works of course when $\alpha^n > 0$ in exactly the same way, so Theorem 1 follows.

Our approach for Theorem 2 is by the perturbation method. Let

$$\gamma(t) = \frac{1}{V} \int_M (\omega_0 + t\Phi)^n, \quad \text{for all } t \in \mathbb{R}.$$

Then $\gamma(t)$ is a smooth function which is positive on $[0, 1)$; and (2.2) is equivalent to $\gamma(1) = 0$. Again, for each $0 \leq t < 1$, by the second author's theorem we have a solution v_t , unique up to a constant, satisfying

$$(2.5) \quad (\omega_0 + t\Phi + dd^c v_t)^n = \gamma(t)\omega_0^n,$$

$$(2.6) \quad \omega_0 + t\Phi + dd^c v_t > 0.$$

To solve (2.3) and (2.4), it suffices to show that there is a smooth limit for a subsequence of $\{v_t\}$ as $t \rightarrow 1$.

Let us normalize the solution by setting

$$(2.7) \quad \int_M v_t \omega_0^n = 0,$$

for each $0 \leq t < 1$. We claim that the unique solution v_t of (2.5), (2.6), and (2.7) depends smoothly on t for $0 \leq t < 1$. This will be proved in the next section. The argument is to apply the implicit function theorem, similar to the proof of the openness in the continuity method (see [7]).

Then we can differentiate the equation (2.5) with respect to t . It turns out that the function $v_t - t\dot{v}_t$ would be important in the *a priori* estimates. Here we denote by $\dot{a} = \partial a / \partial t$. In particular, we obtain that

$$(2.8) \quad \Delta'(v_t - t\dot{v}_t) = C(t) - S.$$

Here Δ' is the Laplacian for $\omega_t = \omega_0 + t\Phi + dd^c v_t$, $C(t)$ is a constant depending on t , and S is the trace of ω_0 with respect to ω_t . On the other hand, motivated by the second author's generalized Schwarz Lemma ([8]), one can estimate S via the Chern-Lu formula ([2] and [4]). Then the curvature condition (*) and the maximum principle will enable us to conclude that S depends only on t . Thus, by (2.8) and solving an ordinary differential equation, we obtain a smooth solution for $t = 1$. This finishes the proof of Theorem 2.

3. A smooth family of solutions

Let v_t be the unique smooth solution satisfying

$$(3.1) \quad (\omega_0 + t\Phi + dd^c v_t)^n = \gamma(t)\omega_0^n,$$

$$(3.2) \quad \omega_0 + t\Phi + dd^c v_t > 0,$$

$$(3.3) \quad \int_M v_t \omega_0^n = 0,$$

for each $0 \leq t < 1$. We would like to show that v_t depends smoothly on t . To see this, we will use the implicit function theorem for Banach spaces. We denote by

$$\mathcal{E}^{k,\alpha} = \left\{ h \in C^{k,\alpha}(M); \int_M h \omega_0^n = 0. \right\}$$

for each positive integer k and $\alpha \in (0, 1)$. Then each $\mathcal{E}^{k,\alpha}$ is itself a Banach space, as a closed subspace in the Hölder space $C^{k,\alpha}(M)$. Let us fix a k and an α . We define

$$\mathcal{M}(t, u) = \frac{(\omega_0 + t\Phi + dd^c u)^n}{\gamma(t)\omega_0^n} - 1,$$

for all $u \in C^{k+2,\alpha}(M)$ and $0 \leq t < 1$. Then $\mathcal{M}(t, u)$ defines a smooth map from $[0, 1) \times \mathcal{E}^{k+2,\alpha}$ to $\mathcal{E}^{k,\alpha}$.

Now for an arbitrary $t \in [0, 1)$ with v_t satisfying (3.1), (3.2), and (3.3), we have $v_t \in \mathcal{E}^{k+2,\alpha}$ and

$$\mathcal{M}(t, v_t) = 0.$$

We need to show that $D_2\mathcal{M}(t, v_t)$, the partial Fréchet derivative of \mathcal{M} at (t, v_t) with respect to v_t , is a linear isomorphism from $\mathcal{E}^{k+2,\alpha}$ to $\mathcal{E}^{k,\alpha}$. Here we identify a Banach space with its the tangent space at a point. Observe that, for any $h \in \mathcal{E}^{k+2,\alpha}$,

$$\begin{aligned} D_2\mathcal{M}(t, v_t)(h) &= \left. \frac{d}{ds} \mathcal{M}(t, v_t + sh) \right|_{s=0} \\ &= \frac{n(\omega_0 + t\Phi + dd^c v_t)^{n-1} \wedge dd^c h}{\gamma(t)\omega_0^n}, \\ &= \Delta' h. \end{aligned}$$

Here Δ' is the Laplacian associated with the metric

$$\omega_t = \omega_0 + t\Phi + dd^c v_t.$$

Note that the kernel of Δ' consists of constant functions; so does its cokernel. Then the former implies the injectivity of $D_2\mathcal{M}(t, v_t)$ on $\mathcal{E}^{k+2,\alpha}$; while the latter, together with the Schauder theory imply the surjectivity of $D_2\mathcal{M}(t, v_t)$. Thus, $D_2\mathcal{M}(t, v_t)$ is a linear isomorphism. It then follows from the implicit function theorem that v_t depends smoothly on t .

4. Estimates

We denote by S the trace of ω_0 with respect to ω_t , where

$$\omega_t = \omega_0 + t\Phi + dd^c v_t, \quad 0 \leq t < 1.$$

By differentiating (3.1) with respect to t we get

$$(4.1) \quad n\omega_t^{n-1} \wedge (\Phi + dd^c \dot{v}_t) = \dot{\gamma}\omega_0^n.$$

Here we denote by $\dot{a} = \partial a / \partial t$. Moreover, observe that

$$(4.2) \quad n\omega_t^{n-1} \wedge (t\Phi + dd^c v_t) = n\omega_t^n - n\omega_t^{n-1} \wedge \omega_0.$$

Combining (4.1) and (4.2) we obtain

$$n\omega_t^{n-1} \wedge dd^c(v_t - t\dot{v}_t) = (n\gamma - t\dot{\gamma})\omega_0^n - n\omega_t^{n-1} \wedge \omega_0.$$

Dividing both sides by ω_t^n yields

$$(4.3) \quad \Delta'(v_t - t\dot{v}_t) = (n - t\dot{\gamma}/\gamma) - S.$$

Next we will derive an estimate for S , motivated by the idea of the generalized Schwarz Lemma by the second author [8].

Proposition 4.1. *Let M be a Kähler manifold of complex dimension n . Let $g_{i\bar{j}}$ and $g'_{i\bar{j}}$ be two Kähler metrics on M , and $R_{i\bar{j}}$, $R'_{i\bar{j}}$ be, respectively, their Ricci curvature tensors. Suppose that*

$$(4.4) \quad R'_{\xi\bar{\xi}} \geq R_{\xi\bar{\xi}}$$

for any type $(1,0)$ tangent vector ξ on M . We assume, in addition, that the curvature tensor of $g_{i\bar{j}}$ satisfies the condition $(*)$ in Theorem 1. That is, for any orthonormal tangent frame $\{e_1, \dots, e_n\}$ at any $x \in M$, and for any nonnegative numbers a_1, \dots, a_n , we have

$$\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0.$$

Let S be the trace of $g_{i\bar{j}}$ with respect to $g'_{i\bar{j}}$. Then

$$\Delta' \log S \geq 0.$$

where Δ' is the Laplacian for $g'_{i\bar{j}}$.

Proof of Proposition 4.1. It is equivalent to show that

$$\Delta' S - \frac{|\nabla' S|^2}{S} \geq 0.$$

Notice that

$$\begin{aligned}
 \Delta' S &= \sum_{k,l} g'^{k\bar{l}} \frac{\partial^2}{\partial z^k \partial \bar{z}^l} \left(\sum_{i,j} g'^{i\bar{j}} g_{i\bar{j}} \right) \\
 &= \sum_{i,j,k,l} g'^{k\bar{l}} g_{i\bar{j}} \frac{\partial^2 g'^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{i,j,k,l} g'^{k\bar{l}} g'^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \\
 (4.5) \quad &+ \sum_{i,j,k,l} g'^{k\bar{l}} \left(\frac{\partial g'^{i\bar{j}}}{\partial z^k} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} + \frac{\partial g'^{i\bar{j}}}{\partial \bar{z}^l} \frac{\partial g_{i\bar{j}}}{\partial z^k} \right).
 \end{aligned}$$

We have, by direct computation,

$$\begin{aligned}
 \frac{\partial g'^{i\bar{j}}}{\partial \bar{z}^l} &= - \sum_{p,q} g'^{i\bar{q}} g'^{p\bar{j}} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}^l}, \\
 \frac{\partial^2 g'^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} &= - \sum_{p,q} g'^{i\bar{q}} g'^{p\bar{j}} \frac{\partial^2 g'_{p\bar{q}}}{\partial z^k \partial \bar{z}^l} \\
 &+ \sum_{a,b,p,q} \left(g'^{i\bar{b}} g'^{a\bar{q}} g'^{p\bar{j}} \frac{\partial g'_{a\bar{b}}}{\partial z^k} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}^l} + g'^{i\bar{q}} g'^{a\bar{j}} g'^{p\bar{b}} \frac{\partial g'_{a\bar{b}}}{\partial z^k} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}^l} \right).
 \end{aligned}$$

Observe that

$$- \sum_{k,l} g'^{k\bar{l}} \frac{\partial^2 g'_{p\bar{q}}}{\partial z^k \partial \bar{z}^l} = R'_{p\bar{q}} - \sum_{a,b,k,l} g'^{a\bar{b}} g'^{k\bar{l}} \frac{\partial g'_{a\bar{q}}}{\partial z^k} \frac{\partial g'_{p\bar{b}}}{\partial \bar{z}^l}.$$

Thus, the first term on the right hand side of (4.5) is given by

$$\begin{aligned}
 \sum_{i,j,k,l} g'^{k\bar{l}} g_{i\bar{j}} \frac{\partial^2 g'^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} &= \sum g'^{i\bar{q}} g'^{p\bar{j}} R'_{p\bar{q}} g_{i\bar{j}} - \sum g_{i\bar{j}} g'^{i\bar{q}} g'^{p\bar{j}} g'^{a\bar{b}} g'^{k\bar{l}} \frac{\partial g'_{a\bar{q}}}{\partial z^k} \frac{\partial g'_{p\bar{b}}}{\partial \bar{z}^l} \\
 &+ \sum \left(g'^{i\bar{b}} g'^{a\bar{q}} g'^{p\bar{j}} + g'^{i\bar{q}} g'^{a\bar{j}} g'^{p\bar{b}} \right) g_{i\bar{j}} g'^{k\bar{l}} \frac{\partial g'_{a\bar{b}}}{\partial z^k} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}^l} \\
 &= \sum g'^{i\bar{q}} g'^{p\bar{j}} R'_{p\bar{q}} g_{i\bar{j}} + \sum g'^{i\bar{q}} g'^{a\bar{j}} g'^{p\bar{b}} g_{i\bar{j}} g'^{k\bar{l}} \frac{\partial g'_{a\bar{b}}}{\partial z^k} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}^l}.
 \end{aligned}$$

Let us now choose a normal coordinate system (z^1, \dots, z^n) at a point of M such that at the point

$$g_{i\bar{j}} = \delta_{ij}, \quad \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} = 0,$$

and

$$g'_{i\bar{j}} = \delta_{ij} g'_{i\bar{i}},$$

for $i, j, k, l = 1, \dots, n$. Then we have

$$\sum_{i,j,k,l} g'^{k\bar{l}} g_{i\bar{j}} \frac{\partial^2 g'^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} = \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} + \sum_{i,p,k} \frac{|\partial g'_{i\bar{p}} / \partial z^k|^2}{(g'_{i\bar{i}})^2 g'_{p\bar{p}} g'_{k\bar{k}}}.$$

The second term on the right of (4.5) is given by

$$\sum_{i,j,k,l} g'^{k\bar{l}} g'^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} = - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{k\bar{k}} g'_{i\bar{i}}}.$$

Therefore, we have

$$(4.6) \quad \begin{aligned} \Delta'(S) &= -\sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}}g'_{k\bar{k}}} + \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} \\ &\quad + \sum_{i,j,k} \frac{|\partial g'_{i\bar{j}}/\partial z^k|^2}{(g'_{i\bar{i}})^2 g'_{j\bar{j}} g'_{k\bar{k}}}. \end{aligned}$$

Observe that

$$\begin{aligned} |\nabla' S|^2 &= \sum_{k,l} g'^{k\bar{l}} \frac{\partial S}{\partial z^k} \frac{\partial S}{\partial \bar{z}^l} \\ &= \sum_k \frac{1}{g'_{k\bar{k}}} \left| \sum_i \frac{\partial g'_{i\bar{i}}/\partial z^k}{(g'_{i\bar{i}})^2} \right|^2 \\ &= \sum_k \frac{1}{g'_{k\bar{k}}} \left| \sum_i \frac{\partial g'_{i\bar{i}}/\partial z^k}{(g'_{i\bar{i}})^{3/2}} \frac{1}{(g'_{i\bar{i}})^{1/2}} \right|^2. \end{aligned}$$

Applying Cauchy–Schwarz inequality yields that

$$(4.7) \quad |\nabla' S|^2 \leq S \sum_{i,k} \frac{|\partial g'_{i\bar{i}}/\partial z^k|^2}{(g'_{i\bar{i}})^3 g'_{k\bar{k}}}.$$

It follows from (4.6), (4.7), and (4.4) that

$$\begin{aligned} \Delta' S - \frac{|\nabla' S|^2}{S} &\geq -\sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}}g'_{k\bar{k}}} + \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} \\ &\geq \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{(g'_{i\bar{i}})^2} - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}}g'_{k\bar{k}}} \\ &= \frac{1}{2} \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{(g'_{i\bar{i}})^2} + \frac{1}{2} \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{(g'_{k\bar{k}})^2} - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}}g'_{k\bar{k}}} \\ &= \frac{1}{2} \sum_{i,k} R_{i\bar{i}k\bar{k}} \left(\frac{1}{g'_{i\bar{i}}} - \frac{1}{g'_{k\bar{k}}} \right)^2. \end{aligned}$$

Now we apply condition (*) to obtain that

$$S\Delta' \log S = \Delta' S - \frac{|\nabla' S|^2}{S} \geq 0.$$

This finishes the proof. \square

In our case, we actually only need $\Delta' S \geq 0$ to conclude that S is a function depending only on t , since M is compact. Here we have assumed that the curvature of ω_0 satisfies (*). By (3.1), we have

$$R'_{i\bar{j}} = R_{i\bar{j}}.$$

Now observe that the right hand side of (4.3) depends only on t . Applying the maximum principle to $v_t - t\dot{v}_t$, we know that both sides of (4.3) will have to equal

zero. Hence, $v_t - t\dot{v}_t$ depends only on t . But we also have the normalization (3.3); differentiating (3.3) with respect to t yields that

$$\int_M \dot{v}_t \omega_0^n = 0.$$

Putting these together, we obtain

$$v_t - t\dot{v}_t = 0 \quad \text{on } M.$$

By solving this ordinary differential equation we conclude that

$$v_t = th,$$

where h is a function on M . This implies that $h \in C^\infty(M)$ with

$$\omega_0 + t\Phi + tdd^c h > 0 \quad \text{on } M,$$

for all $0 \leq t < 1$. Letting $t \rightarrow 1$, we obtain a nonnegative smooth $(1, 1)$ -form

$$\omega_1 = \omega_0 + \Phi + dd^c h$$

which satisfies that

$$\omega_1^n = 0.$$

This completes the proof of Theorem 2. \square

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