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THE SIGNATURE OF THE CHERN COEFFICIENTS OF LOCAL RINGS

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Abstract. This paper considers the following conjecture: If R is an unmixed, equidimensional local ring that is a homomorphic image of a Cohen-Macaulay local ring, then for any ideal J generated by a system of parameters, the Chern coefficient $e_1(J) < 0$ is equivalent to R being non Cohen-Macaulay. The conjecture is established if R is a homomorphic image of a Gorenstein ring, and for all universally catenary integral domains containing fields. Criteria for the detection of Cohen-Macaulayness in equi-generated graded modules are derived.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal. We will consider the set of multiplicative, decreasing filtrations of R ideals, $\mathbf{B} = \{I_n, I_0 = R, I_{n+1} = II_n, n \gg 0\}$, integral over the I-adic filtration. They are conveniently coded in the corresponding Rees algebra and its associated graded ring

$$\mathcal{R}(\mathbf{B}) = \sum_{n\geq 0} I_n t^n, \quad \mathrm{gr}_{\mathbf{B}}(R) = \sum_{n\geq 0} I_n/I_{n+1}.$$

One of our goals is to study cohomological properties of these filtrations. For that we will focus on the role of the Hilbert polynomial of the Hilbert–Samuel function $\lambda(R/I_{n+1})$,

$$H^1_{\mathbf{B}}(n) = P^1_{\mathbf{B}}(n) \equiv \sum_{i=0}^{d} (-1)^i e_i(\mathbf{B}) \binom{n+d-i}{d-i},$$

particularly of the multiplicity, $e_0(\mathbf{B})$, and the *Chern coefficient*, $e_1(\mathbf{B})$. For Cohen-Macaulay rings, many penetrating relationships between these coefficients have been proved, beginning with Northcott's [7]. More recently, similar questions have been examined in general Noetherian local rings and among those pertinent to our concerns are [3], [10] and [12].

Here we extend several of the results of [12] on the meaning of the sign of $e_1(\mathbf{B})$, particularly in the case of I-adic filtrations. Our main results are centered around the following question:

Let J be an ideal generated by a system of parameters. Under which conditions is $e_1(J) < 0$ equivalent to R being not Cohen-Macaulay? We conjecture that this is so

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whenever R is an unmixed, equidimensional local ring that is a homomorphic image of a Cohen-Macaulay local ring.

There are reasons for the interest in these numbers. To make the discussion more direct, we assume that the residue field of R is infinite.

(1) Clarifying the role of the sign of $e_1(J)$ in the Cohen-Macaulay property of R could be used as a *scale* to classify non-CM rings.

(2) In the study of the normalization $\mathbf{B}$ of $R[Jt]$, the expression (see [8], [12])

$$e_1(\mathbf{B}) - e_1(J)$$

bounds the length of certain computations in the construction of normalizations. It highlights the need to look for upper bounds of $e_1(\mathbf{B})$ and for lower bounds of $e_1(J)$.

(3) The value of $e_1(J)$ occurs as a correction term in the extensions of several well-known formulas in the theory of the Hilbert polynomials. We highlight two of them. A classical result of Northcott ([7]) asserts that if R is Cohen-Macaulay, then

$$e_1(I) \geq e_0(I) - \lambda(R/I).$$

For arbitrary Noetherian rings, if J is a minimal reduction of I, Goto and Nishida ([3, Theorem 3.1])proved that

$$e_1(I) - e_1(J) \geq e_0(I) - \lambda(R/I),$$

which gives

$$e_1(\mathbf{B}) - e_1(J) \geq e_0(I) - \lambda(R/\overline{I}),$$

since $e_1(\mathbf{B}) \geq e_1(I)$ for all such ideals I. It is a formula which is relevant to a conjecture of [12], on whether $e_1(\mathbf{B}) \geq 0$.

A different kind of relationship given by Huckaba and Marley ([4, Theorem 4.7]) for Cohen-Macaulay rings,

$$e_1(\mathbf{B}) \leq \sum_{n\geq 1} \lambda(I_n/JI_{n-1}),$$

is extended by Rossi and Valla ([10, Theorem 2.11]) to general filtrations to an expression that replaces $e_1(\mathbf{B})$ in the inequality above by $e_1(\mathbf{B}) - e_1(J)$.

In our main result (Theorem 3.3) we show that the above Conjecture holds if R is a homomorphic image of a Gorenstein ring, or milder extensions that allow an embedding of R into a (small) Cohen-Macaulay module over a possibly larger ring. The proof is a variation of an argument in [12], but turned more abstract. Another result establishes the Conjecture for universally catenary local domains containing a field (Theorem 4.4).

The same question can be asked about filtrations of modules regarding the negativity of the coefficient e_1 of the corresponding associated graded module. In case R is a polynomial ring over a field and M is a graded, torsion–free R-module, this extension has a surprising application to the Cohen-Macaulayness of M. In the special case of modules generated in the same degree, the Hilbert coefficient $e_1(M)$, which may be different from $e_1(\mathrm{gr}_{\mathbf{x}}(M))$, can alone decide whether M is Cohen-Macaulay or not (Corollary 5.4) .

We thank Rodney Sharp and Santiago Zarzuela for sharing with us their expertises on balanced big Cohen-Macaulay modules. We are also grateful to Shiro Goto for

sharing with us a sketch of his proof of a more extended version of Theorem 3.3. The Yokohama Conference on Commutative Algebra, on March 2008, was the setting where some of the questions arose.

2. Preliminaries

We will assemble quickly here some facts about associated graded modules and their Hilbert functions. As a general reference for unexplained terminology and basic results, we shall use [1].

Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal, and M a nonzero finite R–module of dimension d. The associated graded ring $\mathrm{gr}_I(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$ is a standard graded ring with $[\mathrm{gr}_I(R)]_0 = R/I$ Artinian. The associated graded module $\mathrm{gr}_I(M) = \bigoplus_{i=0}^{\infty} I^iM/I^{i+1}M$ of I with respect to M is a finitely generated graded $\mathrm{gr}_I(R)$–module. The *Hilbert–Samuel function* $\chi_M^I(n)$ of M with respect to I is

$$\chi_M^I(n) = \lambda(M/I^{n+1}M) = \sum_{i=0}^{n} \lambda(I^iM/I^{i+1}M).$$

For sufficiently large n, the Hilbert–Samuel function $\chi_M^I(n)$ is of polynomial type :

$$\chi_M^I(n) = \sum_{i=0}^{d} (-1)^i e_i(I, M) \binom{n+d-i}{d-i}.$$

For an R–module M of finite length, we denote the length of M by $\lambda(M)$.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$–primary ideal. Let $0 \to T \to M \to N \to 0$ be an exact sequence of finitely generated R–modules. Assume that M has dimension $d \geq 2$ and that T has finite length. Then $e_1(I, M) = e_1(I, N)$.*

Proof. From the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \cap I^{n+1}M & \longrightarrow & I^{n+1}M & \longrightarrow & I^{n+1}N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

we get an exact sequence

$$0 \longrightarrow T/(T \cap I^{n+1}M) \longrightarrow M/I^{n+1}M \longrightarrow N/I^{n+1}N \longrightarrow 0.$$

By the Artin–Rees Lemma, $T \cap I^{n+1}M = 0$ for all sufficiently large n. Hence we get for all $n \gg 0$

$$\lambda(T) - \lambda(M/I^{n+1}M) + \lambda(N/I^{n+1}N) = 0.$$

Let d' be the dimension of N. There are Hilbert polynomials such that

$$\lambda(M/I^{n+1}M) = \sum_{i=0}^{d} (-1)^i e_i(I, M) \binom{d+n-i}{d-i}$$

and

$$\lambda(N/I^{n+1}N) = \sum_{i=0}^{d'} (-1)^i e_i(I,N)\binom{d'+n-i}{d'-i}$$

for all sufficiently large n. Therefore $d = d'$,

$$e_i(I,M) = e_i(I,N) \quad \text{for all } i = 0,\ldots,d-1 \text{ and } e_d(I,N) = e_d(I,M) + (-1)^{d+1}\lambda(T).$$

In particular, since $d \geq 2$, we have $e_1(I,M) = e_1(I,N)$. □

Let R be a ring, I an R–ideal and M an R–module. An element $h \in I$ is called a *superficial element* of I with respect to M if there exists a positive integer c such that

$$(I^{n+1}M :_M h) \cap I^c M = I^n M$$

for all $n \geq c$. A detailed discussion on superficial elements can be found in several sources, but we especially benefited from the treatment in [9, Theorem 1.5] of the construction of superficial elements. It depends simply on showing that certain finitely generated modules cannot be written as a union of a finite set of proper submodules. The existence of such elements is guaranteed if the residue field of R is infinite. Its usefulness for our purposes is expressed in the following result.

Proposition 2.2. ([6, (22.6)]) *Let $(R,\mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal, and M a nonzero finitely generated R–module of dimension d. Let h be a superficial element of I with respect to M. Then the Hilbert coefficients of M and M/hM satisfy*

$$e_i(I,M) = \begin{cases} e_i(I/(h), M/hM) & \text{for } \; i < d-1. \\ e_{d-1}(I/(h), M/hM) + (-1)^{d-1}\lambda(0 :_M h). & \text{for } \; i = d-1. \end{cases}$$

3. Cohen-Macaulayness versus the vanishing of the Euler number

In this section we develop an abstract approach to the relationship between the signature of e_1 and the Cohen-Macaulayness of a local ring (see [12, Theorem 3.1]). We also give a more general but still self-contained proof of the main result of [12] that avoids the use of big Cohen-Macaulay modules.

Lemma 3.1. *Let $(S,\mathfrak{n})$ be a Cohen–Macaulay local ring of dimension d with infinite residue field and let $R = S/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime ideal of S. Let $x_1,\ldots,x_d$ be a system of parameters of R. Then there exists a system of parameters $a_1,\ldots,a_d$ of S such that $x_i = a_i + \mathfrak{p}$ for each i.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R and let $\mathfrak{p} = (c_1,\ldots,c_s)$. Let $x_1 = b_1 + \mathfrak{p}$ for some $b_1 \in S$ and let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the minimal primes of S different from $\mathfrak{p}$. We claim that there exists $\lambda \in S \setminus \mathfrak{n}$ such that $b_1 + \lambda c_1 + \cdots + \lambda^s c_s \notin \mathfrak{p}_i$ for all $i = 1,\ldots,n$. Suppose not: since $S/\mathfrak{n}$ is infinite, there exist $\lambda_1,\ldots,\lambda_{s+1} \in S \setminus \mathfrak{n}$ such that $\lambda_i + \mathfrak{n} \neq \lambda_j + \mathfrak{n}$ whenever $i \neq j$ and such that $b_1 + \lambda_i c_1 + \cdots + \lambda_i^s c_s \in \mathfrak{p}_k$ for some fixed k. Let A be the Vandermonde matrix determined by λ_i, $1 \leq i \leq s+1$. We have

$$A \begin{bmatrix} b_1 \\ c_1 \\ \vdots \\ c_s \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^s \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{s+1} & \lambda_{s+1}^2 & \cdots & \lambda_{s+1}^s \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ \vdots \\ c_s \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{s+1} \end{bmatrix},$$

where $g_1, \ldots, g_{s+1} \in \mathfrak{p}_k$. Since A is invertible, $c_1, \ldots, c_s \in \mathfrak{p}_k$ so that $\mathfrak{p} \subseteq \mathfrak{p}_k$, a contradiction. Let $a_1 = b_1 + \lambda c_1 + \cdots + \lambda^s c_s$, with $\lambda \in S \setminus \mathfrak{n}$, be such that $a_1 \notin \mathfrak{p}_i$ for all $i = 1, \ldots, n$. Hence a_1 is not contained in any minimal prime ideal of S and $a_1 + \mathfrak{p} = x_1$.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal primes of $a_1 S$ that do not contain $\mathfrak{p}$. Let $x_2 = b_2 + \mathfrak{p}$ for some $b_2 \in S$. Then similarly as shown above, there exists $\tau \in S \setminus \mathfrak{n}$ such that $b_2 + \tau c_1 + \cdots + \tau^s c_s \notin \mathfrak{q}_i$ for all i. Let $a_2 = b_2 + \tau c_1 + \cdots + \tau^s c_s$. Then a_2 is not contained in any minimal prime ideal of $a_1 S$ and $a_2 + \mathfrak{p} = x_2$. Now inductively we show that there exists a system of parameters $a_1, \ldots, a_d$ of S such that $a_i + \mathfrak{p} = x_i$ for all i. □

Lemma 3.2. *Let $(S, \mathfrak{n})$ be a Cohen–Macaulay complete local ring of dimension $d \geq 2$ and let M be a finitely generated S–module of dimension d satisfying Serre's condition (S_1). Then $H^1_{\mathfrak{n}}(M)$ is a finitely generated S–module.*

Proof. Let k denote the residue field of S and ω the canonical module of S. Since M satisfies Serre's condition (S_1), $\operatorname{Ext}_S^{d-1}(M, \omega)$ is zero at each localization at $\mathfrak{p}$ such that $\mathfrak{p} \neq \mathfrak{n}$. Thus it is a module of finite length. By Grothendieck duality ([1, 3.5.8]), we have

$$H^1_{\mathfrak{n}}(M) \simeq \operatorname{Hom}_S(\operatorname{Ext}_S^{d-1}(M, \omega), E(k)).$$

By Matlis duality([1, 3.2.13]), $H^1_{\mathfrak{n}}(M)$ is finitely generated. □

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. The enabling idea is the embedding of R into a Cohen-Macaulay (possibly big Cohen-Macaulay) module E,

$$0 \to R \longrightarrow E \longrightarrow C \to 0.$$

Unfortunately, it may not be always possible to find the appropriate R-module E. Instead we will seek embed R into a Cohen-Macaulay module E over a ring S closely related to R for the purpose of computing associated graded rings of adic-filtrations. The following is our main result.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension $d \geq 2$, which is a homomorphic image of a Cohen-Macaulay local ring $(S, \mathfrak{n})$. If R is not Cohen-Macaulay, then $e_1(J) < 0$ for any R–ideal J generated by a system of parameters.*

Proof. Let $R = S/\mathfrak{p}$. We may assume that S has infinite residue field. If $\operatorname{height}(\mathfrak{p}) \geq 1$, we replace S by S/L, where L is the S–ideal generated by a maximal regular sequence in $\mathfrak{p}$. This means that we may assume that $\dim R = \dim S$, and that $\mathfrak{p}$ is a minimal prime of S. In particular, we have an exact sequence of S-modules

$$0 \longrightarrow R \longrightarrow S \longrightarrow C \longrightarrow 0.$$

Let $J = (x_1, \ldots, x_d)$ be an R–ideal generated by a system of parameters. Then by Lemma 3.1, there exists a system of parameters $a_1, \ldots, a_d$ of S such that $x_i = a_i + \mathfrak{p}$ for each i. Let $I = (a_1, \ldots, a_d)S$. Since $IR = J$, the associated graded ring $\operatorname{gr}_I(R)$ of I with respect to the S–module R is equal to the associated graded ring $\operatorname{gr}_J(R)$ of J. In particular, $e_1(I, R) = e_1(J)$. Therefore, for the purpose of constructing $e_1(J)$, we treat R as an *S-module* and use the I-adic filtration.

Now proceed as in the proof of [12, Theorem 3.1]. Let a be a superficial element for I with respect to the S–module R which is not contained in any associated prime of C distinct from $\mathfrak{n}$. We may assume that $a = a_1$. Reduction modulo aS gives rise to the exact sequence

$$0 \to T = \mathrm{Tor}_1^S(S/aS, C) \longrightarrow R/aR \longrightarrow S/aS \longrightarrow C/aC \to 0,$$

where $T = (0 :_C a) \subset C$. This shows that the associated primes of T contain a. Therefore T is either zero, or T is a non-zero module of finite support.

Let $R' = R/aR$, $I' = I/(a)$, and denote the image of R/aR in S/aS by S'.

Now we use induction on $d \geq 2$ to show that if R is not Cohen–Macaulay, then $e_1(I, R) < 0$. Let $d = 2$. Notice that S' is a Cohen-Macaulay ring of dimension 1. We have that depth $C = 0$, and so $T \neq 0$. As in the proof of [12, Theorem 3.1], we obtain $e_1(I', R') = -\lambda(T)$. Hence by Proposition 2.2, $e_1(I, R) = -\lambda(T) < 0$.

Suppose $d > 2$. Consider the exact sequence of S/aS–modules : $0 \to T \to R' \to S' \to 0$. By Lemma 2.1, we have $e_1(I', R') = e_1(I', S')$ since $\dim(S/aS) = d - 1 \geq 2$. Now it is enough to show that S' is not a Cohen–Macaulay S/aS–module. Then since $\dim(S') = d - 1$, by induction we get $e_1(I', S') < 0$, and we conclude using Proposition 2.2.

Suppose that S' is a Cohen–Macaulay S/aS–module. Let $\mathfrak{n}$ denote the maximal ideal of S/aS as well and let $H^i_{\mathfrak{n}}(\cdot)$ denote the ith local cohomology. We are going to use the argument of [5, Proposition 2.1]. From the exact sequence $0 \to T \to R' \to S' \to 0$, we obtain a long exact sequence:

$$0 \to H^0_{\mathfrak{n}}(T) \to H^0_{\mathfrak{n}}(R') \to H^0_{\mathfrak{n}}(S') \to H^1_{\mathfrak{n}}(T) \to H^1_{\mathfrak{n}}(R') \to H^1_{\mathfrak{n}}(S').$$

Since S' is Cohen–Macaulay of dimension $d-1 \geq 2$, we have $H^0_{\mathfrak{n}}(S') = 0 = H^1_{\mathfrak{n}}(S')$. Since T is a torsion module, we have $H^0_{\mathfrak{n}}(T) = T$ and $H^1_{\mathfrak{n}}(T) = 0$. Therefore $T \simeq H^0_{\mathfrak{n}}(R')$ and $H^1_{\mathfrak{n}}(R') = 0$. Now from the exact sequence of S–modules

$$0 \longrightarrow R \xrightarrow{\cdot a} R \longrightarrow R/aR \longrightarrow 0$$

we obtain the following exact sequence:

$$0 \longrightarrow T \simeq H^0_{\mathfrak{n}}(R') \longrightarrow H^1_{\mathfrak{n}}(R) \xrightarrow{\cdot a} H^1_{\mathfrak{n}}(R) \longrightarrow H^1_{\mathfrak{n}}(R') = 0.$$

Therefore $H^1_{\mathfrak{n}}(R) = aH^1_{\mathfrak{n}}(R)$. Moreover, once R is embedded in S, we may assume that S is a complete local ring. By Lemma 3.2, $H^1_{\mathfrak{n}}(R)$ is finitely generated. By Nakayama Lemma, we have that $H^1_{\mathfrak{n}}(R) = 0$ so that $T = 0$. It follows that $R/aR = R' \simeq S'$, where S' is Cohen–Macaulay. This means that R is Cohen–Macaulay, which is a contradiction. □

Remark 3.4. The proof of Theorem 3.3 can be extended from integral domains to more general local rings, $R = S/L$, where S is Cohen-Macaulay and $\dim R = \dim S$, if R can be embedded into a maximal Cohen-Macaulay S-module. Notice that in order to embed R into a maximal Cohen–Macaulay S-module at a minimum we need to require that R be unmixed and equidimensional.

There is room for the following problem:

Problem 3.5. Let $R = S/L$ (unmixed and equidimensional as above), where S is a Cohen-Macaulay local ring of dimension $\dim R$. Characterize those R that can be embedded into a maximal Cohen–Macaulay S-module. Note that L may be assumed to be a primary ideal: If $L = \cap Q_i$ is a primary decomposition, we have the embedding

$$S/L \hookrightarrow S/Q_1 \oplus \cdots \oplus S/Q_n.$$

We make an elementary observation of what it takes to embed R into a free S-module (see [11, Theorem A.1]).

Proposition 3.6. *Let S be a Noetherian ring and L an ideal of codimension zero without embedded components. If $R = S/L$, there is an embedding $R \to F$ into a free S–module F if and only if $L = 0 : (0 : L)$. In particular, this condition always holds if the total ring of fractions of S is a Gorenstein ring.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a generating set of $0 : L$, and consider the mapping $\varphi : S \to S^n$, $\varphi(1) = (a_1, \ldots, a_n)$; its kernel is isomorphic to $0 : (0 : L)$. This shows that the equality $L = 0 : (0 : L)$ is required for the asserted embedding.

Conversely, given an embedding $\varphi : S/L \to S^n$, let $(a_1, \ldots, a_n) \in S^n$ be the image of a generator of S/L. The ideal $\mathfrak{a}$ these entries generate is annihilated by L, and so $\mathfrak{a} \subset 0 : L$. Since $0 : \mathfrak{a} = L$, we have $0 : (0 : L) \subset 0 : \mathfrak{a} = L$.

If the total ring of fractions of S is Gorenstein, to prove that $0 : (0 : L) \subset L$, it suffices to localize at the associated primes of L, all of which have codimension zero and a localization which is Gorenstein. But the double annihilator property is characteristic of such rings. □

Corollary 3.7. *Let $(R, \mathfrak{m})$ be an unmixed and equidimensional Noetherian local ring of dimension $d \geq 2$, which is a homomorphic image of a Gorenstein local ring. If R is not Cohen-Macaulay, then $e_1(J) < 0$ for any R–ideal J generated by a system of parameters.*

Proof. Let $R = S/L$. If $\text{height}(L) \geq 1$, we replace S by S/L', where L' is the S–ideal generated by a maximal regular sequence in L. So we may assume that $\dim R = \dim S$, and the conclusion follows by Proposition 3.6 and Remark 3.4. □

We give now a family of examples based on a method of [3].

Example 3.8. Let $(S, \mathfrak{m})$ be a regular local ring of dimension four, with an infinite residue field. Let $P_1, \ldots, P_r$ be a family of codimension two Cohen-Macaulay ideals such that for $i \neq j$, $P_i + P_j$ is an $\mathfrak{m}$-primary ideal. Define $R = S/\cap_i P_i$.

Consider the exact sequence of S-modules

$$0 \to R \longrightarrow \bigoplus_i S/P_i \longrightarrow L \to 0.$$

Note that L is a module of finite support; it may be identified with $H^1_{\mathfrak{m}}(R)$. Let $J = (a, b)$ be an ideal of R forming a system of parameters, contained in the annihilator of L.[1] We can assume that $a, b \in S$ form a regular sequence in each S/P_i. We are going to determine $e_1(J)$.

[1]We thank Jugal Verma for this observation.

For each integer n, tensoring by $S/(a,b)^n$ we get the exact sequence

$$0 \to \mathrm{Tor}_1^S(L, S/(a,b)^n) \to R/(a,b)^n \to \bigoplus_i S/(P_i,(a,b)^n) \to L \otimes_S S/(a,b)^n \to 0.$$

For $n \gg 0$, $(a,b)^n L = 0$, so we have that $L \otimes_S S/(a,b)^n = L$ and $\mathrm{Tor}_1^S(L, S/(a,b)^n) = L^{n+1}$, from the Burch-Hilbert $(n+1)\times n$ resolution of the ideal $(a,b)^n$. Since the $R_i = S/P_i$ are Cohen-Macaulay, we obtain the following Hilbert-Samuel polynomial:

$$e_0(J)\binom{n+2}{2} - e_1(J)\binom{n+1}{1} + e_2(J) = (\sum_{i=1}^r e_0(JR_i))\binom{n+2}{2} + (n+2)\lambda(L) - \lambda(L).$$

It gives

$$\begin{aligned} e_0(JR) &= \sum_{i=1}^r e_0(JR_i), \\ e_1(JR) &= -\lambda(L), \\ e_2(JR) &= 0. \end{aligned}$$

4. Embedding into balanced big Cohen-Macaulay modules

Let $(R, \mathfrak{m})$ be a Noetherian local domain. If R has a big Cohen-Macaulay module E, any nonzero element of E allows for an embedding $R \hookrightarrow E$. In fact, one may assume that E is a balanced big Cohen-Macaulay module (see [1, Section 8.5] for a discussion). According to the results of Hochster, if R contains a field, then there is a balanced big Cohen-Macaulay R-module E ([1, 8.4.2]).

To use the argument in [12, Theorem 3.2], in the exact sequence

$$0 \to R \longrightarrow E \longrightarrow C \to 0,$$

we should, given any parameter ideal J of R, pick an element superficial for the purpose of building $\mathrm{gr}_J(R)$ (if $\dim R > 2$) and not contained in any associated prime of C different from $\mathfrak{m}$. This is possible if the cardinality of the residue field is larger than the cardinality of $\mathrm{Ass}(C)$.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local integral domain that is not Cohen-Macaulay and let E be a balanced big Cohen-Macaulay module. If the residue field of R has cardinality larger than the cardinality of a generating set for E, then $e_1(J) < 0$ for any parameter ideal J.*

Let $\mathbf{X}$ be a set of indeterminates of larger cardinality than $\mathrm{Ass}(C)$, and consider $R(\mathbf{X}) = R[\mathbf{X}]_{\mathfrak{m}R[\mathbf{X}]}$. This is a Noetherian ring ([2]), and we are going to argue that if E is a balanced big Cohen-Macaulay R-module, then $R(\mathbf{X}) \otimes_R E$ is a balanced big Cohen-Macaulay module over $R(\mathbf{X})$. S. Zarzuela has kindly pointed out to us the following result:

Theorem 4.2 ([13, Theorem 2.3])**.** *Let $A \to B$ be a flat morphism of local rings $(A, \mathfrak{m}), (B, \mathfrak{n})$ and M a balanced big Cohen-Macaulay A-module. Then, $M \otimes_A B$ is a balanced big Cohen-Macaulay B-module if and only if the following two conditions hold:*

(i) $\mathfrak{n}(M \otimes_A B) \neq M \otimes_A B$ *and*

(ii) *For any prime ideal* $\mathfrak{q} \in \operatorname{supp}_B(M \otimes_A B)$, (1) $\operatorname{height}(\mathfrak{q}/\mathfrak{p}B) = \operatorname{depth}(C_{\overline{\mathfrak{q}}})$ *and* (2) $\operatorname{height}(\mathfrak{q}) + \dim(B/\mathfrak{q}) = \dim(B)$.

Here, we denote by $\operatorname{supp}_A(M)$ *(small support) the set of prime ideals in* A *with at least one non-zero Bass number in the* A*-minimal injective resolution of* M, $\mathfrak{p} = \mathfrak{q} \cap A$, $C = B/\mathfrak{p}B$ *and* $\overline{\mathfrak{q}} = \mathfrak{q}C$.

Moreover, if $\mathfrak{q} \in \operatorname{supp}_B(M \otimes_A B)$ *then* $\mathfrak{p} \in \operatorname{supp}_A(M)$, *and* $\operatorname{height}(A/\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$.

Corollary 4.3. *Let* $(R, \mathfrak{m})$ *be a universally catenary integral domain and let* E *be a balanced big Cohen-Macaulay* R*-module. For any set* $\mathbf{X}$ *of indeterminates and* $B = R(\mathbf{X}) = R[\mathbf{X}]_{\mathfrak{m}[\mathbf{X}]}$, $B \otimes_R E$ *is a balanced big Cohen-Macaulay* B*-module.*

Theorem 4.4. *Let* $(R, \mathfrak{m})$ *be a universally catenary integral domain containing a field. If* R *is not Cohen-Macaulay, then* $e_1(J) < 0$ *for any parameter ideal* J.

Proof. Let E be a balanced big Cohen–Macaulay R–module ([1, 8.4.2]) and consider the exact sequence $0 \to R \to E \to C \to 0$. Let $\mathbf{X}$ be a set of indeterminates of larger cardinality than $\operatorname{Ass}(C)$, and let $R(\mathbf{X}) = R[\mathbf{X}]_{\mathfrak{m}R[\mathbf{X}]}$. By applying Theorem 4.1 to the exact sequence $0 \to R(\mathbf{X}) \to E \otimes R(\mathbf{X}) \to C \otimes R(\mathbf{X}) \to 0$, the assertion is proved. □

5. Filtered modules

The same relationship discussed above between the signature of $e_1(J)$ and the Cohen-Macaulayness of R holds true when modules are examined. Recall that if a Noetherian local ring R is embedded into either a maximal Cohen–Macaulay module ([12, Theorem 3.1]) or a balanced big Cohen–Macaulay module ([12, proof of Theorem 3.2]), then whenever R is not Cohen–Macaulay, we have $e_1(J) < 0$ for any parameter ideal J. Now we use the same arguments as in [12, Theorems 3.1, 3.2] in order to extend the validity of Theorem 3.3 in the following manner.

Theorem 5.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$ *and let* M *be a finitely generated module embedded in a maximal Cohen–Macaulay module* E. *Then* M *is Cohen-Macaulay if and only if* $e_1(J, M) \geq 0$ *for any ideal* J *generated by a system of parameters of* M.

A variation that uses Theorem 4.4 is the following.

Theorem 5.2. *Let* $(R, \mathfrak{m})$ *be a universally catenary integral domain containing a field and let* M *be a finitely generated torsion–free* R*-module. If* M *is not Cohen-Macaulay, then* $e_1(J, M) < 0$ *for any ideal* J *generated by a system of parameters of* M.

Proof. By assumption M is a submodule of a finitely generated free R-module, which can be embedded into a finite direct sum of balanced big Cohen-Macaulay modules. The argument of [12, Theorem 3.2] applies again. □

Let now $R = k[x_1, \ldots, x_d]$ be a ring of polynomials over the field k, and let M be a finitely generated graded R-module. Suppose $\dim M = d$. For $J = (x_1, \ldots, x_d)$ we can apply Theorem 5.1 to M in a manner that uses the Hilbert–Samuel function information of the native grading of M.

Theorem 5.3. *Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_d]$, $d \geq 2$, be a ring of polynomials over the field k, and let M be a finitely generated graded R-module generated in degree 0. If M is torsion–free, then M is a free R-module if and only if $e_1(M) = 0$.*

Proof. Since M is generated in degree 0, $M \simeq \text{gr}_{\mathbf{x}}(M)$. By assumption, M can be embedded in a free R-module E (not necessarily by a homogeneous homomorphism). Now we apply Theorem 5.1. □

If M is generated in degree $a > 0$, we have the equality

$$\lambda(M/(\mathbf{x})^{n+1}M) = \sum_{k=0}^{n} \lambda(M_{a+k}),$$

so the Hilbert coefficients satisfy

$$\begin{aligned} e_0(\text{gr}_{\mathbf{x}}(M)) &= e_0(M[a]) = e_0(M), \\ e_1(\text{gr}_{\mathbf{x}}(M)) &= e_1(M[a]) = e_1(M) - ae_0(M). \end{aligned}$$

Corollary 5.4. *Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_d]$, $d \geq 2$, be a ring of polynomials over the field k, and let M be a finitely generated graded R-module generated in degree $a \geq 0$. If M is torsion–free, then $e_1(M) \leq ae_0(M)$, with equality if and only if M is a free R-module.*

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