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THE ORLIK-TERAO ALGEBRA AND 2-FORMALITY

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Abstract. The Orlik-Solomon algebra is the cohomology ring of the complement of a hyperplane arrangement $\mathcal{A} \subseteq \mathbb{C}^n$; it is the quotient of an exterior algebra $\Lambda(V)$ on $|\mathcal{A}|$ generators. In [9], Orlik and Terao introduced a commutative analog $Sym(V^*)/I$ of the Orlik-Solomon algebra to answer a question of Aomoto and showed the Hilbert series depends only on the intersection lattice $L(\mathcal{A})$. In [6], Falk and Randell define the property of 2-formality; in this note we study the relation between 2-formality and the Orlik-Terao algebra. Our main result is a necessary and sufficient condition for 2-formality in terms of the quadratic component I_2 of the Orlik-Terao ideal I. The key is that 2-formality is determined by the tangent space $T_p(V(I_2))$ at a generic point p.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be an arrangement of complex hyperplanes in $\mathbb{C}^n$. In [7], Orlik and Solomon showed that the cohomology ring of the complement $X = \mathbb{C}^n \setminus \bigcup_{i=1}^d H_i$ is determined by the intersection lattice

$$L(\mathcal{A}) = \{ \bigcap_{H \in \mathcal{A}'} H \mid \mathcal{A}' \subseteq \mathcal{A}\}.$$

The Orlik-Solomon algebra $H^*(X, \mathbb{Z})$ is the quotient of the exterior algebra $E = \bigwedge(\mathbb{Z}^d)$ on generators $e_1, \dots, e_d$ in degree 1 by the ideal generated by all elements of the form

$$\partial e_{i_1 \dots i_r} := \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r},$$

for which $\operatorname{codim} H_{i_1} \cap \cdots \cap H_{i_r} < r$. Throughout this paper, we work with an essential, central arrangement of d hyperplanes; this means we may always assume $L(\mathcal{A})$ has rank n and

$$\mathcal{A} = \bigcup_{i=1}^d V(\alpha_i) \subseteq \mathbb{P}^{n-1},$$

where α_i are distinct homogeneous linear forms such that $H_i = V(\alpha_i)$. Write $[d]$ for $\{1, \dots, d\}$ and let $\Lambda = \{i_1, \dots, i_k\} \subset [d]$. If $codim(\bigcap_{j=1}^k H_{i_j}) < k$, then there are constants c_t such that

$$\sum_{t \in \Lambda} c_t \alpha_t = 0.$$

In [9], Orlik and Terao introduced the following commutative analog of the Orlik-Solomon algebra in order to answer a question of Aomoto.

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Definition 1.1. *For each dependency* $\Lambda = \{i_1, \ldots, i_k\}$*, let* $r_\Lambda = \sum_{j=1}^{k} c_{i_j} y_{i_j} \in R = \mathbb{K}[y_1, \ldots, y_d]$*. Define* $f_\Lambda = \partial(r_\Lambda) = \sum_{j=1}^{k} c_{i_j}(y_{i_1} \cdots \hat{y}_{i_j} \cdots y_{i_k})$*, and let* I *be the ideal generated by the* f_Λ*. The* ***Orlik-Terao algebra*** OT *is the quotient of* $\mathbb{K}[y_1, \ldots, y_d]$ *by* I*. The* ***Artinian Orlik-Terao algebra*** *is the quotient of* OT *by* $\langle y_1^2, \ldots, y_d^2 \rangle$*.*

Orlik and Terao actually study the Artinian version, but for our purposes the OT algebra will turn out to be more interesting. The crucial difference between the Orlik-Solomon algebra and Orlik-Terao algebra(s) is not the difference between the exterior algebra and symmetric algebra, but rather the fact that the Orlik-Terao algebra actually records the "weights" of the dependencies among the hyperplanes. So in any investigation where actual dependencies come into play, the OT algebra is the natural candidate for study.

1.1. 2-Formality.

In [6], Falk and Randell introduced the concept of 2-formality:

Definition 1.2. *For an arrangement* $\mathcal{A}$*, the* relation space $F(\mathcal{A})$ *is the kernel of the evaluation map* ϕ*:*

$$\bigoplus_{i=1}^{d} \mathbb{K}e_i \stackrel{e_i \mapsto \alpha_i}{\longrightarrow} \mathbb{K}[x_1, \ldots, x_n]_1.$$

$\mathcal{A}$ *is* 2-formal *if* $F(\mathcal{A})$ *is spanned by relations involving only three hyperplanes.*

Example 1.3. Suppose we have an arrangement of 4 lines in $\mathbb{P}^2$ given by the linear forms: $\alpha_1 = x_1, \alpha_2 = x_2, \alpha_3 = x_3, \alpha_4 = x_1 + x_2 + x_3$. Obviously any three of the forms are independent, so the only relation is $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$. Hence the OT algebra is

$$\mathbb{K}[y_1, \ldots, y_4]/\langle y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3 \rangle.$$

This arrangement cannot be 2-formal, since there are no relations involving only three lines, whereas $F(\mathcal{A})$ is nonzero.

Many interesting classes of arrangements are 2-formal: in [6], Falk and Randell proved that $K(\pi, 1)$ arrangements and arrangements with quadratic Orlik-Solomon ideal are 2-formal. In [3], Brandt and Terao generalized the notion of 2-formality to $k-$formality, proving that every free arrangement is $k-$formal. Formality of discriminantal arrangements is studied in [1], with surprising connections to fiber polytopes [2]. In [17], Yuzvinsky shows that free arrangements are 2-formal; and gives an example showing that 2-formality does not depend on $L(\mathcal{A})$.

Example 1.4. Consider the following two arrangements of lines in $\mathbb{P}^2$:

$$\mathcal{A}_1 = V(xyz(x+y+z)(2x+y+z)(2x+3y+z)(2x+3y+4z)(3x+5z)(3x+4y+5z))$$
$$\mathcal{A}_2 = V(xyz(x+y+z)(2x+y+z)(2x+3y+z)(2x+3y+4z)(x+3z)(x+2y+3z)).$$

For a graded R–module M, the graded betti numbers of M are

$$b_{ij} = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M, \mathbb{K})_{i+j}.$$

We shall be interested in the case when $M = R/I$. The graded betti numbers of the Orlik-Terao and Artinian Orlik-Terao algebras of $\mathcal{A}_1$ and $\mathcal{A}_2$ are identical; for the

Orlik-Terao algebras the graded betti numbers are:

total	1	26	120	216	190	84	15
0	1	–	–	–	–	–	–
1	–	6	–	–	–	–	–
2	–	20	120	216	190	84	15

The diagram entry in position (i, j) is simply b_{ij}, so for example,

$$\dim_{\mathbb{K}} \mathrm{Tor}_2^R(R/I, \mathbb{K})_4 = 120.$$

Yuzvinsky shows that $\mathcal{A}_1$ is 2-formal, and $\mathcal{A}_2$ is not. The arrangements have the same intersection lattice and appear identical to the naked eye:

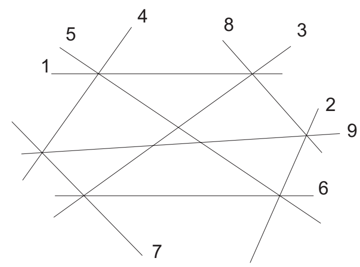

FIGURE 1.

The difference is that the six multiple points of $\mathcal{A}_1$ lie on a smooth conic, while the six multiple points of $\mathcal{A}_2$ do not. The quadratic OT algebra is $\mathbb{K}[y_1, \ldots, y_d]/I_2$, where I_2 consists of the quadratic generators of I. The graded betti numbers of the quadratic OT algebra of $\mathcal{A}_1$ are:

total	1	6	15	20	15	6	1
0	1	–	–	–	–	–	–
1	–	6	–	–	–	–	–
2	–	–	15	–	–	–	–
3	–	–	–	20	–	–	–
4	–	–	–	–	15	–	–
5	–	–	–	–	–	6	–
6	–	–	–	–	–	–	1

while the quadratic OT algebra of $\mathcal{A}_2$ has betti diagram:

total	1	6	20	31	21	5
0	1	–	–	–	–	–
1	–	6	–	–	–	–
2	–	–	20	16	5	–
3	–	–	–	15	16	5

This paper is motivated by the question raised by the previous example: "Is 2-formality determined by the quadratic OT algebra?" Our main result is:

Theorem: Let $\mathcal{A}$ be an arrangement of d hyperplanes of rank n. $\mathcal{A}$ is 2-formal if and only if $codim(I_2) = d - n$.

As noted, the class of 2-formal arrangements is quite large, and includes free arrangements, $K(\pi, 1)$ arrangements, and arrangements with quadratic Orlik-Solomon ideal. The previous theorem gives necessary and sufficient conditions for 2-formality in terms of the Orlik-Terao algebra. We close the introduction with several examples.

Definition 1.5. *Let G be a simple graph on ν vertices, with edge-set E, and let $\mathcal{A}_G = \{z_i - z_j = 0 \mid (i,j) \in \mathsf{E}\}$ be the corresponding arrangement in $\mathbb{C}^\nu$.*

For a graphic arrangement $\mathcal{A}_G$ it is obvious that $d = |\mathsf{E}|$, and easy to show that $\operatorname{rank} \mathcal{A}_G = \nu - 1$. In [16], Tohaneanu showed that a graphic arrangement $\mathcal{A}_G$ is 2-formal exactly when $H_1(\Delta_G) = 0$, where Δ_G is the clique complex of G–a simplicial complex, whose i-faces correspond to induced complete subgraphs on $i + 1$ vertices.

Example 1.6. The clique complex for G as in Figure 2 consists of the four triangles (and all vertices and edges). Clearly $H_i(\Delta_G) = 0$ for all $i \geq 1$, so $\mathcal{A}_G$ is 2-formal.

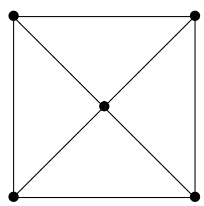

FIGURE 2.

The graded betti numbers of OT are

total	1	5	10	9	3
0	1	–	–	–	–
1	–	4	–	–	–
2	–	1	9	3	–
3	–	–	1	6	3
4	–	–	–	–	15

while the quadratic OT algebra of $\mathcal{A}_G$ has betti diagram:

total	1	4	6	4	1
0	1	–	–	–	–
1	–	4	–	–	- -
2	–	–	6	–	–
3	–	–	–	4	–
4	–	–	–	–	1

Since I_2 is clearly a complete intersection, $codim(I_2) = 4 = 8 - (5 - 1) = d - n$, giving another proof of 2-formality for this configuration.

While the examples of 2-formal arrangements encountered thus far have I_2 a complete intersection, this is generally not the case.

Example 1.7. The Non-Fano arrangement is the unique configuration of seven lines in $\mathbb{P}^2$ having six triple points. It is free, hence 2-formal. The graded betti numbers of OT are

total	1	7	14	12	4
0	1	–	–	–	–
1	–	6	5	–	–
2	–	1	9	12	4

while the quadratic OT algebra has betti diagram:

total	1	6	10	6	1
0	1	–	–	–	–
1	–	6	5	–	–
2	–	–	5	6	–
3	–	–	–	–	1

So I_2 is not a complete intersection, though it is Gorenstein. In general, I_2 need not even be Cohen-Macaulay, and often has multiple components. For the Non-Fano arrangement the primary decomposition of I_2 is

$$I_2 = I \cap \langle y_0, y_1, y_3, y_4 \rangle$$

In particular, if $I \neq I_2$, the primary decomposition is nontrivial.

When $\mathcal{A}$ is 2-formal, it is possible to give a simple necessary and sufficient combinatorial criterion for I_2 to be a complete intersection.

Corollary 1.8. *Let $\mathcal{A}$ be a 2-formal arrangement of rank n, with $|\mathcal{A}| = d$. Then I_2 is a complete intersection iff*

$$b_2 = \binom{d}{2} - d + n$$

Proof. First, since b_2 is the dimension of the Orlik-Solomon algebra in degree two,

$$b_2 = \dim_{\mathbb{K}} \Lambda^2(\mathbb{K}^d) - \dim_{\mathbb{K}} J_2,$$

where J denotes the Orlik-Solomon ideal. Since the Artinian Orlik-Terao algebra has the same Hilbert series as the Orlik-Solomon algebra, this implies

$$b_2 = \dim_{\mathbb{K}}(R/I + \langle y_1^2, \ldots, y_d^2 \rangle)_2$$

Now, clearly

$$\dim_{\mathbb{K}}(R/\langle y_1^2, \ldots, y_d^2 \rangle)_2 = \dim_{\mathbb{K}} \Lambda^2(\mathbb{K}^d).$$

In the next section, it is proved that I is prime. It follows from this that no $y_i^2 \in I_2$. Combining, we obtain that

$$\dim_{\mathbb{K}} I_2 = \dim_{\mathbb{K}} J_2.$$

Now, I_2 is a complete intersection if $\text{codim}(I_2) = \dim_{\mathbb{K}}(I_2)$. By the assumption of 2-formality, this concludes the proof. □

2. The quadratic Orlik-Terao algebra and 2-Formality

We keep the setup of the previous section: $\mathcal{A}$ is an essential, central arrangement of d hyperplanes in $\mathbb{P}^{n-1}$, with relation space $F(\mathcal{A})$. Since $\dim_{\mathbb{K}} F(\mathcal{A}) = d - n$, $\mathcal{A}$ is 2-formal if and only iff $\dim_K(span_{\mathbb{K}}(\{\text{3-relations}\})) = d - n$. Fix defining linear forms α_i so that $H_i = V(\alpha_i)$, let $I \subseteq R = \mathbb{K}[y_1, \ldots, y_d]$ denote the OT ideal, and I_2 the quadratic component of I.

Proposition 2.1. *I is prime, and $V(I)$ is a rational variety of codimension $d - n$.*

Proof. Consider the map

$$R \xrightarrow{\Phi} \mathbb{K}\left[\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_d}\right] = C(\mathcal{A})$$

given by $y_i \mapsto \frac{1}{\alpha_i}$. An easy check shows that $I \subseteq \ker \Phi$. Our assumption that $\mathcal{A}$ is essential implies that

$$\mathbb{K}\left[\frac{1}{y_1}, \ldots, \frac{1}{y_n}\right] \subseteq \mathbb{K}\left[\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_d}\right],$$

hence the field of fractions of $R/\ker(\Phi)$ is $\mathbb{K}(y_1, \ldots, y_n)$, giving rationality and the appropriate dimension (as an affine cone). In [15], Terao proved that the Hilbert series for $C(\mathcal{A})$ is given by

$$HS(C(\mathcal{A}), t) = P\Big(\mathcal{A}, \frac{t}{1-t}\Big).$$

where $P(\mathcal{A}, t)$ is the Poincaré polynomial of $\mathcal{A}$. If H is a hyperplane of $\mathcal{A}$, the *deletion* $\mathcal{A}'$ is the subarrangement $\mathcal{A} \setminus H$ of $\mathcal{A}$, and the *restriction* $\mathcal{A}''$ is the arrangement $\{H' \cap H \mid H' \in \mathcal{A}'\}$, considered as an arrangement in the vector space H, and there is a relation

$$P(\mathcal{A}, t) = P(\mathcal{A}', t) + tP(\mathcal{A}'', t).$$

Thus the Hilbert series of $C(\mathcal{A})$ satisfies the recursion

$$HS(C(\mathcal{A}), t) = HS(C(\mathcal{A}'), t) + \frac{t}{1-t} HS(C(\mathcal{A}''), t)$$

If the quotient R/I satisfies the same recursion, then $I = \ker \Phi$ will follow by induction. Let y_1 be a variable corresponding to $H = H_1$. In [10], Proudfoot and Speyer prove that the broken circuits are a universal Gröbner basis for I, hence in particular a lex basis. Let $R = \mathbb{K}[y_1, \ldots, y_n]$ and $R' = \mathbb{K}[y_2, \ldots, y_n]$, and consider the short exact sequence

$$0 \longrightarrow R(-1)/(in(I) : y_1) \xrightarrow{\cdot y_1} R/in(I) \longrightarrow R/(in(I), y_1) \longrightarrow 0.$$

The initial ideal of I has the form

$$in(I) = \langle f_1, \ldots, f_k, y_1 \cdot g_1, \ldots, y_1 \cdot g_m \rangle,$$

with f_i, g_j not divisible by y_1. Clearly

$$in(I) : y_1 = \langle f_1, \ldots, f_k, g_1, \ldots, g_m \rangle$$

In the rightmost module of the short exact sequence, quotienting by y_1 kills the generator and all relations involving y_1, hence

$$R/(in(I), y_1) \simeq R'/\langle f_1, \dots, f_k \rangle = R'/in(I'),$$

with I' denoting the OT ideal of $\mathcal{A}'$. In the leftmost module, since no relation of $in(I) : y_1$ involves y_1, $in(I) : y_1 \subseteq R'$, so taking into account the degree shift and the fact that y_1 acts freely on the module, we see

$$HS(R(-1)/(in(I) : y_1), t) = \frac{t}{1-t} HS(R'/\langle f_1, \dots, f_k, g_1, \dots, g_m \rangle, t).$$

We now claim that $HS(R'/\langle f_1, \dots, f_k, g_1, \dots, g_m \rangle, t) = HS(C(A''), t)$. To see this, note that the initial monomials f_k express the fact that the dependencies among hyperplanes of $\mathcal{A}'$ continue to hold in $\mathcal{A}''$. The relations g_j represent the "collapsing" that occurs on restricting to H_1. For example, if there is a circuit (123) in $\mathcal{A}$, this means that $H_2|_{H_1} = H_3|_{H_1}$. In OT($\mathcal{A}$), the relation on (123) has initial term $y_1 y_2$, hence $y_2 \in in(I) : y_1$. This reflects the "redundancy" $H_2|_{H_1} = H_3|_{H_1}$ in the restriction. Similar reasoning applies to the relations g_i of degree greater than one, leading to:

$$HS(R'/\langle f_1, \dots, f_k, g_1, \dots, g_m \rangle, t) = HS(C(\mathcal{A}''), t) = P\Big(\mathcal{A}'', \frac{t}{1-t}\Big).$$

Combining this with an induction and Terao's formula shows that $I = \ker \Phi$. □

Corollary 2.2. *The variety $V(I)$ is nondegenerate.*

Proof. Since I is prime, $V(I)$ will be contained in a hyperplane $V(L)$ iff $L \in I$. However, the assumption that the hyperplanes of $\mathcal{A}$ are distinct implies that there are no dependencies involving only two hyperplanes, so that I is generated in degree greater than two. In particular, I contains no linear forms. □

Theorem 2.3. *Let $\mathcal{A}$ be an arrangement of d hyperplanes of rank n. $\mathcal{A}$ is 2-formal if and only if $X = V(I_2) \cap (\mathbb{C}^*)^{d-1}$ has codimension $d-n$.*

Proof. The proof hinges on the fact that for $p \in (\mathbb{C}^*)^{d-1} \subseteq \mathbb{P}^{d-1}$, the Jacobian ideal of I_2, evaluated at p, is (after a linear change of coordinates) exactly the matrix recording the dependencies among triples of hyperplanes.

Let

$$f = a y_2 y_3 + b y_1 y_3 + c y_1 y_2 = \partial(a y_1 + b y_2 + c y_3), a, b, c \neq 0,$$

be a generator of I_2, let $p = (p_1 : \dots : p_d)$ be a point on X, so mp $p_i = 0$. Such points form a dense open subset of $V(I)$ by Corollary 2.2. The Jacobian matrix of the generators of I_2 has a special form when evaluated at $p \in V(I_2) \cap (\mathbb{C}^*)^{d-1}$. Write f_i for $\partial(f)/\partial(y_i)$. For f as above, if $i \geq 4$ then $f_i = 0$, and

$$\begin{aligned} f_1 &= b y_3 + c y_2 \\ f_2 &= a y_3 + c y_1 \\ f_3 &= a y_2 + b y_1 \end{aligned}$$

Evaluating the partials of f at p yields:

$$f_1(p) = p_2 p_3 (b\frac{1}{p_2} + c\frac{1}{p_3}),\ f_2(p) = p_1 p_3 (a\frac{1}{p_1} + c\frac{1}{p_3}),\ f_3(p) = p_1 p_2 (a\frac{1}{p_1} + b\frac{1}{p_2}).$$

Since $f(p) = 0$, $a\frac{1}{p_1} + b\frac{1}{p_2} + c\frac{1}{p_3} = 0$, and simplifying using this relation, we obtain:

$$f_1(p) = -\frac{p_2 p_3}{p_1}a,\ \ f_2(p) = -\frac{p_1 p_3}{p_2}b,\ \ f_3(p) = -\frac{p_1 p_2}{p_3}c.$$

The p_i are nonzero, so rescaling shows that the row of $J_p(I_2)$ corresponding to f is:

$$\left[\frac{a}{p_1^2}, \frac{b}{p_2^2}, \frac{c}{p_3^2}, 0, \ldots, 0\right]$$

Multiplying $J_p(I_2)$ on the right by the diagonal invertible matrix with (i,i) entry p_i^2 yields a matrix whose entries encode the dependencies among triples of the forms l_i. Hence rank $J_p(I_2)$ is exactly the dimension of the space of three relations. □

Theorem 2.4. *$\mathcal{A}$ is 2-formal if and only if codim$(I_2) = d - n$.*

Proof. When $I = I_2$, the result follows from Theorem 2.3. There are two cases to consider. As in Example 1.7, it is possible that $I \neq I_2$ but

$$\dim V(I) = \dim V(I_2).$$

In this case, the theorem holds, simply from the coincidence of dimensions. The second case is that $\dim V(I) < \dim V(I_2)$. In this case, for a smooth point $p \in V(I_2)$, the dimension of $T_p(V(I_2))$ will be greater than $d - n$, hence

$$\operatorname{rank} J_p(I_2) < d - n.$$

The dimension of the tangent space can only increase at singular points, so $\operatorname{rank} J_p(I_2) < d - n$ at all points of $V(I_2)$, and thus the three–relations cannot span the relation space. □

Proposition 2.5. *If $\mathcal{A}$ is supersolvable, then $I = I_2$.*

Proof. Fix the reverse lexicographic order on $R = \mathbb{K}[y_1, \ldots, y_d]$. Suppose C is a circuit with $|C| = p \geq 4$, and that $\partial(C) \notin I_2$. From the set of circuits with $\partial(C) \notin I_2, |C| = p$, select the circuit $C = \{H_{j_1}, \ldots, H_{j_p}\}, j_1 < \cdots < j_p$ which has maximal lead term $M = y_{j_2} \cdots y_{j_p}$. Since $\mathcal{A}$ is supersolvable, there exists $j_r, j_s, 1 \leq r < s \leq p$ and $u > j_s$ such that $D = \{H_{j_r}, H_{j_s}, H_u\}$ is a circuit.

If $u \in \{j_1, j_2, \ldots, j_p\}$ then C contains D, which would contradict the fact that C is a circuit with $|C| \geq 4$. Hence $u \notin \{j_1, j_2, \ldots, j_p\}$. So D gives rise to a dependency:

$$D = b_r\alpha_{j_r} + b_s\alpha_{j_s} + \alpha_u = 0, b_r, b_s \neq 0,$$

yielding an element

$$\partial(D) = y_{j_r}y_{j_s} + b_r y_{j_s}y_u + b_s y_{j_r}y_u \in I_2.$$

Note that C also gives a dependency $a_1\alpha_{j_1} + \cdots + a_p\alpha_{j_p} = 0, a_k \neq 0$ and corresponding element

$$\partial(C) = a_r y_{j_1} \cdots \widehat{y_{j_r}} \cdots y_{j_p} + a_s y_{j_1} \cdots \widehat{y_{j_s}} \cdots y_{j_p} + y_{j_r}y_{j_s}P \in I,$$

with $P = a_1 y_{j_2} \cdots \widehat{y_{j_r}} \cdots \widehat{y_{j_s}} \cdots y_{j_p} + \cdots + a_p y_{j_1} \cdots \widehat{y_{j_r}} \cdots \widehat{y_{j_s}} \cdots y_{j_{p-1}}$. Consider the dependencies $C_r = a_r D - b_r C$ and $C_s = a_s D - b_s C$. Writing $y_{j_r}y_{j_s} = \partial(D) - b_r y_{j_s}y_u - b_s y_{j_r}y_u$ and substituting into the expression for $\partial(C)$ yields

$$\partial(C) = \partial(C_r) + \partial(C_s) + \partial(D)P.$$

Now note that C_r and C_s are circuits of cardinality p with leading terms of $\partial(C_r)$ and $\partial(C_s)$ greater than M, since we replaced the variable y_{j_r} or y_{j_s} by y_u, with $u > j_s > j_r$. The choice of M and C now implies that $\partial(C_r)$ and $\partial(C_s)$ are both in I_2, a contradiction. □

3. Combinatorial Syzygies

Example 1.4 shows that the module of first syzygies on I_2 is not determined by combinatorial data. In this section we study *linear* first syzygies. First, we examine the syzygies which arise from an $X \in L_2(\mathcal{A})$ with $\mu(X) \geq 3$. If $\mu(X) = d-1$, then d hyperplanes $H_1, \ldots, H_d$ pass thru X. To simplify, we localize to the rank two flat, so that $\mathcal{A}$ consists of d points in $\mathbb{P}^1$.

Theorem 3.1. *Suppose $X \in L_2(\mathcal{A})$ has $\mu(X) = d-1 \geq 3$ and let $I \subseteq R = \mathbb{K}[y_1, \ldots, y_d] \subseteq \mathbb{K}[y_1, \ldots, y_n]$ be the subideal of I_2 corresponding to $\mathcal{A}_X$. The ideal I has an Eagon-Northcott resolution*

$$\cdots \to S_2(R^2)^* \otimes \Lambda^4 R(-1)^{d-1} \to (R^2)^* \otimes \Lambda^3 R(-1)^{d-1} \to \Lambda^2 R(-1)^{d-1} \to \Lambda^2 R^2 \to R/I \to 0.$$

In particular, the only nonzero betti numbers are

$$\dim_{\mathbb{K}} Tor_i(R/I, \mathbb{K})_{i+1} = i \cdot \binom{d-1}{i+1}.$$

Proof. Let $X \in L_2(\mathcal{A})$ with $\mu(X) = d-1$. After a change of coordinates, $X = V(x_1, x_2)$, and $X \in H$ iff $l_H \in \langle x_1, x_2 \rangle$. Localization is exact, so without loss of generality we may assume that $\mathcal{A}$ consists of d points in $\mathbb{P}^1$. By Proposition 2.5, I is quadratic, and by Proposition 2.1, I has codimension $d-2$, so $V(I)$ is an irreducible curve in $\mathbb{P}^{d-1}$. Since the irrelevant maximal ideal is not an associated prime, I has depth at least one, and by Corollary 2.2, X is not contained in any hyperplane, so y_d is a nonzero divisor on R/I. This implies that $\deg X = \deg V(I + \langle y_d \rangle)$. Since $\mathcal{A}$ has rank two, any set of three hyperplanes is dependent and thus every triple $\{H_i, H_j, H_k\}$ yields an element of I. Therefore

$$\langle I, y_d \rangle = \langle J, y_d \rangle, \text{ where } J = \langle \{y_i y_j\}_{1 \leq i < j \leq d-1} \rangle.$$

It follows that the primary decomposition of $\langle I, y_d \rangle$ is

$$\langle I, y_d \rangle = \bigcap_{1 \leq i_1 < \cdots < i_{d-2} \leq d-1} \langle y_{i_1}, \ldots, y_{i_{d-2}}, y_d \rangle,$$

so $\deg V(\langle I, y_d \rangle) = d-1$. Any smooth, irreducible nondegenerate curve of degree $d-1$ in $\mathbb{P}^{d-1}$ is a rational normal curve, and has an Eagon-Northcott resolution [4]. So we need only show that $V(I)$ is smooth. Our main theorem implies $\mathcal{A}$ is 2-formal, and the proof of that result shows that

$$\dim T_p(V(I)) = 1$$

for all $p \in V(I)$. Hence $V(I)$ is rational normal curve of degree $d-1$. □

3.1. Graphic arrangements. For a graphic arrangement, all weights of dependencies are ± 1, so the Orlik-Terao ideal I_G has a presentation that is essentially identical to that of the Orlik-Solomon ideal obtained in [12]. The proof of the next lemma is straightforward.

Lemma 3.2. *I_G is minimally generated by the chordless cycles of G.*

Lemma 3.3. *Every K_4 subgraph yields two minimal linear first syzygies on I_G.*

Proof. Let K_4 be the complete graph on $\{1,2,3,4\}$. There are four relations:

$$y_{12}+y_{23}-y_{13},\; y_{12}+y_{24}-y_{14},\; y_{13}+y_{34}-y_{14},\; y_{23}+y_{34}-y_{24}.$$

Let $y_1 = y_{12}, y_2 = y_{13}, y_3 = y_{14}, y_4 = y_{23}, y_5 = y_{24}, y_6 = y_{34}$. I is generated by:

$$\begin{aligned} f_4 &= y_2y_4 + y_1y_2 - y_1y_4 \\ f_3 &= y_3y_5 + y_1y_3 - y_1y_5 \\ f_2 &= y_3y_6 + y_2y_3 - y_2y_6 \\ f_1 &= y_5y_6 + y_4y_5 - y_4y_6 \end{aligned}$$

A direct calculation yields the pair of (independent) linear syzygies:

$$\begin{aligned} &(y_1-y_2)f_1-(y_1+y_5)f_2+(y_2+y_6)f_3-(y_6-y_5)f_4=0 \\ &(y_2-y_3)f_1-(y_4-y_5)f_2+(y_4-y_2)f_3-(y_5-y_3)f_4=0 \end{aligned}$$

A Hilbert function computation as in Corollary 1.8 concludes the proof. □

Theorem 3.4. *If κ_i is the number of induced subgraphs of type $K_{i+1} \subseteq G$, then*

$$Tor_i(R/I_G)_{i+1} = \begin{cases} \kappa_2 & i=1 \\ 2\kappa_3 & i=2 \\ 0 & i \geq 3. \end{cases}$$

Proof. Combine a Hilbert function argument as in Corollary 1.8 with suitable modifications to the proofs of Corollary 6.6 and Lemma 6.9 of [12]. □

3.2. The spaces $R_k(\mathcal{A})$. In [3], Brandt-Terao introduce higher relation spaces.

Definition 3.5. *For $X \in L_2(\mathcal{A})$, let $F(\mathcal{A}_X)$ be the subspace of $F(\mathcal{A})$ generated by the relations associated to circuits of length 3 $\{H_i, H_j, H_k\}$, with $X \subset H_i, H_j, H_k$. The inclusion map $F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$ gives a map*

$$\pi : \bigoplus_{X \in L_2(\mathcal{A})} F(\mathcal{A}_X) \longrightarrow F(\mathcal{A}).$$

The first relation space $\mathcal{R}_3(\mathcal{A}) = \ker \pi$.

The space $\mathcal{R}_3(\mathcal{A})$ captures the dependencies among the circuits of length 3 in $\mathcal{A}$, and $\mathcal{A}$ is 2-formal iff π is surjective. It is clear that $\dim F(\mathcal{A}_X) = \mu(X) - 1$, where μ is the Möbius function.

Proposition 3.6. *Let $X_1, \ldots, X_s \in L_2(\mathcal{A})$ and let $r_1 \in F(\mathcal{A}_{X_1}), \ldots, r_s \in F(\mathcal{A}_{X_s})$ be nonzero relations. If*

$$L_1\partial(r_1) + \cdots + L_s\partial(r_s) = 0, L_i \neq 0$$

is a linear syzygy, then there exist $a_i \in \mathbb{K}$ with $a_1r_1 + \cdots + a_sr_s = 0$.

Proof. Suppose

$$L_1\partial(r_1) + \cdots + L_s\partial(r_s) = 0, L_i \neq 0$$

is a linear syzygy, with $L_i = a^i_1 y_1 + \cdots + a^i_d y_d$. Define $supp(L_i)$ to be the set of indices j for which $a^i_j \neq 0$, and suppose $supp(L_i) \cap supp(r_i) \neq \emptyset$ for some i. In the expression of $L_i\partial(r_i)$ there exists a nonzero monomial of the form $y_u y_v^2$ with $H_u \cap H_v = X_i$. Since in the syzygy this monomial must be cancelled, there exists $k \neq i$ such that $L_k\partial(r_k)$ contains a term $y_u y_v^2$. Since y_v^2 cannot occur in $\partial(r_k)$, we see that $X_k = H_u \cap H_v$, contradicting $X_k \neq X_i$.

Let $\Lambda_i = supp(r_i)$. Substituting $(\frac{1}{y_1}, \ldots, \frac{1}{y_d})$ in the syzygy yields:

$$\frac{f_1}{y_{[d]\setminus\Lambda_1}} \frac{r_1}{y_{\Lambda_1}} + \cdots + \frac{f_s}{y_{[d]\setminus\Lambda_s}} \frac{r_s}{y_{\Lambda_s}} = 0,$$

where $\frac{f_i}{y_{[d]\setminus\Lambda_i}} = L_i(\frac{1}{y_1}, \ldots, \frac{1}{y_d})$ and $f_i \in \mathbb{K}[y_1, \ldots, y_d]$ are square-free non-zero polynomials, possibly divisible by a square-free monomial. Hence $f_1 r_1 + \cdots + f_s r_s = 0$ and $f_1 r_1 \in \langle r_2, \ldots, r_s \rangle$. If $r_1 \notin \langle r_2, \ldots, r_s \rangle$, then $f_1 \in \langle r_2, \ldots, r_s \rangle$ and

$$f_1 = P_2 r_2 + \cdots + P_s r_s.$$

Evaluating this expression at $(\frac{1}{y_1}, \ldots, \frac{1}{y_d})$ and clearing the denominators shows there exists a monomial m such that $mL_1 \in \langle \partial(r_2), \ldots, \partial(r_s) \rangle \subseteq I$. By Proposition 2.1 and Corollary 2.2, I is nondegenerate prime ideal, so this is impossible, and $r_1 \in \langle r_2, \ldots, r_s \rangle$. □

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