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THE JIANG-SU ALGEBRA DOES NOT ALWAYS EMBED

Marius Dadarlat, Ilan Hirshberg, Andrew S. Toms, and Wilhelm Winter

Abstract. We exhibit a unital simple nuclear non-type-I C*-algebra into which the Jiang-Su algebra does not embed unitally. This answers a question of M. Rørdam.

The Jiang-Su algebra, denoted by $\mathcal{Z}$ ([3]), occupies a central position in the structure theory of separable amenable C*-algebras. The property of absorbing the Jiang-Su algebra tensorially is a necessary, and, in considerable generality, sufficient condition for the confirmation of G. A. Elliott's K-theoretic rigidity conjecture for simple separable amenable C*-algebras ([5], [7]). The uniqueness question for this algebra is therefore of great interest. M. Rørdam observed that if $\mathcal{C}$ is a class of unital separable C*-algebras, and $A \in \mathcal{C}$ has the properties that (i) for every $B \in \mathcal{C}$ there is a unital $*$-homomorphism $\gamma : A \to B$ and (ii) every unital $*$-endomorphism of A is approximately inner, then A is the only such algebra, up to isomorphism. (This follows from an application of Elliott's Intertwining Argument.) Every unital $*$-endomorphism of $\mathcal{Z}$ is approximately inner ([3]), and there are no obvious obstructions to the existence of a unital $*$-homomorphism $\gamma : \mathcal{Z} \to A$ for any unital separable C*-algebra A without finite-dimensional quotients. Indeed, such a γ always exists when A has real rank zero, and examples show that the existence of γ is strictly weaker than tensorial absorption of $\mathcal{Z}$–see [1] and [6], respectively. All of this begs the question, first posed by Rørdam: "Does every unital C*-algebra without finite-dimensional quotients admit a unital embedding of $\mathcal{Z}$?", see [1]. We prove that the answer is negative, even when the target algebra is simple and nuclear.

Theorem. *There is a unital simple nuclear infinite dimensional C*-algebra (in fact, an AH algebra) into which the Jiang-Su algebra does not embed unitally.*

In the remainder of the paper we give some background discussion and prove the theorem. For a pair of relatively prime integers $p, q > 1$, we set

$$Z_{p,q} = \{f \in \mathrm{C}([0,1]; \mathrm{M}_p \otimes \mathrm{M}_q) \mid f(0) \in \mathrm{M}_p \otimes 1, f(1) \in 1 \otimes \mathrm{M}_q\}.$$

Each $Z_{p,q}$ is contained unitally in $\mathcal{Z}$. If a unital C*-algebra A admits no unital $*$-homomorphism $\gamma : Z_{p,q} \to A$, then there is no unital embedding of $\mathcal{Z}$ into A.

Let A, B be unital C*-algebras, and let $e, f \in A$ be projections satisfying

$$(n+1)[e] \le n[f]$$

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in the Murray-Von Neumann semigroup $\mathrm{V}(A)$ for some $n \in \mathbb{N}$. It is implicitly shown in the proof of [4, Lemma 4.3] that if $\gamma : Z_{n,n+1} \to B$ is a unital $*$-homomorphism, then $[e \otimes 1_B] \leq [f \otimes 1_B]$ in $\mathrm{V}(A \otimes B)$; tensor products are minimal. Example 4.8 of [2] exhibits a sequence $(B_j)_{j \in \mathbb{N}}$ of unital separable C*-algebras with the following property: there are projections $e, f \in B_1 \otimes B_2$ such that $4[e] \leq 3[f]$, but $[e \otimes 1_{\bigotimes_{j=3}^n B_j}] \not\leq [f \otimes 1_{\bigotimes_{j=3}^n B_j}]$ for any $n \geq 3$. Using Rørdam's result, one concludes that there is no unital $*$-homomorphism $\gamma : Z_{3,4} \to \bigotimes_{j=3}^n B_j$ for any $j \geq 3$. (In fact, there is nothing special about $Z_{3,4}$. A similar construction can be carried out for a wide variety of $Z_{p,q}$s.)

To simplify notation, we renumber the B_js so that $Z_{3,4}$ does not embed into $\bigotimes_{j=1}^n B_j$ for any $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, set $D_i = \bigotimes_{j=1}^i B_j$. We will perturb the canonical embeddings

$$\psi_i := \mathrm{id} \otimes 1_{B_{i+1}} : D_i \longrightarrow D_i \otimes B_{i+1} = D_{i+1}$$

to maps ϕ_i with the property that $(D_i, \phi_i)_{i \in \mathbb{N}}$ has simple limit D. Any such limit, simple or not, fails to admit a unital $*$-homomorphism $\gamma : Z_{3,4} \to D$, and so also fails to admit a unital embedding of $\mathcal{Z}$. Indeed, suppose that such a γ did exist. Then, by the semiprojectivity of $Z_{3,4}$ ([3]), there would exist a unital $*$-homomorphism $\tilde{\gamma} : Z_{3,4} \to D_i$ for some i, contradicting our choice of D_i. We remark that, in particular, $\bigotimes_{j=1}^{\infty} B_j$ admits no unital embedding of $\mathcal{Z}$. This algebra is a continuous field of C*-algebras whose fibres are $\mathcal{Z}$-absorbing – in fact, its fibres are all isomorphic to the CAR algebra (see [2, Example 4.8]).

The B_js have the form $(e_j \oplus f_j)\,(\mathrm{C}(X_j) \otimes \mathcal{K})\,(e_j \oplus f_j)$, where e_j and f_j are rank one projections and $X_j = (\mathrm{S}^2)^{\times m(j)}$. Let $\alpha : X_j \to X_j$ be a homeomorphism homotopic to the identity map, and view B_j as a corner of $\mathrm{C}(X_j) \otimes \mathrm{M}_n$ for some sufficiently large $n \in \mathbb{N}$. The map α induces an automorphism α^* of $\mathrm{C}(X_j) \otimes \mathrm{M}_n$, $\alpha^*(f) = f \circ \alpha$. In general, α^* will not carry B_j into B_j, but this can be corrected. Since α is homotopic to the identity, the projection $e_j \oplus f_j$ is homotopic, and hence unitarily equivalent, to its image under α^*. If u is a unitary implementing this equivalence, then $\overline{\alpha} := (\mathrm{Ad}(u) \circ \alpha^*)|_{B_j}$ is an automorphism of B_j. For our purposes, the salient property of $\overline{\alpha}$ is this: if $f \in B_j$ and $f(x) \neq 0$ for some $x \in X_j$, then $\overline{\alpha}(f)(\alpha^{-1}(x)) \neq 0$.

It remains to construct the ϕ_i, and prove the simplicity of the resulting inductive limit algebra D. Let us set $Y_i := \prod_{j=1}^i X_j$ where $i \in \mathbb{N}$ or $i = \infty$. We endow Y_i with the metric $d(x,y) = \sum_{j=1}^i 2^{-j} d_j(x^j, y^j)$ where d_j is the canonical metric on X_j, normalized so that X_j has diameter equal to one.

Choose a dense sequence $(z_k)_{k \in \mathbb{N}}$ in Y_∞. Fix $z_0 \in Y_\infty$ and for each $k \in \mathbb{N}$ let $\beta_k : Y_\infty \to Y_\infty$ be a cartesian product of isometries of X_js which are homotopic to the identity and such that $\beta_k(z_0) = z_k$. Let $(\alpha_k)_{k \in \mathbb{N}}$ be an enumeration of the set $\{\beta_m \beta_n^{-1} : n, m \in \mathbb{N}\}$. It is easy to see that for any point $x \in Y_\infty$ and any $i \in \mathbb{N}$, the sequence $(\alpha_k(x))_{k \geq i}$ is dense in Y_∞. Note that each α_k is also a cartesian product of isometries α_k^j of X_j homotopic to id_{X_j}. Let us set $\alpha_{k,[i]} = \prod_{j=1}^i \alpha_k^j$. Let $\pi_i : Y_\infty \to Y_i$ be the co-ordinate projection. Then $\pi_i \alpha_k(x) = \alpha_{k,[i]}(\pi_i(x))$ for $x \in Y_\infty$. Therefore for any point $y \in Y_i$, the sequence $(\alpha_{k,[i]}(y))_{k \geq i}$ is dense in Y_i. So is the sequence $((\alpha_{k,[i]})^{-1}(y))_{k \geq i}$ since each $\alpha_{k,[i]}$ is an isometry. By the compactness of Y_i it follows that for any nonempty open set U of Y_i, there is $j \geq i$ such that $Y_i = \bigcup_{k=i}^j (\alpha_{k,[i]})^{-1}(U)$.

For each $i \leq k \in \mathbb{N}$, let $\overline{\alpha_{k,[i]}} : D_i \to D_i$ be the automorphism induced, in the manner described above, by the homeomorphism $\alpha_{k,[i]} : Y_i \to Y_i$.

Observe that the canonical embedding $\psi_i : D_i \to D_{i+1}$ is the direct sum of two non-unital embeddings:

$$\psi_i^{(1)} \stackrel{\text{def}}{=} \mathrm{id} \otimes e_{i+1} : D_i \to D_i \otimes e_{i+1} \subseteq D_{i+1},$$

and

$$\psi_i^{(2)} \stackrel{\text{def}}{=} \mathrm{id} \otimes f_{i+1} : D_i \to D_i \otimes f_{i+1} \subseteq D_{i+1}.$$

Set $\phi_i^{(1)} = \psi_i^{(1)}$, and

$$\phi_i^{(2)} \stackrel{\text{def}}{=} \overline{\alpha_{i,[i]}} \otimes f_{i+1} : D_i \to D_i \otimes f_{i+1} \subseteq D_{i+1}.$$

Define $\phi_i : D_i \to D_{i+1}$ to be $\phi_i^{(1)} \oplus \phi_i^{(2)}$.

Let us now verify that $D = \lim_{i\to\infty}(D_i, \phi_i)$ is simple. It will suffice to prove that for any nonzero $a \in D_i$ there is some $j \geq i$ such that $\phi_{i,j+1}(a) := \phi_j \circ \cdots \circ \phi_i(a)$ is nonzero over every point in the spectrum of D_{j+1}.

For each $\mathbf{v} = (v_i, \ldots, v_j) \in \{1,2\}^{j-i+1}$, set

$$\phi_{i,j+1}^{\mathbf{v}} = \phi_j^{v_j} \circ \phi_{j-1}^{v_{j-1}} \circ \cdots \circ \phi_i^{v_i},$$

and note that $\phi_{i,j+1} = \bigoplus_{\mathbf{v}\in\{1,2\}^{j-i+1}} \phi_{i,j+1}^{\mathbf{v}}$. For $k \in \{i, \ldots, j\}$, let $\mathbf{v}_k \in \{1,2\}^{j-i+1}$ be the vector which is equal to 1 in each co-ordinate except the k^{th} one. We have (with the exception of the cases $k = i, j$ when the formula reads slightly differently)

$$\begin{aligned} \phi_{i,j+1}^{\mathbf{v}_k}(a) &= \left[\, \overline{\alpha_{k,[k]}}(a \otimes e_{i+1} \otimes \cdots \otimes e_k)\right] \otimes f_{k+1} \otimes e_{k+2} \otimes \cdots \otimes e_{j+1} \\ &= \left[\overline{\alpha_{k,[i]}}(a) \otimes \overline{\alpha_k^{i+1}}(e_{i+1}) \otimes \cdots \otimes \overline{\alpha_k^{k}}(e_k))\right] \otimes f_{k+1} \otimes e_{k+2} \otimes \cdots \otimes e_{j+1}. \end{aligned}$$

Since a is nonzero on some nonempty open set U, the formula above shows that $\phi_{i,j+1}^{\mathbf{v}_k}(a)$ is nonzero on $W_{i,j+1}^k := (\alpha_{k,[i]})^{-1}(U) \times X_{i+1} \times \cdots \times X_{j+1}$, for any $j \geq i$. As noticed earlier, there is $j \geq i$ such that $Y_i = \bigcup_{k=i}^{j} (\alpha_{k,[i]})^{-1}(U)$. Therefore $\bigoplus_{k=i}^{j} \phi_{i,j+1}^{\mathbf{v}_k}(f)$ is nonzero on $\bigcup_{k=i}^{j} W_{i,j+1}^k = Y_{j+1} = \widehat{D_{j+1}}$, as required.

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Department of Mathematics, Purdue University, 150 N. University St., West Lafayette, IN, 47907-2067, USA
E-mail address: mdd@math.purdue.edu

Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel
E-mail address: ilan@math.bgu.ac.il

Department of Mathematics and Statistics, York University, 4700 Keele St., Toronto, Ontario, Canada, M3J 1P3
E-mail address: atoms@mathstat.yorku.ca

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom
E-mail address: wilhelm.winter@nottingham.ac.uk