

SCATTERING FOR THE NON-RADIAL 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. Scattering of radial H^1 solutions to the 3D focusing cubic nonlinear Schrödinger equation below a mass-energy threshold $M[u]E[u] < M[Q]E[Q]$ and satisfying an initial mass-gradient bound $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, where Q is the ground state, was established in Holmer-Roudenko [7]. In this note, we extend the result in [7] to non-radial H^1 data. For this, we prove a non-radial profile decomposition involving a spatial translation parameter. Then, in the spirit of Kenig-Merle [10], we control via momentum conservation the rate of divergence of the spatial translation parameter and by a convexity argument based on a local virial identity deduce scattering. An application to the defocusing case is also mentioned.

1. Introduction

We consider the Cauchy problem for the cubic focusing nonlinear Schrödinger (NLS) equation on \mathbb{R}^3 :

$$(1.1) \quad i\partial_t u + \Delta u + |u|^2 u = 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R},$$

$$(1.2) \quad u(x, 0) = u_0 \in H^1(\mathbb{R}^3).$$

It is locally well-posed (e.g., see Cazenave [3]). The equation has 3 conserved quantities; namely, the mass $M[u]$, energy $E[u]$ and momentum $P[u]$:

$$M[u] = \int |u(x, t)|^2 dx = M[u_0],$$

$$E[u] = \frac{1}{2} \int |\nabla u(x, t)|^2 dx - \frac{1}{4} \int |u(x, t)|^4 dx = E[u_0],$$

$$P[u] = \operatorname{Im} \int \bar{u}(x, t) \nabla u(x, t) dx = P[u_0].$$

The scale-invariant Sobolev norm is $\dot{H}^{1/2}$ and the scale-invariant Lebesgue norm is L^3 . Let $u(x, t) = e^{it}Q(x)$; then u solves (1.1) provided Q solves the nonlinear elliptic equation

$$(1.3) \quad -Q + \Delta Q + |Q|^2 Q = 0.$$

This equation has an infinite number of solutions in $H^1(\mathbb{R}^3)$. The solution of minimal mass, hereafter denoted by $Q(x)$, is positive, radial, exponentially decaying, and is called *the ground state*. For further properties of Q , we refer to Weinstein [14], Holmer-Roudenko [7], Cazenave [3].

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In Holmer-Roudenko [7, Theorem 1.1] (see also Holmer-Roudenko [8]), it was proved that under the condition $M[u]E[u] < M[Q]E[Q]$, solutions to (1.1)-(1.2) globally exist if u_0 satisfies

$$(1.4) \quad \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2},$$

and radial solutions with initial data satisfying (1.4) scatter in H^1 in both time directions. This means that there exist $\phi_{\pm} \in H^1$ such that

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta} \phi_{\pm}\|_{H^1} = 0.$$

In this note we extend the scattering result to include non-radial H^1 data.

Theorem 1.1. *Let $u_0 \in H^1$ and let u be the corresponding solution to (1.1) in H^1 . Suppose*

$$(1.5) \quad M[u]E[u] < M[Q]E[Q].$$

If $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$, then u scatters in H^1 .

The argument of [7] in the radial case followed a strategy introduced by Kenig-Merle [9] for proving global well-posedness and scattering for the focusing energy-critical NLS. The argument begins by contradiction: suppose the threshold for scattering is strictly below that claimed. A profile decomposition lemma based on concentration compactness principles (and analogous to that of Keraani [11]) was invoked to prove the existence of a global but nonscattering solution u_c standing exactly at the threshold between scattering and nonscattering. The profile decomposition lemma is again invoked to prove that the flow of u_c is a precompact subset of H^1 , which then implies that u_c remains spatially localized uniformly in time. This uniform localization enabled the use of a local virial identity to establish, with the aid of the sharp Gagliardo-Nirenberg inequality, a strictly positive lower bound on the convexity (in time) of the local mass of u_c . Mass conservation is then violated at a sufficiently large time.

In this paper, we show that the above program carries over to the non-radial setting with the addition of two key ingredients. First, in §2, we introduce a profile decomposition lemma that applies to non-radial H^1 sequences. To compensate for the lack of localization at the origin induced by radially, a spatial translation sequence is needed. We also here adapt the proof given in [7] of the energy Pythagorean expansion (Lemma 2.3) to apply to non-radial sequences; in [7], an inessential application of the compact embedding $H^1_{\text{rad}} \rightarrow L^4$ was used at one point. The profile decomposition and concentration compactness techniques are previously used in works of Keraani [11], Gerard [5], see also Bahouri and Gerard [1]-[2], and originate from P.-L. Lions [12]-[13].

The application of the non-radial profile decomposition to time slices of the flow of the critical solution u_c yields the existence of a continuous time translation parameter $x(t)$ such that the translated flow $u_c(\cdot - x(t), t)$ is precompact in H^1 (Prop. 3.2). This implies the localization of $u_c(\cdot, t)$ near $x(t)$ (as opposed to the radial case, in which localization is obtained near the origin).

Obtaining suitable control on the behavior of $x(t)$ is the main new step beyond [7]. This is done by following a method introduced by Kenig-Merle [10] (who applied it to the energy-critical nonlinear wave equation). First, we argue that by Galilean

invariance, the solution u_c must have zero momentum (see §4). An appropriate selection of the phase shift is possible in our case since our solution belongs to L^2 .¹ This zero-momentum solution is then shown in §5 to have a near-conservation of localized center-of-mass, which provides the desired control on the rate of divergence of $x(t)$ (specifically, $x(t)/t \rightarrow 0$ as $t \rightarrow \infty$).

In §7, we remark on the adaptation of these techniques to the defocusing cubic NLS in 3D.

2. Non-radial profile and energy decompositions

We will make use of the Strichartz norm notation used in [7]. We say that (q, r) is \dot{H}^s Strichartz admissible (in 3D) if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.$$

Let

$$\|u\|_{S(L^2)} = \sup_{\substack{(q,r) L^2 \text{ admissible} \\ 2 \leq r \leq 6, 2 \leq q \leq \infty}} \|u\|_{L_t^q L_x^r}.$$

Define

$$\|u\|_{S(\dot{H}^{1/2})} = \sup_{\substack{(q,r) \dot{H}^{1/2} \text{ admissible} \\ 3 \leq r \leq 6^-, 4^+ \leq q \leq \infty}} \|u\|_{L_t^q L_x^r},$$

where 6^- is an arbitrarily preselected and fixed number < 6 ; similarly for 4^+ .

Lemma 2.1 (Profile expansion). *Let $\phi_n(x)$ be a uniformly bounded sequence in H^1 . Then for each M there exists a subsequence of ϕ_n , also denoted ϕ_n , and*

- (1) *for each $1 \leq j \leq M$, there exists a (fixed in n) profile $\psi^j(x)$ in H^1 ,*
- (2) *for each $1 \leq j \leq M$, there exists a sequence (in n) of time shifts t_n^j ,*
- (3) *for each $1 \leq j \leq M$, there exists a sequence (in n) of space shifts x_n^j ,*
- (4) *there exists a sequence (in n) of remainders $W_n^M(x)$ in H^1 ,*

such that

$$\phi_n(x) = \sum_{j=1}^M e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^M(x).$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$, we have

$$(2.1) \quad \lim_{n \rightarrow +\infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = +\infty.$$

The remainder sequence has the following asymptotic smallness property²:

$$(2.2) \quad \lim_{M \rightarrow +\infty} \left[\lim_{n \rightarrow +\infty} \|e^{it \Delta} W_n^M\|_{S(\dot{H}^{1/2})} \right] = 0.$$

¹It could not be applied in the Kenig-Merle paper [9] on the energy critical NLS since the argument there takes place in \dot{H}^1 .

²We can always pass to a subsequence in n with the property that $\|e^{it \Delta} W_n^M\|_{S(\dot{H}^{1/2})}$ converges. Therefore, we use \lim and not \limsup or \liminf . Similar remarks apply for the limits that appear in the Pythagorean expansion.

For fixed M and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion

$$(2.3) \quad \|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^M \|\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1).$$

Remark 2.2. If the assumption that ϕ_n is uniformly bounded in H^1 is weakened to the assumption that ϕ_n is uniformly bounded in $\dot{H}^{1/2}$, then the above profile decomposition remains valid provided a scaling parameter λ is also involved, similar to the theorem in [11]. However, it is not needed for the results of this note and for simplicity of exposition the proof is omitted.

Proof. The proof is very close to the one of [7, Lemma 5.2]. We also refer to [11] for a similar result in the energy-critical case.

Step 1. Construction of ψ_n^1 . Let $A_1 = \limsup_n \|e^{it\Delta}\phi_n\|_{L_t^\infty L_x^3}$. If $A_1 = 0$, we are done. Indeed, for an arbitrary $\dot{H}^{1/2}$ -admissible couple (q, r) we have

$$\|e^{it\Delta}\phi_n\|_{L_t^q L_x^r} \leq \|e^{it\Delta}\phi_n\|_{L_t^4 L_x^6}^\theta \|e^{it\Delta}\phi_n\|_{L_t^\infty L_x^3}^{1-\theta} \quad \text{with } \theta = \frac{4}{q} \in (0, 1).$$

Noting that $\|e^{it\Delta}\phi_n\|_{L_t^4 L_x^6} \leq C\|\phi_n\|_{\dot{H}^{1/2}}$, we get that $\limsup_n \|e^{it\Delta}\phi_n\|_{S(\dot{H}^{1/2})} = 0$, and we can take $\psi^j = 0$ for all j .

If $A_1 > 0$, let

$$c_1 = \limsup_n \|\phi_n\|_{H^1}.$$

Extracting a subsequence from ϕ_n , we show that there exist sequences t_n^1, x_n^1 and a function $\psi^1 \in H^1$ such that

$$(2.4) \quad e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1 \text{ weakly in } H^1,$$

$$(2.5) \quad Kc_1^4 \|\psi^1\|_{\dot{H}^{1/2}} \geq A_1^5,$$

where $K > 0$ is a constant independent of all parameters.

Let $r = \frac{16c_1^2}{A_1^2}$ and χ_r be a radial Schwartz function such that $\hat{\chi}_r(\xi) = 1$ for $\frac{1}{r} \leq |\xi| \leq r$, and $\text{supp } \chi_r \subset [\frac{1}{2r}, 2r]$. By the arguments of [7], there exists sequences t_n^1, x_n^1 such that

$$|\chi_r * e^{it_n^1 \Delta} \phi_n(x_n^1)| \geq \frac{A_1^3}{32c_1^2}.$$

Pass to a subsequence so that $e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ weakly in H^1 . In [7] the functions ϕ_n are radial, and thus, by the radial Gagliardo-Nirenberg inequality, one can show that x_n^1 is bounded in n , which is not necessarily the case here. As in [7], the estimate $\|\chi_r\|_{\dot{H}^{-1/2}} \leq r$ yields, together with Plancherel and Cauchy-Schwarz inequalities, the estimate (2.5).

Next, define $W_n^1(x) = \phi_n(x) - e^{-it_n^1 \Delta} \psi^1(x - x_n^1)$. Since $e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ in H^1 , expanding $\|W_n^1\|_{\dot{H}^s}^2$ as an inner product and using the definition of W_n^1 , we obtain

$$\lim_{n \rightarrow \infty} \|W_n^1\|_{\dot{H}^s}^2 = \lim_{n \rightarrow \infty} \|e^{it_n^1 \Delta} \phi_n\|_{\dot{H}^s}^2 - \|\psi^1\|_{\dot{H}^s}^2, \quad 0 \leq s \leq 1,$$

which yields (2.3) for $M = 1$.

Step 2. Construction of ψ^j for $j \geq 2$. We construct the functions ψ^j inductively, applying Step 1 to the sequences (in n) W_n^{j-1} . Let $M \geq 2$. Assuming that ψ^j, x_n^j, t_n^j and W_n^j are known for $j \in \{1, \dots, M-1\}$, we consider

$$A_M = \limsup_n \|W_n^{M-1}\|_{L_t^\infty L_x^3}.$$

If $A_M = 0$, we take, as in Step 1, $\psi^j = 0$ for $j \geq M$. Assume $A_M > 0$. Applying Step 1 to the sequence W_n^{M-1} , we obtain, extracting if necessary, sequences x_n^M, t_n^M and a function $\psi^M \in H^1$ such that

$$(2.6) \quad e^{it_n^M \Delta} W_n^{M-1}(\cdot + x_n^M) \rightharpoonup \psi^M \text{ weakly in } H^1,$$

$$(2.7) \quad K c_M^4 \|\psi^M\|_{\dot{H}^{1/2}} \geq A_M^5, \text{ where } c_M = \limsup_n \|W_n^{M-1}\|_{H^1}.$$

We then define $W_n^M(x) = W_n^{M-1}(x) - e^{-it_n^M \Delta} \psi^M(x - x_n^M)$.

We next show (2.1) and (2.3) by induction. Assume that (2.3) holds at rank $M-1$. Expanding

$$\|W_n^M\|_{\dot{H}^s}^2 = \left\| e^{it_n^M \Delta} W_n^M(\cdot + x_n^M) \right\|_{\dot{H}^s}^2 = \left\| e^{it_n^M \Delta} W_n^{M-1}(\cdot + x_n^M) - \psi^M \right\|_{\dot{H}^s}^2$$

and using the weak convergence (2.6), we obtain directly (2.3) at rank M .

Assume that the condition (2.1) holds for $j, k \in \{1, \dots, M-1\}$. Let $j \in \{1, \dots, M-1\}$. Then (here, $W_n^0 = \phi_n$),

$$-e^{it_n^j \Delta} W_n^{M-1}(x + x_n^j) + e^{it_n^j \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x) = \sum_{k=j+1}^{M-1} e^{i(t_n^j - t_n^k) \Delta} \psi^k(x + x_n^j - x_n^k).$$

By the orthogonality condition (2.1), the right hand side converges to 0 weakly in H^1 as n tends to infinity. Furthermore, by the definition of W_n^{j-1} ,

$$e^{it_n^j \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x) \xrightarrow{n \rightarrow +\infty} 0 \text{ weakly in } H^1.$$

Thus, $e^{it_n^j \Delta} W_n^{M-1}(x + x_n^j)$ must go to 0 weakly in H^1 . From (2.6), we deduce, if $\psi^M \neq 0$

$$\lim_{n \rightarrow +\infty} |x_n^j - x_n^M| + |t_n^j - t_n^M| = +\infty,$$

which shows that (2.1) must also holds for $k = M$.

It remains to show (2.2). Note that by (2.3), $c_M \leq c_1$ for all M . If for all M , $A_M > 0$, we have by (2.3)

$$\sum_{M \geq 1} A_M^{10} \leq K^2 c_1^8 \sum_{n \geq 1} \|\psi^M\|_{\dot{H}^{1/2}}^2 \leq K^2 c_1^8 \limsup \|\phi_n\|_{\dot{H}^{1/2}}^2 < \infty,$$

which shows that A_M tends to 0 as M goes to ∞ , yielding (2.2) and concluding the proof of Lemma 2.1. \square

Lemma 2.3 (Energy Pythagorean expansion). *In the situation of Lemma 2.1, we have*

$$(2.8) \quad E[\phi_n] = \sum_{j=1}^M E[e^{-it_n^j \Delta} \psi^j] + E[W_n^M] + o_n(1).$$

Proof. According to (2.3), it suffices to establish for all $M \geq 1$,

$$(2.9) \quad \|\phi_n\|_{L^4}^4 = \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|_{L^4}^4 + \|W_n^M\|_{L^4}^4 + o_n(1).$$

Step 1. Pythagorean expansion of a sum of orthogonal profiles. We show that if $M \geq 1$ is fixed, orthogonality condition (2.1) implies

$$(2.10) \quad \left\| \sum_{j=1}^M e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j) \right\|_{L^4}^4 = \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|_{L^4}^4 + o_n(1).$$

By reindexing, we can arrange so that there is $M_0 \leq M$ such that

- For $1 \leq j \leq M_0$, we have that t_n^j is bounded in n .
- For $M_0 + 1 \leq j \leq M$, we have that $|t_n^j| \rightarrow \infty$ as $n \rightarrow \infty$.

By passing to a subsequence, we may assume that for each $1 \leq j \leq M_0$, t_n^j converges (in n), and by adjusting the profiles ψ^j we can take $t_n^j = 0$.

Note that

$$(2.11) \quad M_0 + 1 \leq k \leq M \implies \lim_{n \rightarrow +\infty} \|e^{-it_n^k \Delta} \psi^k\|_{L^4} = 0.$$

Indeed, in this case $|t_n^k| \rightarrow \infty$ as $n \rightarrow \infty$. For a function $\tilde{\psi} \in \dot{H}^{3/4} \cap L^{4/3}$, from Sobolev embedding and the L^p space-time decay estimate of the linear flow, we obtain

$$\|e^{-it_n^k \Delta} \psi^k\|_{L^4} \leq c \|\psi^k - \tilde{\psi}\|_{\dot{H}^{3/4}} + \frac{c}{|t_n^k|^{3/4}} \|\tilde{\psi}\|_{L^{4/3}}.$$

By approximating ψ^k by $\tilde{\psi} \in C_c^\infty$ in $\dot{H}^{3/4}$ and sending $n \rightarrow \infty$, we obtain (2.11).

By (2.1), if $1 \leq j < k \leq M_0$, $\lim_n |x_n^j - x_n^k| = +\infty$, and thus, it implies

$$\left\| \sum_{j=1}^{M_0} \psi^j(\cdot - x_n^j) \right\|_{L^4}^4 = \sum_{j=1}^{M_0} \|\psi^j\|_{L^4}^4 + o_n(1),$$

which yields, together with (2.11), expansion (2.10).

Step 2. End of the Proof. We first note

$$(2.12) \quad \lim_{M_1 \rightarrow +\infty} \left(\lim_{n \rightarrow +\infty} \|W_n^{M_1}\|_{L^4} \right) = 0.$$

Indeed,

$$\begin{aligned} \|W_n^{M_1}\|_{L_x^4} &\leq \|e^{it\Delta} W_n^{M_1}\|_{L_t^\infty L_x^4} \\ &\leq \|e^{it\Delta} W_n^{M_1}\|_{L_t^\infty L_x^3}^{1/2} \|e^{it\Delta} W_n^{M_1}\|_{L_t^\infty \dot{H}_x^1}^{1/2} \\ &\leq \|e^{it\Delta} W_n^{M_1}\|_{L_t^\infty L_x^3}^{1/2} \sup_n \|\phi_n\|_{H^1}^{1/2}. \end{aligned}$$

By (2.2), we get (2.12).

Let $M \geq 1$ and $\varepsilon > 0$. Note that $\{\phi_n\}_n$ is uniformly bounded in L^4 , since it is uniformly bounded in H^1 by the hypothesis; furthermore, by (2.12) $\{W_n^M\}_n$ is also

uniformly bounded in L^4 . Thus, we can choose $M_1 \geq M$ and N_1 such that for $n \geq N_1$, we have

$$(2.13) \quad \left| \|\phi_n - W_n^{M_1}\|_{L^4}^4 - \|\phi_n\|_{L^4}^4 \right| + \left| \|W_n^M - W_n^{M_1}\|_{L^4}^4 - \|W_n^M\|_{L^4}^4 \right| \leq C \left[\left(\sup_n \|\phi_n\|_{L^4}^3 + \sup_n \|W_n^M\|_{L^4}^3 \right) \|W_n^{M_1}\|_{L^4} + \|W_n^{M_1}\|_{L^4}^4 \right] \leq \varepsilon.$$

By (2.10), we get $N_2 \geq N_1$ such that for $n \geq N_2$,

$$(2.14) \quad \left| \|\phi_n - W_n^{M_1}\|_{L^4}^4 - \sum_{j=1}^{M_1} \|e^{-it_n^j \Delta} \psi^j\|_{L^4}^4 \right| \leq \varepsilon.$$

Using the definition of W_n^j , expand $W_n^M - W_n^{M_1}$ to obtain

$$W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} e^{-it_n^j \Delta} \psi^j(\cdot - x_j).$$

By (2.10) there exists $N_3 \geq N_2$ such that for $n \geq N_2$,

$$(2.15) \quad \left| \|W_n^M - W_n^{M_1}\|_{L^4}^4 - \sum_{j=M+1}^{M_1} \|e^{-it_n^j \Delta} \psi^j\|_{L^4}^4 \right| \leq \varepsilon.$$

By (2.13), (2.14) and (2.15), we obtain that for $n \geq N_3$,

$$\left| \|\phi_n\|_{L^4}^4 - \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|_{L^4}^4 - \|W_n^M\|_{L^4}^4 \right| \leq 4\varepsilon,$$

which concludes the proof of (2.9). □

3. Outline of the proof of the main result

Let $u(t)$ be the corresponding H^1 solution to (1.1)-(1.2). By Theorem 1.1(1)(a) in [7] the solution is globally well-posed, so our goal is to show that

$$(3.1) \quad \|u\|_{S(\dot{H}^{1/2})} < \infty.$$

This combined with Proposition 2.2 from [7] will give H^1 scattering. We will use the strategy of [9]. We shall say that $SC(u_0)$ holds if (3.1) is true for the solution $u(t)$ generated from u_0 .

By the small data theory there exists $\delta > 0$ such that if $M[u]E[u] < \delta$ and $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$, then (3.1) holds. For each $\delta > 0$ define the set S_δ to be the collection of all such initial data in H^1 :

$$S_\delta = \{u_0 \in H^1 \text{ with } M[u]E[u] < \delta \text{ and } \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}\}.$$

Next define $(ME)_c = \sup\{\delta : u_0 \in S_\delta \implies SC(u_0) \text{ holds}\}$. If $(ME)_c = M[Q]E[Q]$, then we are done, so we assume

$$(3.2) \quad (ME)_c < M[Q]E[Q].$$

Then there exists a sequence of solutions u_n to (1.1) with H^1 initial data $u_{n,0}$ (rescale all of them to have $\|u_n\|_{L^2} = 1$ for all n) such that $\|\nabla u_{n,0}\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$ and $E[u_n] \searrow (ME)_c$ as $n \rightarrow +\infty$, for which $SC(u_{n,0})$ does not hold for any n .

The next proposition gives the existence of an H^1 solution u_c to (1.1) with initial data $u_{c,0}$ such that $\|u_{c,0}\|_{L^2}\|\nabla u_{c,0}\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ and $M[u_c]E[u_c] = (ME)_c$ for which $SC(u_{c,0})$ does not hold. This will imply that $K = \{u_c(\cdot - x(t), t) \mid 0 \leq t < +\infty\}$ is precompact in H^1 (Proposition 3.2). As a consequence (see Corollary (3.3)) we obtain that for each $\epsilon > 0$, there is an $R > 0$ such that, uniformly in t , we have

$$(3.3) \quad \int_{|x+x(t)|>R} |\nabla u_c(t, x)|^2 dx \leq \epsilon.$$

This together with the hypothesis of zero momentum (which can always be achieved by Galilean invariance – see §4) provides a control on the growth of $x(t)$ (Lemma 5.1). Finally, the rigidity theorem (Theorem 6.1), which appeals to this control on $x(t)$ and the uniform localization (3.3), will lead to a contradiction that such critical element exists (unless it is identically zero) which will conclude the proof.

Proposition 3.1 (Existence of a critical solution). *Assume (3.2). Then there exists a global ($T^* = +\infty$) solution u_c in H^1 with initial data $u_{c,0}$ such that $\|u_{c,0}\|_{L^2} = 1$,*

$$E[u_c] = (ME)_c < M[Q]E[Q],$$

$$\|\nabla u_c(t)\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2} \quad \text{for all } 0 \leq t < +\infty,$$

and

$$\|u_c\|_{S(\dot{H}^{1/2})} = +\infty.$$

Proof. The proof closely follows the proof of [7, Prop 5.4]. □

Proposition 3.2 (Precompactness of the flow of the critical solution). *With u_c as in Proposition 3.1, there exists a continuous path $x(t)$ in \mathbb{R}^3 such that*

$$K = \{u_c(\cdot - x(t), t) \mid t \in [0, +\infty)\} \subset H^1$$

is precompact in H^1 (i.e., \bar{K} is compact).

Proof. For convenience, we write $u = u_c$. We argue by contradiction. By the arguments in Appendix A, we can assume that there exists $\eta > 0$ and a sequence t_n such that for all $n \neq n'$,

$$(3.4) \quad \inf_{x_0 \in \mathbb{R}^3} \|u(\cdot - x_0, t_n) - u(\cdot, t_{n'})\|_{H^1} \geq \eta.$$

Take $\phi_n = u(t_n)$ in the profile expansion lemma (Lemma 2.1). The remainder of the argument closely follows the proof of [7, Prop 5.5]. □

Corollary 3.3 (Precompactness of the flow implies uniform localization). *Let u be a solution to (1.1) such that*

$$K = \{u(\cdot - x(t), t) \mid t \in [0, +\infty)\}$$

is precompact in H^1 . Then for each $\epsilon > 0$, there exists $R > 0$ so that

$$\int_{|x+x(t)|>R} |\nabla u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^4 dx \leq \epsilon, \quad \text{for all } 0 \leq t < +\infty.$$

Proof. If not, then there exists $\epsilon > 0$ and a sequence of times t_n such that

$$\int_{|x+x(t_n)|>n} |\nabla u(x, t_n)|^2 + |u(x, t_n)|^2 + |u(x, t_n)|^4 dx \geq \epsilon,$$

or, by changing variables,

$$(3.5) \quad \int_{|x|>n} |\nabla u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^4 dx \geq \epsilon.$$

Since K is precompact, there exists $\phi \in H^1$ such that, passing to a subsequence of t_n , we have $u(\cdot - x(t_n), t_n) \rightarrow \phi$ in H^1 . By (3.5)

$$\forall R > 0, \quad \int_{|x|>R} |\nabla \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^4 \geq \epsilon,$$

which is a contradiction with the fact that $\phi \in H^1$. □

4. Zero momentum of the critical solution

Proposition 4.1. *Assume (3.2) and let u_c be the critical solution constructed in Section 3. Then its conserved momentum $P[u_c] = \text{Im} \int \bar{u}_c \nabla u_c dx$ is zero.*

Proof. Consider for some $\xi_0 \in \mathbb{R}^3$ the transformed solution

$$w_c(x, t) = e^{ix \cdot \xi_0} e^{-it|\xi_0|^2} u_c(x - 2\xi_0 t, t).$$

We compute

$$\|\nabla w_c\|_{L^2}^2 = |\xi_0|^2 M[u_c] + 2\xi_0 \cdot P[u_c] + \|\nabla u_c\|_{L^2}^2.$$

Observe that $M[w_c] = M[u_c]$ and

$$E[w_c] = \frac{1}{2} |\xi_0|^2 M[u_c] + \xi_0 \cdot P[u_c] + E[u_c].$$

To minimize $E[w_c]$, we take $\xi_0 = -P[u_c]/M[u_c]$.

Assume $P[u_c] \neq 0$. Choose $\xi_0 = -\frac{P[u_c]}{M[u_c]}$. Then $P[w_c] = 0$ and

$$(4.1) \quad M[w_c] = M[u_c], \quad E[w_c] = E[u_c] - \frac{1}{2} \frac{P[u_c]^2}{M[u_c]}, \quad \|\nabla w_c\|_{L^2}^2 = \|\nabla u_c\|_{L^2}^2 - \frac{P[u_c]^2}{M[u_c]}.$$

Thus, $M[w_c]E[w_c] < M[u_c]E[u_c]$, $\|w_c\|_{L^2} \|\nabla w_c\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$. By Proposition 3.1, $\|u_c\|_{S(\dot{H}^{1/2})} = +\infty$, and hence, $\|w_c\|_{S(\dot{H}^{1/2})} = +\infty$, which contradicts the definition of u_c . □

5. Control of the spatial translation parameter

Observe that

$$(5.1) \quad \frac{\partial}{\partial t} \int x |u(x, t)|^2 dx = 2 \text{Im} \int \bar{u} \nabla u dx = 2P[u].$$

Since $P[u_c] = 0$ (see Prop. 4.1), it follows that $\int x |u_c(x, t)|^2 dx = \text{const}$, provided it is finite. We will replace this identity with a version localized to a suitably large radius $R > 0$. Provided the localization R is taken large enough over an interval $[t_0, t_1]$ to envelope the entire path $x(t)$ over $[t_0, t_1]$, we can exploit the localization of u_c in H^1 around $x(t)$ (induced by the precompactness of the translated flow $u_c(\cdot - x(t), t)$)

and the zero-momentum property to prove that the localized center of mass is *nearly* conserved. The parameter $x(t)$ is then constrained from diverging too quickly to $+\infty$ by the localization of u_c in H^1 around $x(t)$ and the near conservation of localized center of mass. We refer to [10, Lemma 5.5] for a similar proof in the case of the energy-critical non-radial wave equation.

Lemma 5.1. *Let u be a solution of (1.1) defined on $[0, +\infty)$ such that $P[u] = 0$ and $K = \{u(\cdot - x(t), t) \mid t \in [0, \infty)\}$ is precompact in H^1 , for some continuous function $x(\cdot)$. Then*

$$(5.2) \quad \frac{x(t)}{t} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Proof. Assume that (5.2) does not hold. Then there exists a sequence $t_n \rightarrow +\infty$ such that $|x(t_n)|/t_n \geq \epsilon_0$ for some $\epsilon_0 > 0$. Without loss of generality we may assume $x(0) = 0$. For $R > 0$, let

$$t_0(R) = \inf\{t \geq 0 : |x(t)| \geq R\},$$

i.e., $t_0(R)$ is the first time when $x(t)$ reaches the boundary of the ball of radius R . By continuity of $x(t)$, the value $t_0(R)$ is well-defined. Moreover, the following properties hold: (1) $t_0(R) > 0$; (2) $|x(t)| < R$ for $0 \leq t < t_0(R)$; and (3) $|x(t_0(R))| = R$.

Define $R_n = |x(t_n)|$ and $\tilde{t}_n = t_0(R_n)$. Note that $t_n \geq \tilde{t}_n$, which combined with $|x(t_n)|/t_n \geq \epsilon_0$ gives $R_n/\tilde{t}_n \geq \epsilon_0$. Since $t_n \rightarrow +\infty$ and $|x(t_n)|/t_n \geq \epsilon_0$, we have $R_n = |x(t_n)| \rightarrow +\infty$. Thus, $\tilde{t}_n = t_0(R_n) \rightarrow +\infty$. At this point, we can forget about t_n ; we will work on the time interval $[0, \tilde{t}_n]$ and the only data that we will use in the remainder of the proof is:

- (1) for $0 \leq t < \tilde{t}_n$, we have $|x(t)| < R_n$;
- (2) $|x(\tilde{t}_n)| = R_n$;
- (3) $\frac{R_n}{\tilde{t}_n} \geq \epsilon_0$ and $\tilde{t}_n \rightarrow +\infty$.

By the precompactness of K and Corollary 3.3, it follows that for any $\epsilon > 0$ there exists $R_0(\epsilon) \geq 0$ such that for any $t \geq 0$,

$$(5.3) \quad \int_{|x+x(t)| \geq R_0(\epsilon)} (|u|^2 + |\nabla u|^2) \, dx \leq \epsilon.$$

We will select $\epsilon > 0$ appropriately later.

For $x \in \mathbb{R}$, let $\theta(x) \in C_c^\infty(\mathbb{R})$ be such that $\theta(x) = x$, for $-1 \leq x \leq 1$, $\theta(x) = 0$ for $|x| \geq 2^{1/3}$, $|\theta(x)| \leq |x|$, $\|\theta'\|_\infty \leq 4$, and $\|\theta\|_\infty \leq 2$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $\phi(x) = (\theta(x_1), \theta(x_2), \theta(x_3))$. Then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi\|_\infty \leq 2$. For $R > 0$, set $\phi_R(x) = R\phi(x/R)$. Let $z_R : \mathbb{R} \rightarrow \mathbb{R}^3$ be the truncated center of mass given by

$$z_R(t) = \int \phi_R(x) |u(x, t)|^2 \, dx.$$

Then $z'_R(t) = ([z'_R(t)]_1, [z'_R(t)]_2, [z'_R(t)]_3)$, where

$$[z'_R(t)]_j = 2 \operatorname{Im} \int \theta'(x_j/R) \partial_j u \bar{u} \, dx.$$

Note that $\theta'(x_j/R) = 1$ for $|x_j| \leq 1$. By the zero momentum property,

$$\operatorname{Im} \int_{|x_j| \leq R} \partial_j u \bar{u} = -\operatorname{Im} \int_{|x_j| > R} \partial_j u \bar{u},$$

and thus,

$$[z'_R(t)]_j = -2 \operatorname{Im} \int_{|x_j| \geq R} \partial_j u \bar{u} dx + 2 \operatorname{Im} \int_{|x_j| \geq R} \theta'(x_j/R) \partial_j u \bar{u} dx,$$

from which we obtain by Cauchy-Schwarz,

$$(5.4) \quad |z'_R(t)| \leq 5 \int_{|x| \geq R} (|\nabla u|^2 + |u|^2).$$

Set $\tilde{R}_n = R_n + R_0(\epsilon)$. Note that for $0 \leq t \leq \tilde{t}_n$ and $|x| > \tilde{R}_n$, we have $|x + x(t)| \geq \tilde{R}_n - R_n = R_0(\epsilon)$, and thus, (5.4) and (5.3) give

$$(5.5) \quad |z'_{\tilde{R}_n}(t)| \leq 5\epsilon.$$

Now we obtain an upper bound for $z_{\tilde{R}_n}(0)$ and a lower bound for $z_{\tilde{R}_n}(t)$.

$$z_{\tilde{R}_n}(0) = \int_{|x| < R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2 dx + \int_{|x+x(0)| \geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2 dx,$$

and hence, by (5.3), we have

$$(5.6) \quad |z_{\tilde{R}_n}(0)| \leq R_0(\epsilon)M[u] + 2\tilde{R}_n\epsilon.$$

For $0 \leq t \leq \tilde{t}_n$, we split $z_{\tilde{R}_n}(t)$ as

$$\begin{aligned} z_{\tilde{R}_n}(t) &= \int_{|x+x(t)| \geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u(x,t)|^2 dx + \int_{|x+x(t)| \leq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u(x,t)|^2 dx \\ &= \text{I} + \text{II} \end{aligned}$$

To estimate I, we note that $|\phi_{\tilde{R}_n}(x)| \leq 2\tilde{R}_n$ and use (5.3) to obtain $|\text{I}| \leq 2\tilde{R}_n\epsilon$. For II, we first note that $|x| \leq |x+x(t)| + |x(t)| \leq R_0(\epsilon) + R_n = \tilde{R}_n$, and thus $\phi_{\tilde{R}_n}(x) = x$. We now rewrite II as

$$\begin{aligned} \text{II} &= \int_{|x+x(t)| \leq R_0(\epsilon)} (x+x(t)) |u(x,t)|^2 dx - x(t) \int_{|x+x(t)| \leq R_0(\epsilon)} |u(x,t)|^2 dx \\ &= \int_{|x+x(t)| \leq R_0(\epsilon)} (x+x(t)) |u(x,t)|^2 dx - x(t)M[u] + x(t) \int_{|x+x(t)| \geq R_0(\epsilon)} |u(x,t)|^2 dx \\ &= \text{IIA} + \text{IIB} + \text{IIC}. \end{aligned}$$

Trivially, $|\text{IIA}| \leq R_0(\epsilon)M[u]$, and by (5.3), $|\text{IIC}| \leq |x(t)|\epsilon \leq \tilde{R}_n\epsilon$. Thus,

$$\begin{aligned} |z_{\tilde{R}_n}(t)| &\geq |\text{IIB}| - |\text{I}| - |\text{IIA}| - |\text{IIC}| \\ &\geq |x(t)|M[u] - R_0(\epsilon)M[u] - 3\tilde{R}_n\epsilon. \end{aligned}$$

Taking $t = \tilde{t}_n$, we get

$$(5.7) \quad |z_{\tilde{R}_n}(\tilde{t}_n)| \geq \tilde{R}_n(M[u] - 3\epsilon) - R_0(\epsilon)M[u].$$

Combining (5.5), (5.6), and (5.7), we have

$$\begin{aligned} 5\epsilon \tilde{t}_n &\geq \int_0^{\tilde{t}_n} |z'_{\tilde{R}_n}(t)| dt \geq \left| \int_0^{\tilde{t}_n} z'_{\tilde{R}_n}(t) dt \right| \geq |z_{\tilde{R}_n}(\tilde{t}_n) - z_{\tilde{R}_n}(0)| \\ &\geq \tilde{R}_n(M[u] - 5\epsilon) - 2R_0(\epsilon)M[u]. \end{aligned}$$

Dividing by \tilde{t}_n and using that $\tilde{R}_n \geq R_n$ (assume $\epsilon \leq \frac{1}{5}M[u]$), we obtain

$$5\epsilon \geq \frac{R_n}{\tilde{t}_n}(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}.$$

Since $R_n/\tilde{t}_n \geq \epsilon_0$, we have

$$5\epsilon \geq \epsilon_0(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}.$$

Take $\epsilon = M[u]\epsilon_0/16$ (assume $\epsilon_0 \leq 1$), and then send $n \rightarrow +\infty$. Since $\tilde{t}_n \rightarrow +\infty$, we get a contradiction. \square

6. Rigidity theorem

We now prove the following rigidity, or Liouville-type, theorem.

Theorem 6.1 (Rigidity). *Suppose $u_0 \in H^1$ satisfies $P[u_0] = 0$,*

$$(6.1) \quad M[u_0]E[u_0] < M[Q]E[Q]$$

and

$$(6.2) \quad \|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}.$$

Let u be the global H^1 solution of (1.1) with initial data u_0 and suppose that

$$K = \{u(\cdot - x(t), t) \mid t \in [0, +\infty)\} \text{ is precompact in } H^1.$$

Then $u_0 = 0$.

Before beginning the proof, we recall in Lemma 6.2 below a few basic facts proved in [7]. These facts are consequences of the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \leq c_{GN}\|u\|_{L^2}\|\nabla u\|_{L^2}^3$$

with the sharp value of c_{GN} expressed as

$$c_{GN} = \frac{4}{3\|Q\|_2\|\nabla Q\|_2}.$$

One also uses the relation

$$M[Q]E[Q] = \frac{1}{6}\|Q\|_{L^2}^2\|\nabla Q\|_{L^2}^2,$$

which is a consequence of the Pohozaev identities.

Lemma 6.2. *If $M[u]E[u] < M[Q]E[Q]$ and $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then for all t ,*

$$(6.3) \quad \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq \omega\|Q\|_{L^2}\|\nabla Q\|_{L^2},$$

where $\omega = \left(\frac{M[u]E[u]}{M[Q]E[Q]}\right)^{1/2}$. We also have the bound, for all t

$$(6.4) \quad 8\|\nabla u(t)\|^2 - 6\|u(t)\|_{L^4}^4 \geq 8(1 - \omega)\|\nabla u(t)\|_{L^2}^2 \geq 16(1 - \omega)E[u].$$

We remark that under the hypotheses here, $E[u] > 0$ unless $u \equiv 0$. In fact, one has the bound $E[u] \geq \frac{1}{6} \|\nabla u_0\|_{L^2}^2$.

Proof of Theorem 6.1. In the proof below, all instances of a constant c refer to some absolute constant. Let $\varphi \in C_0^\infty$ be radial with

$$\varphi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}.$$

For $R > 0$, define

$$z_R(t) = \int R^2 \varphi\left(\frac{x}{R}\right) |u(x, t)|^2 dx.$$

Then, by direct calculation,

$$z'_R(t) = 2 \operatorname{Im} \int R \nabla \varphi\left(\frac{x}{R}\right) \cdot \nabla u(t) \bar{u}(t) dx.$$

By the Hölder inequality,

$$(6.5) \quad |z'_R(t)| \leq cR \int_{|x| \leq 2R} |\nabla u(t)| |u(t)| dx \leq cR \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}.$$

Also by direct calculation, we have the local virial identity

$$z''_R(t) = 4 \sum_{j,k} \int \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \left(\frac{x}{R}\right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int (\Delta^2 \varphi) \left(\frac{x}{R}\right) |u|^2 - \int (\Delta \varphi) \left(\frac{x}{R}\right) |u|^4.$$

Since φ is radial, we have

$$(6.6) \quad z''_R(t) = \left(8 \int |\nabla u|^2 - 6 \int |u|^4\right) + A_R(u),$$

where

$$A_R(u) = 4 \sum_j \int \left((\partial_{x_j}^2 \varphi) \left(\frac{x}{R}\right) - 2 \right) \left| \frac{\partial u}{\partial x_j} \right|^2 + 4 \sum_{j \neq k} \int_{R \leq |x| \leq 2R} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \left(\frac{x}{R}\right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int (\Delta^2 \varphi) \left(\frac{x}{R}\right) |u|^2 - \int \left((\Delta \varphi) \left(\frac{x}{R}\right) - 6 \right) |u|^4.$$

From this expression, we obtain the bound

$$(6.7) \quad |A_R(u(t))| \leq c \int_{|x| \geq R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^4 \right) dx.$$

We want to examine $z_R(t)$, for R chosen suitably large, over a suitably chosen time interval $[t_0, t_1]$, where $1 \ll t_0 \ll t_1 < \infty$. By (6.6) and (6.4), we have

$$(6.8) \quad |z''_R(t)| \geq 16(1 - \omega)E[u] - |A_R(u(t))|.$$

Set $\epsilon = \frac{1-\omega}{c} E[u]$ in Corollary 3.3 to obtain $R_0 \geq 0$ such that $\forall t$,

$$(6.9) \quad \int_{|x+x(t)| \geq R_0} (|\nabla u|^2 + |u|^2 + |u|^4) \leq \frac{(1 - \omega)}{c} E[u].$$

If we select $R \geq R_0 + \sup_{t_0 \leq t \leq t_1} |x(t)|$, then (6.8) combined with the bounds (6.7) and (6.9) will imply that, for all $t_0 \leq t \leq t_1$,

$$(6.10) \quad |z_R''(t)| \geq 8(1 - \omega)E[u].$$

By Lemma 5.1, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, we have $|x(t)| \leq \eta t$, with $\eta > 0$ to be selected later. Thus, by taking $R = R_0 + \eta t_1$, we obtain that (6.10) holds for all $t_0 \leq t \leq t_1$. Integrating (6.10) over $[t_0, t_1]$, we obtain

$$(6.11) \quad |z_R'(t_1) - z_R'(t_0)| \geq 8(1 - \omega)E[u](t_1 - t_0).$$

On the other hand, for all $t_0 \leq t \leq t_1$, by (6.5) and (6.3), we have

$$(6.12) \quad \begin{aligned} |z_R'(t)| &\leq cR\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq cR\|Q\|_{L^2}\|\nabla Q\|_{L^2} \\ &\leq c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1). \end{aligned}$$

Combining (6.11) and (6.12), we obtain

$$8(1 - \omega)E[u](t_1 - t_0) \leq 2c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1).$$

Recall that ω and R_0 are constants depending only upon $(M[u]E[u])/(M[Q]E[Q])$, while $\eta > 0$ is yet to be specified and $t_0 = t_0(\eta)$. Put $\eta = (1 - \omega)E[u]/(c\|Q\|_2\|\nabla Q\|_2)$ and then send $t_1 \rightarrow +\infty$ to obtain a contradiction unless $E[u] = 0$ which implies $u \equiv 0$. \square

To complete the proof of Theorem 1.1, we just apply Theorem 6.1 to u_c constructed in Proposition 3.1, which by Propositions 3.2 and 4.1, meets the hypotheses in Theorem 6.1. Thus $u_{c,0} = 0$, which contradicts the fact that $\|u_c\|_{S(\dot{H}^{1/2})} = \infty$. We have thus obtained that if $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ and $M[u]E[u] < M[Q]E[Q]$, then SC(u_0) holds, i.e. $\|u\|_{S(\dot{H}^{1/2})} < \infty$. By Proposition 2.2 [7], H^1 scattering holds.

7. Remarks on the defocusing equation

One may use the above arguments to show H^1 -scattering of solutions of the defocusing equation

$$(7.1) \quad i\partial_t u + \Delta u - |u|^2 u = 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R},$$

$$(7.2) \quad u(x, 0) = u_0 \in H^1(\mathbb{R}^3).$$

In this case, scattering is already known, as a consequence of Morawetz [6], or interaction Morawetz [4] inequalities.

We argue by contradiction. If scattering does not hold, there exists a critical solution u_c , which does not scatter, and such that $M[u_c]E[u_c]$ is minimal for non-scattering solutions of (7.1). As before, one shows that $P[u_c] = 0$, and that there exists $x(t)$ such that the set $K = \{u_c(t, \cdot - x(t)), t \in \mathbb{R}\}$ is precompact in H^1 . Note that because of the defocusing sign of the non-linearity, we do not need to assume $M[u_c]E[u_c] < M[Q]E[Q]$ and $\|u_c(0)\|_{L^2}\|\nabla u_c(0)\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$. The control of the spatial translation $x(t)$ works as in Section 5, and one concludes as in Section 6, by a localized virial argument, using that in the defocusing case, the second derivative of the localized variance $z_R(t)$ is

$$z_R''(t) = \left(8 \int |\nabla u|^2 + 6 \int |u|^4 \right) + B_R(u(t)),$$

where B_R satisfies the bound

$$|B_R(u(t))| \leq c \int_{|x| \geq R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^4 \right) dx.$$

Note that the use of the virial identity is potentially more robust since one might be able to handle variants of the NLS equation (for example with a linear potential) that might be out of reach for Morawetz based proofs.

Appendix A. A lifting lemma

In this appendix, we discuss some basic analysis facts needed in the very beginning of the proof of Prop. 3.2.

Let $G \cong \mathbb{R}^3$ act on H^1 by translation, i.e., $(x_0 \cdot \phi)(x) = \phi(x - x_0)$. Write $G \backslash H^1$ for the quotient space endowed with the quotient topology. We represent elements of $G \backslash H^1$ (the equivalence classes) by $[\phi]$, and let $\pi : H^1 \rightarrow G \backslash H^1$ be the natural projection.

Lemma A.1. *$G \backslash H^1$ is metrizable with metric*

$$d([\phi], [\psi]) = \inf_{x_0 \in \mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1}.$$

With respect to this metric, $G \backslash H^1$ is complete. (Caution that $G \backslash H^1$ is not a vector space, however.)

Proof. First, we establish that the orbits of G are closed in H^1 . The orbit of 0 is 0. Suppose $\phi \neq 0$, $\{x_n\} \subset \mathbb{R}^3$ and $\phi(\cdot - x_n)$ converges to ψ in H^1 . Then we claim that x_n converges. Indeed, if not, then either x_n is unbounded and there is a subsequence x_n such that $|x_n| \rightarrow \infty$, or x_n is bounded and there are two subsequences $x_n \rightarrow x_0$ and $x_{n'} \rightarrow x'_0$. In the first case, we obtain that $\psi = 0$ (by examining, for fixed $R > 0$, the convergence on $B(0, R)$), which implies $\phi = 0$, a contradiction. In the second case, we obtain that $\phi(\cdot - x_0) = \phi(\cdot - x'_0)$, only possible if $\phi = 0$, a contradiction.

Next, we verify that d is a metric. Suppose $d([\phi], [\psi]) = 0$. Then $\inf_{x_0 \in \mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1} = 0$, and thus ψ is a point of closure (in H^1) of the orbit of ϕ . But since the orbits are closed, ψ belongs to this orbit, and thus, $[\phi] = [\psi]$. The triangle inequality is a straightforward exercise dealing with infima, and symmetry is obvious.

Suppose $[\phi_n]$ is a Cauchy sequence; to show that it converges, it suffices to show that a subsequence converges. We can pass to a subsequence $[\phi_n]$ so that $d([\phi_n], [\phi_{n+1}]) \leq 2^{-n}$. Take $x_1 = 0$. Construct a sequence x_n inductively as follows: given x_{n-1} , select x_n so that $\|\phi_{n-1}(\cdot - x_{n-1}) - \phi_n(\cdot - x_n)\|_{H^1} \leq 2^{-n+1}$. Then $\phi_n(\cdot - x_n)$ is a Cauchy sequence in H^1 , and hence, converges to some ϕ . It is then clear that $[\phi_n] \rightarrow [\phi]$ in $G \backslash H^1$.

It can be checked that for each $\phi \in H^1$ and $r > 0$, $\pi(B(\phi, r)) = B([\phi], r)$. Therefore, the topology induced by the metric d on $G \backslash H^1$ is the quotient topology. \square

The following two lemmas will reduce Prop. 3.2 to proving that the set

$$\pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$$

is precompact in $G \backslash H^1$.

Lemma A.2. *Let K be a precompact subset of $G \setminus H^1$. Assume*

$$(A.1) \quad \exists \eta > 0 \text{ such that } \forall \phi \in \pi^{-1}(K), \quad \eta \leq \|\phi\|_{H^1}.$$

Then there exists \tilde{K} precompact in H^1 such that $\pi(\tilde{K}) = K$.

Proof. Let $B(0, 1)$ be the unit ball in \mathbb{R}^3 . We first show by contradiction that there exists $\varepsilon > 0$ such that for all p in K , there exists $\psi = \psi(p) \in \pi^{-1}(p)$ such that

$$(A.2) \quad \|\psi(p)\|_{H^1(B(0,1))} \geq \varepsilon.$$

If not, there exists a sequence ϕ_n in $\pi^{-1}(K)$ such that

$$(A.3) \quad \sup_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0)\|_{H^1(B(0,1))} \leq \frac{1}{n}.$$

The precompactness of K implies, extracting a subsequence from ϕ_n if necessary, that there exists $\phi \in H^1$ such that $\pi(\phi_n) \rightarrow p$ in $G \setminus H^1$. In other words, if ϕ is fixed in $\pi^{-1}(p)$, $\inf_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0) - \phi\|_{H^1}$ tends to 0 as n tends to infinity. Thus, one may find a sequence x_n in \mathbb{R}^3 such that

$$(A.4) \quad \|\phi_n(\cdot - x_n) - \phi\|_{H^1} \xrightarrow{n \rightarrow +\infty} 0.$$

Now, by (A.3), for all $x_0 \in \mathbb{R}^3$, $\|\phi_n(\cdot - x_0 - x_n)\|_{H^1(B(0,1))} \leq \frac{1}{n}$. Hence, by (A.4), for all x_0 , ϕ vanishes on $B(x_0, 1)$. But then $\phi = 0$, which contradicts assumption (A.1), concluding the proof of the existence of $x(\phi)$.

Let $\tilde{K} = \{\psi(p) \mid p \in K\}$, where $\psi(p)$ satisfies (A.2). Of course, $\pi(\tilde{K}) = K$. By the definition of $x(\phi)$,

$$(A.5) \quad \forall \phi \in \pi^{-1}(K), \quad \|\phi\|_{H^1(B(0,1))} \geq \varepsilon.$$

Let us show that \tilde{K} is precompact. Let ϕ_n be a sequence in \tilde{K} . Then by the precompactness of K , there exists (extracting subsequences) $\phi \in H^1$ and a sequence x_n of \mathbb{R}^3 , such that

$$(A.6) \quad \lim_{n \rightarrow +\infty} \|\phi_n(\cdot - x_n) - \phi\|_{H^1} = 0.$$

Note that K being precompact, ϕ_n is bounded in H^1 , thus, we may assume (extracting again)

$$(A.7) \quad \lim_{n \rightarrow +\infty} \|\phi_n\|_{H^1} = \ell \in (0, +\infty).$$

Let us show that x_n is bounded. If not, we may assume that $|x_n| \rightarrow +\infty$. By (A.5) and (A.7), we have

$$\limsup_{n \rightarrow \infty} \|\phi_n(\cdot - x_n)\|_{H^1(B(0, |x_n| - 1))} \leq \ell - \varepsilon.$$

As $|x_n| \rightarrow \infty$, we conclude that $\|\phi\|_{H^1} \leq \ell - \varepsilon$, contradicting (A.7). Therefore, x_n is bounded. Extracting if necessary, we may assume that x_n converges, which shows by (A.6) that ϕ_n converges. This concludes the proof of the precompactness of \tilde{K} . \square

Lemma A.3. *Let u be a global H^1 solution to (1.1). Suppose*

$$\pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$$

is precompact in $G \setminus H^1$. Then there exists $x(t)$, a continuous path in \mathbb{R}^3 , such that

$$\{u(\cdot - x(t), t) \mid t \in [0, +\infty)\}$$

is precompact in H^1 .

Proof. By taking $K = \pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$ in Lemma A.2, we obtain \tilde{K} precompact in H^1 such that $\pi(\tilde{K}) = K$. For each N , the map $u : [N, N+1] \rightarrow H^1$ is uniformly continuous. Thus, for each N , there exists $\delta_N > 0$ such that if $t, t' \in [N, N+1]$ and $|t - t'| \leq \delta_N$, then $\|u(t, \cdot) - u(t', \cdot)\|_{H^1} \leq 1/N$. Let t_n be the increasing sequence of times $\rightarrow +\infty$ defined to include evenly spaced elements with density δ_N in $[N, N+1]$ for each N . (Thus, t_n is an increasing sequence with possibly more elements per unit interval as we move out to $+\infty$.) For each n , select $x(t_n) \in \mathbb{R}^3$ such that $u(\cdot - x(t_n), t_n) \in \tilde{K}$. Now define $x(t)$ to be the continuous function that connects $x(t_n)$ to $x(t_{n+1})$ by a straight line in \mathbb{R}^3 .

We claim that $\{u(\cdot - x(t), t) \mid t \in [0, +\infty)\}$ is precompact in H^1 . Indeed, let s_k be a sequence in $[0, +\infty)$. Then there exists a subsequence (also labeled s_k) such that either s_k converges to some finite s_0 or $s_k \rightarrow +\infty$. In the first case, $u(\cdot - x(s_k), s_k) \rightarrow u(\cdot - x(s_0), s_0)$ by the continuity of $u(t)$ and $x(t)$. In the second case, for each k , obtain the unique index $n(k)$ such that $t_{n(k)-1} \leq s_k < t_{n(k)}$. By the precompactness of \tilde{K} , we can pass to a subsequence (in k) such that both $u(\cdot - x(t_{n(k)-1}), t_{n(k)-1})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge in H^1 . By the density of the t_n sequence and uniform continuity of u , we obtain that $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ converges and that it suffices to show that $u(\cdot - x(s_k), t_{n(k)})$ has a convergent subsequence. But since both $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge, we have that $x(t_{n(k)-1}) - x(t_{n(k)})$ converges. Recall that $x(s_k)$ lies on the line segment joining $x(t_{n(k)-1})$ and $x(t_{n(k)})$, and thus, $x(s_k) - x(t_{n(k)-1})$ converges (after passing to a subsequence). Hence, $u(\cdot - x(s_k), t_{n(k)})$ converges in H^1 . \square

Thus, to prove Prop. 3.2, it suffices to prove that

$$(A.8) \quad \pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$$

is precompact in $G \setminus H^1$. Since $G \setminus H^1$ is complete, if we assume that (A.8) is not precompact in $G \setminus H^1$, then there exists a sequence $\{[u(t_n)]\}$ in $G \setminus H^1$ and $\eta > 0$ such that $d([u(t_n)], [u(t_{n'})]) \geq \eta$, or equivalently, (3.4) in the proof of Prop 3.2 holds.

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