

**IRREGULAR BALL-QUOTIENT SURFACES WITH
NON-POSITIVE KODAIRA DIMENSION**

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Dedicated to Rolf-Peter Holzapfel on occasion of his retirement

ABSTRACT. Let $\Gamma \subset \mathbf{PU}(2,1)$ be a lattice. Γ then acts on the complex open unit ball $\mathbf{B} \subset \mathbb{C}^2$. Suppose that Γ is not cocompact and ‘sufficiently’ neat. Then there exists a compactification $X = \overline{\Gamma \backslash \mathbf{B}}$ with the property that X is a smooth projective surface and $D = X \setminus (\Gamma \backslash \mathbf{B})$ is a disjoint sum of elliptic curves. The main result is: $kod(X) \leq 0$, $h^1(\mathcal{O}_X) > 0 \Leftrightarrow X$ admits an abelian surface Y as minimal model, and Y carries a divisor $S = \sum S_i$ which consists of elliptic curves such that $4 \cdot |sing S| = \sum |S_i \cap sing S|$.

1. Introduction

1.1. Motivation. Let U be a complex manifold which has the complex open unit-ball $\mathbf{B} \subset \mathbb{C}^2$ as universal holomorphic covering. If U is compact then it is of general type. The situation changes if one considers surfaces X which are toroidal compactifications of non-compact unramified ball-quotients $U \subset X$. Then the compactification divisor $D = X \setminus U$ is a disjoint sum of elliptic curves and X may be of special type. The article is concerned with this problem. It marks the beginning of a classification of surfaces $X = U \cup D$ by purely geometric methods. For simplicity, the class of such surfaces will be denoted by \mathcal{T} .

Recall that $\mathbf{PU}(2,1)$ is identified with the group of holomorphic automorphisms of

$$\mathbf{B} = \{[x_0 : x_1 : x_2] \in \mathbf{P}^2; |x_0|^2 + |x_1|^2 < |x_2|^2\}$$

in a natural way. Therefore, surfaces in \mathcal{T} typically arise from the minimal singularity resolution of Baily-Borel compactifications of ball-quotients $U = \Gamma \backslash \mathbf{B}$ by neat subgroups $\Gamma \subset \mathbf{PU}(2,1)$ which are arithmetic but not cocompact. Natural candidates are subgroups $\Gamma \subset \mathbf{PU}(2,1; \mathcal{O}_{\mathbf{K}})$ with \mathbf{K} an imaginary quadratic field, and their quotients serve as a natural pendant to Hilbert modular surfaces. More generally, however, any discrete $\Gamma \subset \mathbf{PU}(2,1)$ contains a neat subgroup of finite index inducing a surface in \mathcal{T} , provided that $\Gamma \backslash \mathbf{B}$ is not compact and admits a fundamental domain with finite Bergman-volume. Hence, \mathcal{T} forms in fact the class of ‘generic’ compactified ball-quotients.

The french mathematician E. Picard discovered the ball 125 years ago as co-domain

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of multivalued hypergeometric integrals and also noticed the analogy to the j -line in the theory of elliptic functions (cf. [9]). Systematic research in this area was, among others, started by Holzapfel. He called the surfaces, which arise from subgroups $\Gamma \subset \mathbf{PU}(2, 1; \mathcal{O}_{\mathbf{K}})$, **Picard modular surfaces**. Already in the middle of the 1980's, Holzapfel was able to boast an amount of classification results for these special ball-quotients. We refer to [4] and to the new book [8] for an overview and a bibliography. These results, however, are based on an arithmetic theory of ball-lattices and are proved using arithmetically defined lattice invariants. By way of contrast, the fundamental work of Deligne-Mostow [2] shows that also non-arithmetic ball-lattices should be considered. Moreover, even for the arithmetic case it is not clear *to what extent* there are general obstructions for ball-quotient surfaces with respect to the Enriques-Kodaira classification.

1.2. Statement and discussion of the main result. In the hypotheses of our **main theorem** all complex surfaces are smooth and compact.

Theorem 1.1. *Let Y be a relatively minimal surface with non-positive Kodaira dimension. Assume that Y is irregular, that is, $h^1(\mathcal{O}_Y) > 0$. Then the following are equivalent:*

- (1) Y is birationally dominated by a surface in \mathcal{T} .
- (2) Y is an abelian surface. It supports a reduced elliptic divisor $S = \sum S_i$ satisfying the proportionality condition

$$4|M| = \sum s_i$$

where $M = \text{sing } S$ is the set of singular points of the curve S and $s_i = |S_i \cap M|$.

If (2) holds then Y is isogenous to a product $C \times C$ of an elliptic curve C , and we have: $s_i > 0$ for each i ; $s_x = |\{S_i; x \in S_i\}| \geq 3$ for $x \in M$; blowing up once along each $x \in M$ produces a surface in \mathcal{T} with compactification divisor the proper transform of S .

The proof relies on two steps. We will show that an $X \in \mathcal{T}$ with Kodaira dimension $\text{kod}(X) \leq 0$ admits only abelian minimal models or is regular. This assertion is completely new. Then, up to the claim about s_i, s_x (which results from Cor. 3.2 and Lemma 3.3), the gap from the latter result to the stated theorem is closed by the following criterion of Holzapfel [7, Thm. 2.5]:

Theorem 1.2 (Holzapfel). *Let Y be an abelian surface admitting a reduced divisor $S = \sum S_i$ of elliptic curves. Let $M = \text{sing } S$ be the set of singular point of S and set $s_i = |S_i \cap M|$. Then Y is the minimal model of an $X \in \mathcal{T}$ with compactification divisor D the proper transform of S if and only if the porportionality condition*

$$4|M| = \sum s_i$$

holds with $s_i > 0$ for each i . More precisely, if the conditions are satisfied then X results from blowing up once along each point $x \in M$ and Y is isogenous to a product $C \times C$ of an elliptic curve C .

The value of Thm. 1.1 and Thm. 1.2 arises from the following modular intepretation. For $m \geq 2$, the vector space $[\Gamma, m]$ of Γ -modular forms of weight $m \geq 2$ is naturally identified with $H^0(X, m(K_X + D))$. Now, together with the surfaces X and Y , to

a large extend the corresponding rings of automorphic forms can also be effectively compared to each other. The automorphic forms belonging to a square $C \times C$ of an elliptic curve C may be expressed by means of the classical theta function, and hence the Γ -automorphic forms coming from the ball-lattice Γ corresponding to X may be ‘transferred’ into such ‘theta-expressions’. In [6], Holzapfel laid the foundation for this approach by constructing ‘appropriate’ quotients of elliptic theta functions and illustrated the advantages of such representations for the determination of X and the ring of Γ -automorphic forms. In [5], he also calculated the dimension of $[\Gamma, m]$ in the abelian case. The proof given there readily extends to the general case. To be more precise, we have

$$\dim_{\mathbb{C}}[\Gamma, m] = \frac{3}{2}m(m-1)e(X) + \chi(X) + h_X \quad (m \geq 2)$$

where $e(X)$ is the topological Euler number, $\chi(X) = \sum (-1)^i h^i(X, \mathcal{O}_X)$ and h_X is the number of components of the compactification divisor D .

However, it is known that ‘most’ surfaces in \mathcal{T} are of general type. The log pair (X, D) is logarithmically of general type, i.e. $\text{kod}(X, K_X + D) = 2$. Thus our structure result seems to deal with exceptional cases. And in fact, up to isogeny only few commensurability classes of surfaces in \mathcal{T} of special type are known yet; all of them admit abelian minimal models (cf. [7]).

1.3. Organization of the proof. The proof of Thm. 1.1 is essentially based on classification theory of algebraic surfaces. In order to make our result more accessible for researchers who, for instance, come from arithmetic but are no experts in geometric methods, we repeat in Section 2 some general facts from the geometry of surfaces. In Section 3 we state first properties of surfaces in $X \in \mathcal{T}$ which are needed in all parts of the proof. In the two remaining sections we prove our main result. After the preparations, each step is rather easy.

2. Short review of surface theory

2.1. From the classification theory of algebraic surfaces. We focus on the category of algebraic surfaces over \mathbb{C} which are **non-singular**, **integral** and **projective**. Throughout, a ‘surface’ will always mean a two dimensional \mathbb{C} -variety of this type as long as the converse is not explicitly stated. A surface X is called **relatively minimal** if it does not contain a (-1) -curve L , i.e. a rational curve with self-intersection $= -1$. It is a **minimal surface** if all relatively minimal models in the birational class of X are pairwise isomorphic. The latter holds automatically if the Kodaira dimension $\text{kod}(X)$ is non-negative. Starting with an arbitrary smooth surface X , one obtains a birational morphism to a relatively minimal model by successively blowing down arbitrary (-1) -curves. For this and what follows, we refer to [1]. A **geometrically ruled surface** is by definition a smooth surface X together with a \mathbf{P}^1 -bundle $\pi : X \rightarrow C$ over an irreducible smooth curve C . More generally, a fibration that factors through a geometrically ruled surface is called a **ruling**, and a **ruled surface** is a surface together with a ruling. By classification theory for algebraic surfaces we have:

- (1) If $\text{kod}(X) = -\infty$ and $q(X) > 0$ then X admits a ruling over a curve C with the property that $q(X) = g_C$, the genus of C . In this case, X is birational to $\mathbf{P}^1 \times C$. Moreover, $\text{Num}(X)$, the group of divisors modulo numerical equivalence, is free of rank 2 and generated by a fibre F and an arbitrary section C_o of π (cf. [3, Ch. V, Prop. 2.3]).
- (2) If X is a minimal surface so that $\text{kod}(X) = 0$ then K_X is numerically trivial. If $q(X) > 0$ then a minimal surface X is either abelian, or a hyperelliptic surface. In the second case, there exists an étale covering $A \rightarrow X$ by an abelian surface A , and $b_2(X) = 2$.

2.2. A result on ruled surfaces. We will need the following easy lemma.

Lemma 2.1. *Let $\pi : X \rightarrow C$ be a ruled surface. Let $C_o \subset X$ be a section of π . Then there exists a diagram $X \xrightarrow{\phi} Y \xrightarrow{\tilde{\pi}} C$ such that ϕ is an isomorphism in an open neighbourhood of C_o and $\tilde{\pi} : Y \rightarrow C$ is a geometrically ruled surface.*

Proof. Let $F = \sum E_k$ be a reducible π -fibre. Since $FC_o = 1$, there is precisely one E_{k_o} having non-trivial intersection with C_o . Therefore, we have to show that one can find a succession of blowing downs of the E_k 's with the property that the image of E_{k_o} becomes a smooth fibre. By induction on the number n of components E_k , all we have to verify is the following: As long as $n \geq 2$, $F = \sum E_k$ supports one (-1) -curve E_{k_1} with $E_{k_1}C_o = 0$. The assertion is certainly clear if $n = 2$. Assume the assertion holds for a $n \geq 2$, and let F_n be a fibre consisting of n components. Blowing up once along $x \in F_n$, the resulting (-1) -curve satisfies our requirement if $x \notin C_o$. If $x \in C_o$ let $E_{k_1} \neq E_{k_o}$ be a (-1) -curve not intersecting C_o . Then the proper transform of E_{k_1} on the new fibre F_{n+1} is still a (-1) -curve not intersecting (the proper transform of) C_o . The assertion follows, because each fibre with $n + 1$ components is a blown up F_n . \square

3. Fundamental properties of surfaces in \mathcal{T}

We return to toroidally compactified ball-quotient surfaces. The most important properties of surfaces in \mathcal{T} are summarized in the following theorem. It is a special case of Tian-Yau's inequality [10, Thm. 3.1].

Theorem 3.1. *Let X be a complex projective surface together with a normal-crossing divisor $D \subset X$. Assume:*

- (1) $K_X + D$ is nef and $K_X + D$ is ample modulo D .
- (2) $3c_2(X) = (K_X + D)^2 > 0$.

Then $X \setminus D$ admits the open unit ball $\mathbf{B} \subset \mathbb{C}^2$ as universal holomorphic covering. Conversely, an unramified ball-quotient surface $X \setminus D$ with compactification divisor D which is a sum of elliptic curves fulfills (1) and (2).

In (1), 'ample modulo D ' means that $\mathcal{O}_C(K_X + D)$ is positive for each curve $C \subset X$ not supported in D . (1) is equivalent to the fact that for large m the sheaf $\mathcal{L}^{\otimes m}$, where $\mathcal{L} = \mathcal{O}_X(K_X + D)$, is generated by global sections and provides an embedding of $X \setminus D$ into projective space (cf. [10, Rem. p. 613]).

Corollary 3.2. *Let X be a surface in \mathcal{T} with compactification divisor D . Then D is contractible such that the contraction space is a normal projective variety. In particular, each component D_i of D has a negative self-intersection number $D_i^2 < 0$.*

Although readily verified, the next two observations are indispensable ingredients in the proof of our main result.

Lemma 3.3. *Let X be in \mathcal{T} with compactification divisor D and consider a rational curve $L \subset X$. Then $|L \cap D| \geq 3$.*

Proof. Assume $|L \cap D| = 2$. Then $L \setminus D$ is up to isomorphism \mathbb{C}^* , and admits the complex line a universal holomorphic covering. Similarly, if $|L \cap D| = 1$ then $L \setminus D \simeq \mathbb{C}$. Proceeding to universal coverings we would thus obtain a holomorphic map from \mathbb{C} to \mathbf{B} , which is not trivial. This contradicts Liouville’s theorem. \square

Lemma 3.4. *Let $\pi : X \rightarrow C$ be a fibration of an element in \mathcal{T} over an elliptic curve C . Assume that $\phi(D) = C$. Then there exists a commutative diagram*

$$\begin{CD} \tilde{X} @>\tilde{\pi}>> \tilde{C} \\ @V\beta VV @VVgV \\ X @>\pi>> C \end{CD}$$

such that β and g are etale; that is, \tilde{X} is a surface in \mathcal{T} with compactification divisor $\tilde{D} = \beta^{-1}(D)$ and fibred over \tilde{C} , and with the additional property that each component of \tilde{D} is a section of $\tilde{\pi}$, as soon as it dominates \tilde{C} .

Proof. Let F be a general fibre of π . The proof is by induction on $n(X) := \max \{D_i F\}$, where D_i ranges over all curves in the compactification divisor dominating C . We suppose w.l.o.g. that $n(X) = D_1 F > 1$, since otherwise there is nothing to prove. By general properties of the fibre product, the projection

$$\pi_1 := pr_{D_1} : X_1 = X \times_C D_1 \rightarrow C_1 = D_1$$

has a copy $T_1 \subset X_1$ of D_1 as a section. One checks that the projection $\beta_1 : X_1 \rightarrow X$ is etale, and that a general fibre of π is dominated by $n(X) = \deg(D_1 \rightarrow C)$ fibres of π_1 . Therefore, X_1 is smooth, $\deg \beta_1 = n(X)$, and $\beta_1^{-1}(D_1)$ is of the form $T_1 + B$ with a divisor B . More precisely, noting that $\beta_1^{-1}(X \setminus D)$ must have the ball \mathbf{B} as universal holomorphic covering, one recognizes that X_1 is a surface in \mathcal{T} with compactification part $\beta_1^{-1}(D)$. Hence, B consists of a disjoint sum of elliptic curves contained in the compactification divisor of X_1 . Fix a D_i dominating C and let D_{ij} range over the curves in $\beta_1^{-1}(D_i)$. We claim:

$$(3.1) \quad \sum_j \deg(D_{ij} \rightarrow C_1) = \deg(D_i \rightarrow C) \quad (i \text{ fixed})$$

We have $\deg(D_1 \rightarrow C) = \deg(C_1 \rightarrow C)$. Thus, an easy calculation yields for all j :

$$(3.2) \quad \deg(D_{ij} \rightarrow D_i) \cdot \deg(D_i \rightarrow C) = \deg(D_{ij} \rightarrow C_1) \cdot \deg(D_1 \rightarrow C).$$

Moreover, recalling that $n(X) = \deg(D_1 \rightarrow C) = \deg \beta_1$, we get

$$(3.3) \quad n(X) = \deg(\beta_1^{-1}(D_i) \rightarrow D_i) = \sum_j \deg(D_{ij} \rightarrow D_i)$$

where D_{ij} varies over all curves dominating D_i . Combining (3.2) and (3.3), we receive (3.1). Proceeding this way, we obtain a sequence X_n of fibred etale covers. (3.1) shows that for $n \gg 0$ the surface $\tilde{X} = X_n$ and the fibration $\tilde{\pi} = \pi_n$ satisfy the requirement. \square

4. The case ‘ $\text{kod}(X) = -\infty, \mathbf{q}(X) > 0$ ’

In this section we are going to prove that no irregular surface in \mathcal{T} has negative Kodaira dimension. We proceed by way of contradiction. Thus, assume that $X \in \mathcal{T}$ has negative Kodaira dimension. Then there exists a birational morphism $\phi : X \rightarrow Y$ onto a relatively minimal model such that Y is geometrically ruled over a curve C . Here, the genus of C is the irregularity of Y (cf. Subsection 2.1). So, C is elliptic, because C is dominated by components of D . We define S_i, S to be $\phi(D_i), \phi(D)$ respectively. The next is clear from Section 3.

Claim 4.1. It is sufficient to derive a contradiction in the case where the curves S_i are all sections of the ruling.

We assume the hypotheses of Claim 4.1. We define h_X to be the number of components of D . Note that by Lemma 3.2, $h_X = SF \geq 3$ where F is a general fibre of π . Next, we derive from Subsection 2.2:

Claim 4.2. We can assume that D_{h_X} is not affected during the transition from X to Y , i.e. that $S_{h_X}^2 = D_{h_X}^2 < 0$ and $S_i S_{h_X} = 0$ for all $i < h_X$.

With respect to numerical equivalence it holds then

$$(4.1) \quad S_i \equiv S_{h_X} - S_{h_X}^2 F \quad (i < h_X)$$

In fact, by Subsection 2.1 $Num(Y) = \mathbb{Z}S_{h_X} \oplus \mathbb{Z}F$, and we have $S_i \equiv S_{h_X} + b_i F$ with a $b_i \in \mathbb{Z}$. Hence, $b_i = S_i S_{h_X} - S_{h_X}^2 = -S_{h_X}^2$. Note that this is up to now the second place where we use that the S_i ’s are sections. In particular, $S_i F = 1$ and

$$(4.2) \quad S_i^2 = S_{h_X} S_i - S_{h_X}^2 = -S_{h_X}^2 > 0 \quad (i < h_X)$$

We also need to use the fact that $K_Y^2 = c_2(Y) = 0$ (cf. [1, p.244]). Together with Tian-Yau’s theorem 3.1 we find

$$(4.3) \quad -D^2 = 4s,$$

where s is the number of blow-ups in the transition from Y to X . Recall that $K_Y S_i + S_i^2 = 0$ for all i . Now, let x vary over all points that are centers of blowing ups in the transition from Y to X , and write s_x for the multiplicity of the proper transform of S at x . We apply [3, Ch. V, Exmpl. 3.9.2], the adjunction formula and Claim 4.2, and receive

$$0 = K_X D + D^2 = K_Y S + S^2 - \sum_x s_x (s_x - 1) = \sum_{1 \leq i \neq j < h_X} S_i S_j - \sum_x s_x (s_x - 1).$$

Combining with (4.1) and (4.2), and letting $t = -D_{h_X}^2 = -S_{h_X}^2$, we deduce

$$(4.4) \quad (h_X - 1)(h_X - 2)t = \sum_x s_x^2 - \sum_x s_x.$$

On the other hand, $\sum_{i < h_X} S_i^2 = (h_X - 1)t$ and for each i the multiplicity $s_{i,x}$ of the proper transform of S_i at x equals 0 or 1. By [3, Ch. V, Prop. 3.6], $D_i^2 = S_i^2 - \sum_x s_{i,x}^2$. With this and because of (4.3), we get

$$4s = -D_{h_X}^2 - \sum_{i < h_X} D_i^2 = t + \left(\sum_{i,x} s_{i,x}^2 - (h_X - 1)t \right) = -(h_X - 2)t + \sum_x s_x,$$

i.e.

$$(4.5) \quad (h_X - 2)t = \sum_x (s_x - 4).$$

Now, define m_3 to be the number of points x such that $s_x = 3$, and let m_4 be the number of points x with $s_x = 4$. Note that always $s_x \geq 3$, as results from Lemma 3.3 and the fact that the ϕ -exceptional divisor consists of rational curves. So,

$$(h_X - 2)t = \sum_{x, s_x \geq 5} s_x - m_3,$$

that is,

$$4m_3 + 4m_4 + (h_X - 2)t = \sum_x s_x \stackrel{(4.5)}{=} 4s + (h_X - 2)t.$$

We conclude that $m_3 + m_4 = s$, so $3 \leq s_x \leq 4$ for all x . For the final step, we combine (4.4) with (4.5), and obtain

$$\sum_x (s_x^2 + 4(h_X - 1) - h_X s_x) = 0.$$

As $3 \leq s_x \leq 4$, this yields

$$m_3(5 + h_X) + 12m_4 = 0,$$

an obvious contradiction. We are done with this case.

5. The Case ‘ $\text{kod}(\mathbf{X}) = 0, \mathbf{q}(\mathbf{X}) > 0$ ’

We start by proving:

Lemma 5.1. *Let Y be a complex irregular minimal surface with $\text{kod}(Y) = 0$. Then any two distinct elliptic curves $S_1, S_2 \subset Y$ intersect each other transversally (if at all).*

Proof. We know from Subsection 2.1 that Y is either abelian or it is a hyperelliptic surface. If Y is hyperelliptic then it admits an abelian etale covering, and we are therefore reduced to the case where Y is abelian. Let $x \in S_1 \cap S_2$. We may assume that x serves as the neutral element of the underlying abstract groups on S_1, S_2 and Y . If $\iota_i : S_i \rightarrow Y$ denotes the respective embedding for $i = 1, 2$ then the Lie functor yields a linear mapping

$$\text{Lie}_{\iota_i} : \text{Lie}(S_i) = \mathbb{C} \rightarrow \text{Lie}(Y) = \mathbb{C}^2.$$

A priori, the lines $Lie_{\iota_1}(\mathbb{C})$ and $Lie_{\iota_2}(\mathbb{C})$ are equal or intersect transversally in $0 \in \mathbb{C}^2$. By assumption, the second possibility holds true. The result is now immediate. \square

With this in mind, let Y be a minimal model of an irregular surface in $X \in \mathcal{T}$ and assume that $kod(Y) = 0$. By Subsection 2.1, Y is either abelian or it is a hyperelliptic surface. By way of contradiction, assume the second mentioned possibility: Y is hyperelliptic. Note that $X \neq Y$. Namely, K_Y is numerically trivial and therefore Y does not contain elliptic curves with negative self-intersection, whereas X does (Cor. 3.2). Denote by S_i the image curve of D_i .

Claim 5.2. Each curve S_i is smooth elliptic.

Proof. Let $D = \sum D_i$ be the compactification divisor of X and consider a birational morphism $\phi : X \rightarrow Y$. Y admits an etale covering $\alpha : \tilde{Y} \rightarrow Y$ by an abelian surface \tilde{Y} , and α induces an etale covering $\beta : \tilde{X} \rightarrow X$. Let $\tilde{D} = \beta^{-1}(D)$. Then, (\tilde{X}, \tilde{D}) is a \mathcal{T} -pair, because $\tilde{X} \setminus \tilde{D}$ must have \mathbf{B} as universal holomorphic covering. Let \tilde{S} be the respective divisor on \tilde{Y} , i.e. $\tilde{S} = \alpha^{-1}(S)$. We know from Thm. 1.2 that \tilde{S} is a sum of elliptic curves $\sum \tilde{S}_{ij}$ satisfying $4\tilde{s} = \sum \tilde{s}_{ij}$ with $\tilde{s} = |\text{sing } \tilde{S}|$ and $\tilde{s}_{ij} = |\tilde{S}_{ij} \cap \text{sing } \tilde{S}|$ (the first index means that \tilde{S}_{ij} dominates S_i). Let U be a small ball around a point $x \in \text{sing } \tilde{S}$. It results from Lemma 5.1 that the sheets $S \cap U$ become disjoint after blowing up once along x . Thus, it follows from Lemma 3.2 that \tilde{X} results from \tilde{Y} by blowing-up once at each point $x \in \text{sing } \tilde{S}$; in particular, \tilde{D} always intersects transversally with an exceptional curve. Therefore,

$$K_{\tilde{X}}^2 = K_{\tilde{Y}}^2 - \tilde{s} = -\tilde{s}$$

and

$$c_2(\tilde{X}) = c_2(\tilde{Y}) + \tilde{s} = \tilde{s},$$

so that $-\tilde{D}^2 = 4\tilde{s}$ by Thm. 3.1. Moreover, since β is etale, the single blowing ups along $M = \text{sing } D$ produce X , and \tilde{D} always intersects transversally with an exceptional curve.

Now, suppose that for some i a point $x \in S_i$ is a singularity of S with the property that for a small open ball $U \subset X$ around x a sheet of $S_i \cap U$ admits a cusp singularity at x . Then the exceptional curve $L \subset X$ arising from the blow-up at x does not intersect D_i transversally. As β is etale, neither do a \tilde{D}_{ij} and an exceptional curve $\tilde{L} \subset \alpha^{-1}(L)$. By contraposition, no sheet of $S_i \cap U$ has cusp singularities. Therefore, if $\nu_i : D_i \rightarrow S_i$ is the restriction of ϕ to D_i , then

$$D_i^2 = S_i^2 - |\nu_i^{-1}(S_i \cap M)|.$$

It still remains to show that each curve S_i is indeed smooth. But, clearly, for $n = \deg \alpha$ we have $n \cdot \sum_i |\nu_i^{-1}(S_i \cap M)| = \sum_{ij} \tilde{s}_{ij}$. Since $nD^2 = \tilde{D}^2 = 4\tilde{s}$, we deduce

$$-4 \cdot \frac{\tilde{s}}{n} = D^2 = \sum_i (S_i^2 - |\nu_i^{-1}(S_i \cap M)|) = \sum_i S_i^2 - \frac{\sum_{ij} \tilde{s}_{ij}}{n}.$$

Recalling that $4\tilde{s} = \sum \tilde{s}_{ij}$ and noting that *a priori* $(K_Y + S_i)S_i = S_i^2 \geq 0$, we conclude $(K_Y + S_i)S_i = 0$. So, the claim follows from adjunction formula: $p_a(S_i) = 1$. \square

As already noticed above, X results from A by single blowing-ups along the points in $\text{sing } S$. Since each component of S is smooth by Lemma 5.1, it follows from Lemma 3.3 that there exists a point $y \in Y$ which is the intersection point of three distinct elliptic curves S_1, S_2 , and S_3 . On the other hand, the rank of $\text{Num}(Y)$ is $\leq b_2(Y) = 2$. Thus, for some numbers $a, b \in \mathbb{Q}$, we receive a numerical equivalence

$$S_1 \equiv aS_2 + bS_3.$$

Using that $S_1^2 = S_2^2 = S_3^2 = 0$, one readily derives a contradiction.

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