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A NOTE ON STEIN FILLINGS OF CONTACT MANIFOLDS

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Abstract. In this note we construct infinitely many distinct simply connected Stein fillings of a certain infinite family of contact 3–manifolds.

1. Introduction

Given a contact 3–manifold (M, ξ) it is useful to understand the types of fillings it has. In particular, Stein and strong fillings of (M, ξ) can be used in symplectic cut-and-paste procedures. Recall that X is a *Stein filling* of (M, ξ) if X is the sublevel set of a plurisubharmonic function on a Stein surface and the contact structure induced on ∂X by complex tangencies is contactomorphic to (M, ξ). Many early results concerning Stein fillings pointed to a finiteness result. For example, Eliashberg [4] showed the tight contact structure on S^3 has a unique Stein filling. McDuff [12] and Lisca [11] have shown that the tight contact structures on Lens spaces that are covered by the standard tight contact structure on S^3 have a finite number of Stein fillings. Several other uniqueness results are known [13, 14, 17]. However, Ozbagci and Stipsicz [15] and independently Smith [18] have shown that certain contact structures have infinitely many Stein fillings. The fillings in these examples had non-trivial fundamental group (they were distinguished by torsion in the first homology group). As various constructions in 4–manifold topology are simpler when the fundamental group is not controlled it was natural to ask if there is a similar non-finiteness result for simply connected Stein fillings. Our main result shows there is.

Theorem 1.1. *There is a sequence of distinct contact manifolds (M_i, ξ_i), each of which has an infinite number of homeomorphic, but non-diffeomorphic, simply connected Stein fillings.*

These examples are a simple consequence of Fintushel and Stern's work [7] but fill a gap in the literature concerning Stein fillings of contact structures. It follows from the construction here and work of Ozbagci and Stipsicz [15], for example, that the manifolds (M_i, ξ_i) each have infinitely many non-homeomorphic Stein fillings, as well as infinitely many homeomorphic but non-diffeomorphic Stein fillings. One issue that is tantalizingly left open is whether or not there is a contact 3–manifold whose Stein fillings realize infinitely many values for their Euler characteristics χ, or realize infinitely many pairs of values for (χ, σ), where σ is the signature. Given any natural number n it is not hard to check that the construction below will provide a contact manifold (M, ξ) whose Stein fillings realize at least n distinct pairs (χ, σ).

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2. Knot Sugery and Lefschetz Fibrations on $E(n)_K$

2.1. Lefschetz fibrations on $E(n)$. Recall that the elliptic surface $E(n)$ can be described as the desingularization of a double branched cover. Specifically, consider the double branched cover of $\mathbb{C}P^1 \times \mathbb{C}P^1$ whose branch set $B_{2,n}$ is the union of four disjoint copies of $\mathbb{C}P^1 \times \{\text{pt}\}$ and $2n$ disjoint copies of $\{\text{pt}\} \times \mathbb{C}P^1$. This branched cover has $8n$ singular points corresponding to the intersections of horizontal and vertical lines in the branch set $B_{2,n}$. Each singular point is easily seen to be a cone on $\mathbb{R}P^3$. After desingularizing the above space by removing a neighbourhood of each singular point and replacing it by the unit cotangent bundle of S^2, which is a D^2-bundle over S^2 with Euler number -2, one obtains $E(n)$.

The horizontal and vertical fibrations of $\mathbb{C}P^1 \times \mathbb{C}P^1$ pull back to give fibrations of $E(n)$ over $\mathbb{C}P^1$. A generic fiber of the vertical fibration is the double cover of S^2, branched over 4 points. Thus a generic fiber is a torus and the fibration is an elliptic fibration on $E(n)$. The generic fiber of the horizontal fibration is the double cover of S^2, branched over $2n$ points, which is a genus $n-1$ surface. The resulting genus $n-1$ fibration has 4 singular fibers which correspond to preimages of the four copies of $S^2 \times pt$'s in the branch set together with the spheres of self-intersection -2 coming from the desingularization. While this fibration is not a Lefschetz fibration it was shown in [7] that it can be slightly deformed near the singular fibers to become a Lefschetz fibration. The deformation amounts to a Morsification of the singularities of the singular fibers and is akin to semistable reduction in algebraic geometry (here no base change is required); the details of the local model are not relevant for our construction. Notice that the generic fiber of the horizontal fibration Σ_{n-1} intersects a generic fiber F of the elliptic fibration in two points.

2.2. Lefschetz Fibrations on $E(n)_K$. Let K be a fibered knot of genus g and let F be a generic torus fiber of $E(n)$. Denote by $N(K)$ an open neighbourhood of K in S^3. Then a knot surgered elliptic surface is a manifold

$$E(n)_K = (E(n) \setminus (F \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)),$$

where each normal 2-disk to F is replaced by a fiber of the fibration of $S^3 \setminus N(K)$ over S^1. This does not necessarily determine $E(n)_K$ up to diffeomorphism since there is still ambiguity in the gluing map; however, the Seiberg-Witten invariants are determined [6] and we let $E(n)_K$ denote the manifold obtained by any gluing map satisfying the above criteria. Since F intersects each generic horizontal fiber twice, we get an induced fibration

$$h : E(n)_K \to \mathbb{C}P^1$$

with fiber genus $2g + n - 1$.

Lemma 2.1. [7] *Let K be a fibered knot of genus g. The manifold $E(n)_K$ admits a locally holomorphic fibration (over $\mathbb{C}P^1$) of genus $2g + n - 1$ which has exactly four singular fibers. Furthermore, this fibration can be deformed locally to be Lefschetz with no reducible fibers.*

We also note the following fact about these Lefschetz fibrations.

Lemma 2.2. *Let K be a fibered knot of genus g. Then for any n, the Lefschetz fibration $E(n)_K$ admits two disjoint sphere sections each with self-intersection -2. Moreover, if S denotes one such section, then $E(n)_K \backslash S$ is simply connected.*

Proof. Let S' be one of the vertical spheres $\{\mathrm{pt}\} \times \mathbb{C}P^1$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ that is part of the branch locus in the description of $E(n)$ above. In the double branched cover S' lifts to a sphere S'' and after desingularizing to obtain $E(n)$ we get a sphere S that intersects each of the four desingularizing D^2 bundles over S^2 in a fiber. One can easily check (cf. [7]) that S has self-intersection -2 and is a section of the genus $n-1$ Lefschetz fibration of $E(n)$ over $\mathbb{C}P^1$. Moreover, since S intersects one of the desingularizing S^2's in a point, its meridian is null-homotopic in the complement of S. Thus, as $E(n)$ is simply connected, the fundamental group of the complement of S, which is normally generated by its meridian, is trivial.

When normal summing $S^1 \times S^3$ to $E(n)$ to construct $E(n)_K$, the sphere S and the desingularizing S^2 are not affected. Thus S is still a section with simply connected complement. Lastly, taking another vertical sphere in the branch locus provides a section that is disjoint from S. □

3. Extensions of diffeomorphisms over plumbings

Let $X_{g,n}$ be the 4-manifold obtained by plumbing $\Sigma_g \times D^2$ to a D^2-bundle over S^2 with Euler number $-n$. The main result of this section is the following extension result.

Lemma 3.1. *Any orientation preserving diffeomorphism of $\partial X_{g,n}$ extends over $X_{g,n}$.*

Proof. The case when $n = 0$ is clear since $X_{g,0}$ is the boundary sum of $2g$ copies of $S^1 \times D^3$.

If $n \neq 0$ and $g > 0$ then $\partial X_{g,n}$ is a Seifert fibered space over Σ_g with one singular fiber over the point $p \in \Sigma_g$. Let ϕ be a diffeomorphism of $\partial X_{g,n}$. Let $C = \{c_1, \ldots, c_g\}$ and $C' = \{c'_1, \ldots, c'_g\}$ be collections of curves on Σ_g such that the curves within each collection are disjoint and $c_i \cap c'_j$ is exactly one point if $i = j$ and empty otherwise. We moreover assume that no curve in C or C' goes through the point p and C and C' are chosen so that $\Sigma_g \setminus (C \cup C')$ is a planar surface. Finally choose a curve C'' on Σ_g so that C'' is disjoint from the curves in C', intersects each curve in C in one point, is disjoint from the point p and $\Sigma_g \setminus (C \cup C' \cup C'')$ consists of two disks. Let T and T' be the preimages of C and C' in $\partial X_{g,n}$. These are collections of incompressible tori. Thus $\phi(T)$ is also a collection of incompressible tori. It is well known in this situation that the collection $\phi(T)$ can be isotoped to be vertical (that is a union of fibers in the Seifert fibration), [8, 9]. By extending this isotopy to all of $\partial X_{g,n}$ we can assume $\phi(T)$ is a collection of vertical tori. Each component of T' in $\partial X_{g,n} \setminus T$ is an incompressible and boundary incompressible annulus. The images of these components will also be such annuli and thus can be isotoped to be horizontal or vertical. Since a horizontal annulus in $\partial X_{g,n} \setminus T$ would be part of a horizontal torus in $\partial X_{g,n}$, which does not exist, we can further isotop ϕ so that $\phi(T')$ and $\phi(T)$ are collections vertical tori. We can similarly isotop ϕ so that $\phi(T'')$ is a vertical torus, where T'' is the preimage of C'' in $\partial X_{g,n}$. At this point it is clear that ϕ can be further isotoped to preserve the Seifert fibration on $T \cup T' \cup T''$ and even on a neighborhood of $T \cup T' \cup T''$.

As the tori in $T \cup T' \cup T''$ consist of regular fibers we can assume they are contained in the $\Sigma_g \times D^2$ part of $X_{g,n}$. Each such fiber in this part of the Seifert fibration bounds an obvious disk in $X_{g,n}$. We can now extend ϕ over those disks whose boundaries are fibers in a neighborhood of $T \cup T' \cup A$, where A is the union of all but one of the components of T'' in $\partial X_{g,n} \setminus (T \cup T')$. It remains to extend ϕ over the D^2-bundle over S^2. The boundary of this bundle is $L(n,1)$. In [2] it was shown that, up to isotopy, there is precisely one non-identity diffeomorphism of $L(n,1)$. Recall $L(n,1)$ is the double branched cover of S^3 with branch set the boundary of the unknotted annulus or Möbius band with $-n$ half twists. The nontrivial diffeomorphism of $L(n,1)$ is simply the covering automorphism of this two fold cover, see [2]. Since the disk bundle bounded by $L(n,1)$ is the two fold branched cover of B^4 branched over the interior of the above mentioned annulus or Möbius band with its interior pushed into the interior of B^4 it is clear that this diffeomorphism extends over the disk bundle. Thus we may extend ϕ over the rest of $X_{g,n}$. □

4. Stein fillings

In this section we construct our examples of contact structures with infinitely many distinct, simply connected Stein fillings all of which are homeomorphic. We begin by recalling a well-known relation between Lefschetz fibrations and Stein manifolds.

Lemma 4.1. *Let $f : X \to \mathbb{C}P^1$ be a Lefschetz fibration of a 4-manifold with irreducible fibers that admits a section S of square $-n$. If F is a generic fiber of the fibration then $X - (F \cup S)$ admits the structure of a Stein manifold. Moreover, the Stein manifold fills the contact 3-manifold (Y, ξ) where ξ is supported by an open book decomposition with page $F \setminus \nu(S \cap F)$, where $\nu(S \cap F)$ is a neighborhood of $S \cap F$ in F, and monodromy n-right handed Dehn twists about a curve parallel to the boundary.*

This well known lemma follows from the handlebody description of Lefschetz fibrations and the relation between open books and contact structures, see [1, 5, 16].

With the above preparation we are now ready to prove our main result.

Theorem 4.2. *Consider the contact manifold defined by the open book decomposition with page a surface of genus $g > 4$ having one boundary component and monodromy $2m$ positive Dehn twists about a curve parallel to the boundary. This contact manifold has infinitely many homeomorphic but non-diffeomorphic simply connected Stein fillings.*

Proof. Notice that the manifold defined by the open book decomposition in the theorem is the Seifert fibered space $M(g;m)$ over a surface of genus g with one singular fiber of multiplicity $2m$. Let $\xi_{g,m}$ denote the contact structure on $M(g;m)$ that is supported by this open book.

From Lemma 2.2 we know that for a genus g fibered knot K in S^3, $E(n)_K$ has a genus $2g+n-1$ Lefschetz fibration with a section S of square -2, the complement of which is simply connected. It is clear that $X_{2g+n-1,2}$ embeds in $E(n)_K$ as a neighborhood of a fiber and S. From Lemma 4.1 we know that $S_{n,K} = E(n)_K \setminus X_{2g+n-1,2}$ is a Stein manifold filling the contact structure $\xi_{2g+n-1,1}$ described in the theorem. Moreover, $S_{n,K}$ is seen to be simply connected by the existence of a section of $E(n)_K$ disjoint from S.

If K_i is a sequence of fibered genus g knots in S^3 with distinct Alexander polynomials, which by [10] always exists if $g > 1$, then the $E(n)_{K_i}$'s are all mutually non-diffeomorphic. Indeed, according to a Theorem of [6], they are distinguished by their Seiberg-Witten invariants. Thus by Lemma 3.1 we conclude that the S_{n,K_i} are all mutually non-diffeomorphic Stein fillings of $\xi_{2g+n-1,1}$. It follows easily from [3, Proposition 0.5] that these manifolds are all homeomorphic.

The existence of infinitely many examples of contact 3-manifolds, as asserted in the theorem, now follows by taking fiber sums (see [16]) of the $E(n)_{K_i}$'s. □

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