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ON THE HYPOTHESIS K^* OF HARDY, LITTLEWOOD AND HOOLEY AND ITS RELATION WITH DISCRETE FRACTIONAL OPERATORS

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Abstract. In this paper, we revisited the relation between the Hypothesis K^* of Hardy, Littlewood and Hooley and the boundedness of the discrete fractional operator

$$I_{\lambda,k}f(n) = \sum_{m=1}^{\infty} \frac{f(n-m^k)}{m^\lambda},$$

with $k \in \mathbb{N}$, $k \geq 2$, in order to obtain that, for every $\varepsilon > 0$, and every $2 \leq r < 1/(1-\lambda)$,

$$\int_0^1 |m_{\lambda,k}(\theta)|^r |\theta - a|^{(\frac{1-\lambda}{k}+\varepsilon)r-1} d\theta < \infty,$$

uniformly in a, where

$$m_{\lambda,k}(\theta) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m^k \theta}}{m^\lambda}.$$

We recall that the Hypothesis K^* is equivalent to the fact that $m \in L^{2k}$, for every $\lambda > 1/2$.

1. Introduction

In the last years, considerable interest has been shown in discrete harmonic analysis, although is not as developed as the continuous case, probably because exponential sums are usually more difficult to estimate than oscillatory integrals. See for example [3], [4], [2] and [5] where, in particular, the operator

$$I_{\lambda,2}f(n) = \sum_{m=1}^{\infty} \frac{f(n-m^2)}{m^\lambda} \tag{1.1}$$

has been deeply studied, and it is known that, if $0 < \lambda < 1$, $I_\lambda : \ell^p \to l^q$, whenever $1 \leq p < q \leq \infty$, $1/q = 1/p - (1-\lambda)/2$, $p < 1/(1-\lambda)$ and $q > 1/\lambda$.

If instead of (1.1), we consider the operator

$$I_{\lambda,k}f(n) = \sum_{m=1}^{\infty} \frac{f(n-m^k)}{m^\lambda}$$

with $k \in \mathbb{N}$, $k > 2$, the boundedness properties (ℓ^p, ℓ^q) is an open question although in the above papers is naturally conjectured that $I_{\lambda,k} : \ell^p \to l^q$ is bounded, whenever

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$1 \leq p < q \leq \infty$, $p < 1/(1-\lambda)$, $q > 1/\lambda$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{1-\lambda}{k}\cdot$$

However, this problem seems to be a difficult one since the techniques used in [4], [2] and [5], adapted to this new situation would imply the Hypothesis K* of Hardy, Littlewood and Hooley to be true ([1]). This Hypothesis states that if $r_k(l)$ denotes the number of ways one can represent the integer l as the sum of k^{th}-powers, $l = n_1^k + n_2^k + ... + n_k^k$, with k summands, then, for every ε,

$$\sum_{l=1}^{N} r_k(l)^2 = O(N^{1+\varepsilon}),$$

as N tends to infinity. This Hypothesis is solved in the case $k = 2$ and open for $k > 3$. In [4], it is proved that this hypothesis is equivalent to the fact that, for every $\lambda > 1/2$,

$$m_{\lambda,k}(\theta) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m^k \theta}}{m^{\lambda}} \tag{1.2}$$

belongs to $L^{2k}(\mathbb{T})$. Observe, and this is the conection between the two problems, that the Fourier transform of $I_{\lambda,k} f$ satisfies:

$$(I_{\lambda,k} f)\hat{}(\theta) = m_{\lambda,k}(\theta)\hat{f}(\theta). \tag{1.3}$$

In fact, in [4], the boundedness of $I_{\lambda,2} : \ell^p \to \ell^q$ with p and q as before and $\lambda > 1/2$, is done by proving that the corresponding function $m_{\lambda,2} \in L^{2/(1-\lambda),\infty}(\subset L^4(\mathbb{T}))$ and applying trivial computations of classical Fourier Analysis.

In any case it is clear, using (1.3), that the boundedness property of $I_{\lambda,k}$ is related to the size of $m_{\lambda,k}$ and hence related to the Hypothesis K^*. This makes the problem itself very interesting.

The purpose of this paper is to give some new estimates on the function $m_{\lambda,k}$ which may give some new hints on the solution of the Hypothesis K^*. In fact, our result can be applied to a more general exponential sum than (1.2); namely, to the function

$$m_{\lambda,\mu}(\theta) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \mu_m \theta}}{m^{\lambda}}$$

where $\mu = (\mu_m)_m$ is a sequence of positive integers such that, for some positive constants A and B

$$A m^{k-1} \leq \mu_m - \mu_{m-1} \leq B m^{k-1}. \tag{1.4}$$

To give some estimate of this function we start by giving a boundedness result for the operator

$$I_{\lambda,\mu} f(n) = \sum_{m=1}^{\infty} \frac{f(n-\mu_m)}{m^{\lambda}}\cdot$$

Through out the paper, C will represent an universal constant independent of all the parameters involved, $E \lesssim F$ means that there exists an universal constant C such that $E \leq CF$ and $E \approx F$ means that $E \lesssim F$ and $F \lesssim CE$.

2. Boundedness result for $I_{\lambda,\mu}$

Let $(\mu_m)_m$ be a sequence of positive integers satisfying (1.4). Clearly, $(\mu_m)_m$ is an increasing sequence and

$$\mu_m \approx m^k. \tag{2.1}$$

Given $\varepsilon > 0$ and $q \geq 1$, set

$$\alpha = \frac{1}{q} + \frac{1-\lambda}{k} + \varepsilon$$

and let us consider the space

$$\ell^\infty((1+|n|^\alpha)) = \Big\{(a_n)_n;\ ||a||_{\ell^\infty((1+|n|^\alpha))} = \sup_n |a_n|(1+|n|^\alpha) < \infty\Big\}.$$

Theorem 2.1. *a) Let μ and α as before and $q > \lambda$. Then*

$$I_{\lambda,\mu} : \ell^\infty((1+|n|^\alpha)) \longrightarrow \ell^q(\mathbb{Z})$$

is bounded.
b) If $q > \lambda$ and

$$\frac{1}{p} = \frac{1}{q} + \frac{1-\lambda}{k} \tag{2.2}$$

then

$$I_{\lambda,\mu} : \ell^\infty((1+|n|^{1/p})) \longrightarrow \ell^{q,\infty}(\mathbb{Z})$$

is bounded.

Proof. a) First of all, we observe that if $n \in \mathbb{Z}^-$, then

$$\begin{aligned} |(I_\lambda a)(n)| &\leq ||a||_{\ell^\infty((1+|n|^\alpha))} \sum_{m=1}^\infty \frac{m^{-\lambda}}{1+|\mu_m+|n||^\alpha} \\ &\leq ||a||_{\ell^\infty((1+|n|^\alpha))} \frac{1}{1+|n|^{\frac{1}{q}+\frac{\varepsilon}{2}}} \sum_{m=1}^\infty \frac{m^{-\lambda}}{\mu_m^{\frac{1-\lambda}{k}+\frac{\varepsilon}{2}}}, \end{aligned}$$

and by (2.1) the above sum in finite and hence $I_\lambda a \in \ell^q(\mathbb{Z}^-)$.

Now, given $n \in \mathbb{N}$, $n \geq 1$, let $m_0 = m_0(n) \in \mathbb{N}$ be such that

$$\mu_{m_0} = \sup_m \{\mu_m;\ \mu_m \leq n\},$$

where we assume by simplicity that $\mu_1 = 1$. Then, using (2.1) we have that $\mu_{m_0(n)} \approx n$.

Then, given $a=(a_n)_n\in\ell^\infty((1+|n|^\alpha))$, with $||a||_{\ell^\infty((1+|n|^\alpha))}=1$,

$$\begin{aligned}
&|(I_\lambda a)(n)|\le\sum_{m=1}^{\infty}\frac{m^{-\lambda}}{1+|\mu_m-n|^\alpha}\\
=&\sum_{m<m_0(n)}\frac{m^{-\lambda}}{1+|\mu_m-n|^\alpha}+\left(\frac{m_0(n)^{-\lambda}}{1+(n-\mu_{m_0(n)})^\alpha}+\frac{(m_0(n)+1)^{-\lambda}}{1+(\mu_{m_0(n)+1}-n)^\alpha}\right)\\
+&\sum_{m>m_0(n)+1}\frac{m^{-\lambda}}{1+|\mu_m-n|^\alpha}=I+II+III\cdot
\end{aligned}$$

Estimate of I:

$$\sum_{m<m_0(n)}\frac{m^{-\lambda}}{1+(n-\mu_m)^\alpha}\lesssim\sum_{j=1}^{m_0(n)-1}\frac{(m_0(n)-j)^{-\lambda}}{(n-\mu_{m_0(n)-j})^\alpha}\lesssim\sum_{j=1}^{m_0(n)-1}\frac{(m_0(n)-j)^{-\lambda}}{(\mu_{m_0(n)}-\mu_{m_0(n)-j})^\alpha}$$

Now,

$$\begin{aligned}
\mu_{m_0}-\mu_{m_0-j}&=\sum_{i=0}^{j-1}(\mu_{m_0-i}-\mu_{m_0-i-1})\ge A\sum_{i=0}^{j-1}(m_0-i)^{k-1}\\
&\ge Aj(m_0-j)^{k-1}\cdot
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{m<m_0(n)}\frac{m^{-\lambda}}{1+(n-\mu_m)^\alpha}&\lesssim\sum_{j=1}^{m_0(n)-1}\frac{1}{j^\alpha(m_0(n)-j)^{(k-1)\alpha+\lambda}}\approx m_0(n)^{-\lambda-\alpha k+1}\\
&\approx n^{\frac{1-\lambda}{k}-\alpha}=n^{-\frac{1}{q}-\varepsilon}\in\ell^q(\mathbb{N}).
\end{aligned}$$

Estimate of III:

$$III=\sum_{m>m_0(n)+1}\frac{m^{-\lambda}}{1+(\mu_m-n)^\alpha}\lesssim\sum_{j=2}^{\infty}\frac{(m_0+j)^{-\lambda}}{(\mu_{m_0+j}-n)^\alpha}\lesssim\sum_{j=2}^{\infty}\frac{(m_0+j)^{-\lambda}}{(\mu_{m_0+j}-\mu_{m_0+1})^\alpha}\cdot$$

Then, since

$$\mu_{m_0+j}-\mu_{m_0+1}\ge A\sum_{l=2}^{j}(m_0+l)^{k-1}\ge CAj^{k-1}\max(j,m_0)$$

we obtain

$$III\lesssim\sum_{j=2}^{\infty}\frac{(m_0+j)^{-\lambda}}{(j^{k-1}\max(j,m_0))^\alpha}\approx m_0^{1-\alpha k-\lambda}\approx n^{-\frac{1}{q}-\varepsilon}\in\ell^q(\mathbb{N})\cdot$$

Estimates of II: We have to prove that

$$\sum_{n\ge2}\left(\frac{m_0(n)^{-\lambda}}{1+(n-\mu_{m_0(n)})^\alpha}\right)^q\approx\sum_{n\ge2}\frac{n^{\frac{-\lambda q}{k}}}{1+(n-\mu_{m_0(n)})^{\alpha q}}<\infty,$$

and similarly

$$\sum_{n\geq 2} \frac{n^{\frac{-\lambda q}{k}}}{1+(\mu_{m_0(n)+1}-n)^{\alpha q}} < \infty.$$

We shall prove the first one since the second follows the same pattern. To do this, let us define, for each m,

$$I_m = \{n \geq 1; m_0(n) = m\}\cdot$$

Then,

$$\sum_{n\geq 2} \frac{n^{\frac{-\lambda q}{k}}}{1+(n-\mu_{m_0(n)})^{\alpha q}} = \sum_{m=1}^{\infty}\sum_{n\in I_m} \frac{n^{\frac{-\lambda q}{k}}}{1+(n-\mu_{m_0(n)})^{\alpha q}}$$

$$\approx \sum_{m=1}^{\infty} m^{-\lambda q} \sum_{n\in I_m} \frac{1}{1+(n-\mu_m)^{\alpha q}} \lesssim \sum_{m=1}^{\infty} m^{-\lambda q} \sum_{j\geq 1} \frac{1}{j^{\alpha q}} < \infty$$

since $\alpha > 1/q$ and $q\lambda > 1$.

b) The proof of b) follows from the same computations.

□

Remarks 2.2.

i) Observe that, if p is as in (2.2), then $\ell^\infty(1+|n|^\alpha) \subset \ell^p$, and hence, in the original case $\mu_m = m^2$, our result is consequence of the boundedness of $I_{\lambda,2}$ proved in ([4]). In the case $k > 2$, the above result is weaker that the open Stein and Wainger's conjecture mentioned before on the boundedness of $I_{\lambda,k} : \ell^p \longrightarrow \ell^q$. However, it will have some interesting consequences concerning estimates for the function $m_{\lambda,k}$.

ii) The result is false if we only assume the condition $\mu_m \approx m^k$ on the sequence $(\mu_m)_m$. To see this let us take

$$\mu_n = 2^{2(j+1)} \qquad \text{if } n \in [2^j, 2^{j+1}).$$

Then,

$$I_{\lambda,\mu} f(n) \approx \sum_{j\geq 0} 2^{j(1-\lambda)} f(n-2^{2(j+1)}),$$

and taking the sequence $f(n) = 1$ if $n = 0$ and $f(n) = 0$ if $n \neq 0$, we have $I_{\lambda,\mu}f(n) = 0$ if $n \neq 2^{2(j+1)}$ for some j, and if $n = 2^{2(j+1)}$, $I_{\lambda,\mu}f(n) = 2^{j(1-\lambda)}$. Consequently, $(I_{\lambda,\mu}f(n))_n \notin \ell^q$.

3. Estimates for the $m_{\lambda,\mu}$ function

Given $\varepsilon > 0$ and $\bar{h} = (h_n)_n$ with $|h_n| \leq 1$, let us define

$$W_h(\theta) = \sum_{n>0} \frac{h_n}{n^{\frac{1}{q}+\frac{1-\lambda}{k}+\varepsilon}} e^{2\pi i n\theta}.$$

Then, as a first consequence of Theorem 2.1 we get the following.

Theorem 3.1. *For every $\varepsilon > 0$ and every $\frac{1}{\lambda} < q \leq 2$, $m_{\lambda,\mu}W_h \in L^{q',q}$.*

Proof. The result follows inmediately using Theorem 2.1 and Hausdorf-Young Theorem, since

$$m_{\lambda,\mu}(\theta)W_h(\theta) = \sum_{n\in\mathbb{Z}} (I_\lambda w_h)(n)e^{2\pi in\theta},$$

where

$$w_h(n) = \hat{W}_h(n) \in \ell^\infty((1+|n|^\alpha))$$

with $\alpha = \frac{1}{q} + \frac{1-\lambda}{k} + \varepsilon$.

□

Similarly, if we take $\varepsilon = 0$ in the definition of W_h, we get:

Theorem 3.2. *For every $\frac{1}{\lambda} < q \le 2$, $m_{\lambda,\mu}W_h \in L^{q'\infty}$.*

Now, it is known (see [6], page 70) that, for every $0 < \alpha < 1$ there exists C_α and C'_α such that for every $0 < x \le 1$,

$$\sum_{n=1}^\infty \frac{e^{2\pi inx}}{n^\alpha} = C_\alpha x^{\alpha-1} + C'_\alpha(1-x)^{\alpha-1} + O(1)$$

Hence, if we take $a \in (0,1)$ and $h = (e^{-2\pi ina})_n$, one can easily see that the function $W_h(\theta)$ is bounded in any interval out of a and blows up like $|\theta-a|^{\alpha-1}$ near a. Therefore, as a consequence of Theorems 3.1 and 3.2, we obtain our main estimate:

Theorem 3.3. *a) For every $\varepsilon > 0$, every $a \in (0,1)$, and every $2 \le r < \frac{1}{1-\lambda}$,*

$$\int_0^1 |m_{\lambda,\mu}(\theta)|^r |\theta-a|^{(\frac{1-\lambda}{k}+\varepsilon)r-1} d\theta < \infty, \tag{3.1}$$

uniformly in a.
b) For every $2 \le r < \frac{1}{1-\lambda}$ and every $a \in (0,1)$,

$$|m_{\lambda,\mu}(\theta)||\theta-a|^{\frac{1-\lambda}{k}-\frac{1}{r}} \in L^{r,\infty}.$$

By taking ε such that $\frac{1-\lambda}{k} + \varepsilon = \frac{1}{2k}$, we get that

Theorem 3.4. *For $2 \le r < \frac{1}{1-\lambda}$ and every $a \in (0,1)$,*

$$\int_0^1 |m_{\lambda,k}(\theta)|^r |\theta-a|^{\frac{r}{2k}-1} d\theta < \infty. \tag{3.2}$$

uniformly in a.

Remarks 3.5.
1.- The above result is not true for $r = \frac{1}{1-\lambda}$, since in this case, taking $\lambda = 1/2$ and $r = 2$, we get that $m_{\lambda,k} \in L^2(\mathbb{T})$ which is trivially not true, by Parseval's formula.

2.- Observe that in the original case $\mu_m = m^2$ it was proved in Proposition 1 of [4] that $m_{\lambda,2} \in L^{2/(1-\lambda)}$ and hence $(m_{\lambda,2})^*(t) \lesssim t^{\frac{1-\lambda}{2}}$. Therefore, it follows that

$$\int_0^1 |m_{\lambda,2}(\theta)|^r |\theta-a|^{(\frac{1-\lambda}{2}+\varepsilon)r-1} d\theta \le \int_0^1 (m_{\lambda,2})^*(t)^r \frac{1}{t^{1-(\frac{1-\lambda}{2}+\varepsilon)r}} dt \lesssim \int_0^1 \frac{dt}{t^{1-\varepsilon r}} < \infty$$

and hence (3.1) can be deduced from Proposition 1 of [4], in this particular case. Moreover, in Proposition 3 of [4], it was also proved that for $\lambda < 1$ but big enough $m_{\lambda,k} \in L^{k/(1-\lambda)}$ and hence (3.1) is also consequence of this fact for such $\lambda' s$.

3.- Observe that, for every $0 < t \leq 1$,

$$\frac{1}{t}\int_0^t |m_{\lambda,k}(\theta)|^r d\theta \leq t^{1-r\left(\frac{1-\lambda}{k}+\varepsilon\right)} \frac{1}{t}\int_0^t |m_{\lambda,k}(\theta)|^r \theta^{r\left(\frac{1-\lambda}{k}+\varepsilon\right)-1} d\theta \leq C t^{-r\left(\frac{1-\lambda}{k}+\varepsilon\right)},$$

and hence, since $\lambda > 1/2$, we can choose ε appropriately to get that, for every $1 \leq r < \frac{1}{1-\lambda}$,

$$\left(\frac{1}{t}\int_0^t |m_{\lambda,\mu}(\theta)|^r d\theta\right)^{\frac{1}{r}} \in L^{2k}.$$

In fact, we have that, for every $0 < a < 1$,

$$\left(\frac{1}{t-a}\int_a^t |m_{\lambda,\mu}(\theta)|^r d\theta\right)^{\frac{1}{r}} \in L^{2k}(a, a+1).$$

Final Observation: Let $r_\mu(l)$ be the number of ways that the integer l can be represented in the form

$$l = \mu_{n_1} + \mu_{n_2} + \cdots + \mu_{n_k}.$$

Let us formulate a natural Hipótesis μ^* stating that, for every $\varepsilon > 0$,

$$\sum_{l=1}^{N} r_\mu(l)^2 = O(N^{1+\varepsilon}), \qquad N \to \infty. \tag{3.3}$$

Then, by a complete similar argument that the one in [4], we have that

$$\text{(3.3) holds if and only if } m_{\lambda,\mu} \in L^{2k} \quad \forall \lambda > 1/2.$$

To see this, let us assume that (3.3) holds and let us define

$$S_y(\theta) = \sum_{n=1}^{\infty} h_n e^{-\pi\mu_n(y-2i\theta)}$$

with $y \in (0,1]$ and $(h_n)_n$ some positive numbers satisfying $h_n \approx 1$ to be choosen later on. Then,

$$\begin{aligned}
\int_0^1 S_y(\theta)\frac{dy}{y^{1-\lambda/k}} &= \sum_{n=1}^{\infty} h_n e^{2\pi i\mu_n\theta}\int_0^1 e^{-\pi\frac{\mu_n}{n^k}n^k y}\frac{dy}{y^{1-\lambda/k}} \\
&= \sum_{n=1}^{\infty} h_n e^{2\pi i\mu_n\theta}\frac{k}{n^\lambda}\int_0^n e^{-\pi\frac{\mu_n}{n^k}z^k}\frac{dz}{z^{1-\lambda}} \\
&= \sum_{n=1}^{\infty} e^{2\pi i\mu_n\theta} h_n\left(\frac{C_n}{n^\lambda} - k\frac{\int_n^\infty e^{-\pi\frac{\mu_n}{n^k}z^k}\frac{dz}{z^{1-\lambda}}}{n^\lambda}\right),
\end{aligned}$$

where

$$C_n = \int_0^\infty e^{-\pi\frac{\mu_n}{n^k}z^k}\frac{dz}{z^{1-\lambda}}$$

and, since $\frac{\mu_n}{n^k} \approx 1$, we have that $C_n \approx 1$ and hence taking $h_n = \frac{1}{C_n}$, we obtain

$$m_{\lambda,\mu}(\theta) = \int_0^1 S_y(\theta)\frac{dy}{y^{1-\lambda/k}} + O(1).$$

Therefore,

$$||m_{\lambda,\mu}||_{L^{2k}} \lesssim 1 + \int_0^1 ||S_y||_{L^{2k}}\frac{dy}{y^{1-\lambda/k}}.$$

Now,

$$S_y(\theta)^k = \sum_{n_1,n_2,\cdots,n_k}\left(\prod_{j=1}^k h_{n_j}\right)e^{-\pi(y-2i\theta)(\mu_{n_1}+\mu_{n_2}+\cdots+\mu_{n_k})} = \sum_{l=1}^{\infty} c_\mu(l)e^{-\pi(y-2i\theta)l},$$

with $c_\mu(l) = \sum_{\mu_{n_1}+\mu_{n_2}+\cdots+\mu_{n_k}=l}\prod_{j=1}^k h_{n_j} \approx r_\mu(l)$.

Hence,

$$||S_y||_{L^{2k}} = ||S_y^k||_{L^2}^{1/k} \approx \left(\sum_{l=1}^{\infty} r_\mu(l)^2e^{-2\pi yl}\right)^{1/2k}.$$

Since

$$\begin{aligned}\sum_{l=1}^{\infty} r_\mu(l)^2e^{-2\pi yl} &= (1-e^{-2\pi y})\sum_{l=1}^{\infty} r_\mu(l)^2\sum_{j=l}^{\infty}e^{-2\pi yj}\\ &= (1-e^{-2\pi y})\sum_{j=1}^{\infty}e^{-2\pi yj}\sum_{l=1}^{j} r_\mu(l)^2\\ &\lesssim (1-e^{-2\pi y})\sum_{j=1}^{\infty}e^{-2\pi yj}j^{1+\varepsilon} \lesssim \frac{1-e^{-2\pi y}}{y}\frac{1}{y^{1+\varepsilon}} \lesssim \frac{1}{y^{1+\varepsilon}},\end{aligned}$$

we obtain,

$$||S_y||_{L^{2k}} \lesssim \frac{1}{y^{\frac{1+\varepsilon}{2k}}},$$

and for $\lambda > 1/2$, we can conclude

$$||m_{\lambda,\mu}||_{L^{2k}} \lesssim 1 + \int_0^1 \frac{1}{y^{\frac{1+\varepsilon}{2k}+1-\frac{\lambda}{k}}}dy < \infty.$$

Conversely, if $m_{\lambda,\mu} \in L^{2k}$ for every $\lambda > 1/2$, then

$$m_{\lambda,\mu}(\theta)^k = \sum_{n_1,n_2,\cdots,n_k}\frac{e^{2\pi i(\mu_{n_1}+\mu_{n_2}+\cdots+\mu_{n_k})\theta}}{n_1^\lambda n_2^\lambda\cdots n_k^\lambda} = \sum_{l=1}^{\infty} C_l e^{2\pi il\theta} \in L^2,$$

with

$$C_l = \sum_{\mu_{n_1}+\mu_{n_2}+\cdots+\mu_{n_k}=l}\frac{1}{n_1^\lambda n_2^\lambda\cdots n_k^\lambda}.$$

Since $n_j^k \approx \mu_{n_j} \leq l$ we have that $n_1^\lambda n_2^\lambda\cdots n_k^\lambda \leq l^\lambda$ and hence $C_l \geq \frac{r_\mu(l)}{l^\lambda}$. Since $(C_l)_l \in \ell^2$, we finally obtain that, for every $\lambda > 1/2$,

$$\frac{1}{N^{2\lambda}}\sum_{l=1}^{N} r_\mu(l)^2 \leq \sum_{l=1}^{\infty}\frac{r_\mu(l)^2}{l^{2\lambda}} < \infty,$$

as we wanted to see.

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