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A BIAS PHENOMENON ON THE BEHAVIOR OF DEDEKIND SUMS

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Abstract. In this paper we present a bias phenomenon on the behavior of Dedekind sums at visible points in a dilated region. Our results indicate that in more than three quarters of the time the Dedekind sum increases as one moves from one visible point to the next.

1. Introduction

Various bias phenomena in number theory have been studied for a long time, see for example the work of Dirichlet [13] (page 9), Knapowski and Turán [30], [31], [35], Rubinstein and Sarnak [34], Bombieri [10], Kaczorowski [29], Feuerverger and Martin [15], Ford and Hudson [16], Ford and Konyagin [17], [18], [19], Granville and Martin [24], Granville, Shiu and Shiu [25], Granville, Shparlinski and one of the authors [26].

In the present paper we provide another interesting example of bias, which appears in the behavior of Dedekind sums. Dedekind sums occur naturally in the functional equation of the η-function. The reciprocity laws of Dedekind sums and their generalizations have been studied by a number of authors, including Rademacher and Grosswald ([33]) and Berndt ([7], [8], [9]). Various distribution properties of Dedekind sums were studied by Hickerson [28], Bruggeman [11], Conrey, Fransen, Klein and Scott [12], Girstmair [20], [21], [22], Girstmair and Schoissengeier [23], Vardi [36]. Here we are concerned with the following problem. For each large positive real number X consider the set $\mathcal{A}(X)$ of integer points (a,b) with relatively prime coordinates in the first quadrant whose distance to the origin is $\leq X$, and order them increasingly with respect to the angles through origin measured in the counterclockwise direction. Next, we compute the Dedekind sum $s(a,b)$ at each such point and ask the following question: what is the proportion of points (a,b) for which $s(a,b) < s(a',b')$? Here (a,b) and (a',b') are consecutive in $\mathcal{A}(X)$. We will see that the limit

$$\lim_{X\to\infty} \frac{\#\left\{(a,b)\in\mathcal{A}(X) : s(a,b) < s(a',b')\right\}}{\#\left(\mathcal{A}(X)\right)}$$

exists, and it is strictly larger than $1/2$.

More generally, let us choose a simple, closed, continuous curve C in the plane. Then $\mathbb{R}^2\setminus C$ has two connected components, and we denote the bounded component by Ω. We will also assume that the origin $(0,0)$ lies in $\Omega\bigcup C$, and that $\Omega\bigcup C$ is

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star-shaped with respect to the origin, in the sense that for any point $(x, y) \in \Omega \bigcup C$, the line segment joining $(0,0)$ to (x, y) is contained in $\Omega \bigcup C$. We denote the set of all plane regions Ω that arise in this way by $\mathscr{M}$. For each such region $\Omega \in \mathscr{M}$ and any large $X > 0$, let us denote by $\mathcal{A}_\Omega(X)$ the finite sequence of visible points in $X\Omega$ ordered counterclockwise as above. Our aim is to estimate the proportion of points $(a, b) \in \mathcal{A}_\Omega(X)$ for which $s(a, b) < s(a', b')$, where (a, b) and (a', b') are consecutive in $\mathcal{A}_\Omega(X)$. We will show that for each fixed $\Omega \in \mathscr{M}$, this proportion has a limit as X tends to infinity. Surprisingly, for regions $\Omega \in \mathscr{M}$ contained in the upper half plane, this limit is independent of Ω, and equals $\frac{4}{\sqrt{5}} - 1 \approx 0.788854....$ This shows that in the upper half plane, in more than 3/4 of the time the Dedekind sum increases as we move from one visible point to the next one.

Next, for each fixed h we ask a similar question for all possible orders among values of the Dedekind sum at h-tuples of consecutive visible points. Let $\Omega \in \mathscr{M}$, $X > 0$, and order $\mathcal{A}_\Omega(X)$ as

$$\mathcal{A}_\Omega(X) = \{P_i = (a_i, b_i) : 1 \leq i \leq N = N(X, \Omega)\}.$$

Then define a function $\underline{v}_h : \mathcal{A}_\Omega(X) \longrightarrow \{0,1\}^h$ as $\underline{v}_h(P_i) = \underline{v}_h(a_i, b_i) = (v_1, \ldots, v_h)$, where

$$v_j = \begin{cases} 0 & : \quad s(a_{i+j-1}, b_{i+j-1}) < s(a_{i+j}, b_{i+j}), \\ 1 & : \quad s(a_{i+j-1}, b_{i+j-1}) \geq s(a_{i+j}, b_{i+j}), \end{cases} \quad 1 \leq j \leq h.$$

Here $\underline{v}_h(P_i)$ is not defined for i satisfying $i + h > N$, but the number of such i is at most h and they are negligible as $N \to \infty$. We are concerned with the problem of the existence of the limit

$$\lim_{X \to \infty} \frac{\#\{P \in \mathcal{A}_\Omega(X) : \underline{v}_h(P) = \underline{v}\}}{\#(\mathcal{A}_\Omega(X))}.$$

In order to state our main result in full generality, we introduce more notation. For any region $\Omega \subset \mathbb{R}^2$, denote by $A(\Omega)$ its area and let

$$\Omega_{y \geq 0} := \{(x, y) \in \Omega : y \geq 0\}, \quad \Omega_{y \leq 0} := \{(x, y) \in \Omega : y \leq 0\}.$$

Denote

$$\mathscr{T} = \{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x, y \leq 1, x + y > 1\},$$

and consider for each $(x, y) \in \mathscr{T}$, the sequence $(L_j(x, y))_{j \geq 0}$ defined by $L_0(x, y) = x, L_1(x, y) = y$ and recursively, for $j \geq 0$,

$$L_{j+2}(x, y) = \left[\frac{1 + L_j(x, y)}{L_{j+1}(x, y)}\right] L_{j+1}(x, y) - L_j(x, y),$$

where [.] denotes the integer part function. Let

$$\lambda_1 = \frac{3 - \sqrt{5}}{2}, \quad \lambda_2 = \frac{3 + \sqrt{5}}{2},$$

and for $j \geq 1$, define

$$\mathscr{T}^j(0) = \left\{(x, y) \in \mathscr{T} : \lambda_1 < \frac{L_{j-1}(x, y)}{L_j(x, y)} < \lambda_2\right\}, \quad \mathscr{T}^j(1) = \mathscr{T} \setminus \mathscr{T}^j(0).$$

For any $\underline{v} = (v_1, \dots, v_h) \in \{0,1\}^h$, denote

$$\mathscr{T}_{\underline{v}} = \bigcap_{j=1}^{h} \mathscr{T}^j(v_j). \tag{1}$$

Theorem 1. *For any $\Omega \in \mathscr{M}$, any positive integer h and any vector $\underline{v} \in \{0,1\}^h$,*

$$\lim_{X\to\infty} \frac{\#\{P \in \mathcal{A}_\Omega(X) : \underline{v}_h(P) = \underline{v}\}}{\#(\mathcal{A}_\Omega(X))} = 2 \cdot \frac{A(\Omega_{y\geq 0})A(\mathscr{T}_{\underline{v}}) + A(\Omega_{y\leq 0})A(\mathscr{T}_{\underline{v}+\underline{1}})}{A(\Omega)},$$

where $\underline{1} = (1, \dots, 1) \in \{0,1\}^h$, and the addition $\underline{v} + \underline{1}$ is taken modulo 2.

Corollary 1. *For any $\Omega \in \mathscr{M}$ such that $\Omega \subset \mathbb{R}^2_{y\geq 0} = \{(x,y) : y \geq 0\}$, any positive integer h and any vector $\underline{v} \in \{0,1\}^h$,*

$$\lim_{X\to\infty} \frac{\#\{P \in \mathcal{A}_\Omega(X) : \underline{v}_h(P) = \underline{v}\}}{\#(\mathcal{A}_\Omega(X))} = 2A(\mathscr{T}_{\underline{v}}).$$

We remark that for $h = 1$ one has $2A(\mathscr{T}_0) = \frac{4}{\sqrt{5}} - 1$, as mentioned above. For $h = 2$, we find that $2A(\mathscr{T}_{(0,0)}) = -\frac{1291}{399} + \frac{166}{19\sqrt{5}} \approx 0.671646....$ This means that when one chooses randomly three consecutive visible points $(a,b), (a',b'), (a'',b'')$ in $X\Omega$, the probability that one has $s(a,b) < s(a',b') < s(a'',b'')$ is 0.671646... in the limit as $X \to \infty$. Note that this is larger than the product between the probability that $s(a,b) < s(a',b')$ and the probability that $s(a',b') < s(a'',b'')$, which equals $\left(\frac{4}{\sqrt{5}} - 1\right)^2 \approx 0.622291....$

The key ingredients in the proof of Theorem 1 are the reciprocity law of Dedekind sums, in the form of Lemma 1, the technique of counting visible points in various regions by employing Kloosterman sums, in the form of Lemma 2, and the properties of the area preserving map T, which provides us with an explicit way of producing chains of consecutive Farey fractions. All these are reviewed briefly in Section 2 below. Then in Section 3 we combine them to obtain certain asymptotic formulas, which are further used in Section 4 in order to complete the proof of Theorem 1.

2. Dedekind sums and visible points

For any real number x, let $((x))$ be the sawtooth function defined as

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For positive integers h, k the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{s \pmod{k}} \left(\left(\frac{s}{k}\right)\right)\left(\left(\frac{hs}{k}\right)\right),$$

where the notation $s \pmod{k}$ means that s runs over a complete residue system modulo k. Since the sawtooth function has period one, $s(h,k)$ is a periodic function of h with period k. The following lemma follows from formula (38) on page 29 of [33], which is in term a consequence of the reciprocity law of Dedekind sums.

Lemma 1. *For any positive integers a, b, c, d satisfying $ad - bc = 1$, one has*

$$s(a,c) - s(b,d) = -\frac{1}{4} + \frac{1}{12}\left(\frac{d}{c} + \frac{c}{d} + \frac{1}{cd}\right).$$

Next we recall some results on Farey fractions. For an exposition of their basic properties, the reader is referred to [27]. Let $\mathscr{F}_Q = \{\gamma_1, \dots, \gamma_{N(Q)}\}$ denote the Farey sequence of order Q with $1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1$. It is well-known that

$$N(Q) = \sum_{j=1}^{Q} \phi(j) = \frac{3Q^2}{\pi^2} + O(Q \log Q).$$

Write $\gamma_i = a_i/q_i$ in reduced form, i.e., $a_i, q_i \in \mathbb{Z}, 1 \le a_i \le q_i \le Q, \gcd(a_i, q_i) = 1$. For any two consecutive Farey fractions $a_i/q_i < a_{i+1}/q_{i+1}$, one has $a_{i+1}q_i - a_i q_{i+1} = 1$ and $q_i + q_{i+1} > Q$. Conversely, if q and q' are two coprime integers in $\{1, \dots, Q\}$ with $q + q' > Q$, then there are unique $a \in \{1, \dots, q\}$ and $a' \in \{1, \dots, q'\}$ for which $a'q - aq' = 1$, and $a/q < a'/q'$ are consecutive Farey fractions of order Q. Therefore, the pairs of coprime integers (q, q') with $q + q' > Q$ are in one-to-one correspondence with the pairs of consecutive Farey fractions of order Q. Moreover, the denominator q_{i+2} of γ_{i+2} can be expressed (cf. [1]) by means of the denominators of γ_i and γ_{i+1} as

$$q_{i+2} = \left[\frac{Q + q_i}{q_{i+1}}\right] q_{i+1} - q_i,$$

where $[\ .\ ]$ denotes the greatest integer part function. By induction, for any $j \ge 2$, the denominator q_{i+j} of γ_{i+j} can be expressed in terms of the denominators of γ_i, γ_{i+1}. More precisely, let $\mathscr{T}$ denote the Farey triangle

$$\mathscr{T} = \{(x, y) \in [0,1]^2 : x + y > 1\},$$

and consider, for each $(x, y) \in \mathscr{T}$, the sequence $(L_i(x,y))_{i \ge 0}$ defined by $L_0(x,y) = x, L_1(x,y) = y$ and recursively, for $i \ge 2$,

$$L_i(x,y) = \left[\frac{1 + L_{i-2}(x,y)}{L_{i-1}(x,y)}\right] L_{i-1}(x,y) - L_{i-2}(x,y).$$

Then for all $i, j \ge 0$ with $i + j \le N(Q)$, we have

$$\frac{q_{i+j}}{Q} = L_j\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\right).$$

Such formulas prove to be very useful in the study of various questions on the distribution of Farey fractions (see, for example [1],[2],[3]). The bijective, piecewise smooth and area preserving map $T : \mathscr{T} \longrightarrow \mathscr{T}$ defined by ([2])

$$T(x,y) = \left(y, \left[\frac{1+x}{y}\right] y - x\right)$$

also plays an important role in recent developments of the subject.

Let us consider the set of visible lattice points in the plane,

$$\mathbb{Z}^2_{vis} = \{(a,b) \in \mathbb{Z}^2 : \gcd(a,b) = 1\},$$

and for each region Ω in $\mathbb{R}^2$ and each C^1 function $f : \Omega \longrightarrow \mathbb{C}$, let us denote

$$||f||_{\infty,\Omega} = \sup_{(x,y)\in\Omega} |f(x,y)|,$$

and

$$||Df||_{\infty,\Omega} = \sup_{(x,y)\in\Omega} \left(\left|\frac{\partial f}{\partial x}(x,y)\right| + \left|\frac{\partial f}{\partial y}(x,y)\right| \right).$$

We need the following variation of a result from [2]. For any subinterval $\mathbf{I} = [\alpha, \beta]$ of $[0,1]$, let $\mathbf{I}_a = [(1-\beta)a, (1-\alpha)a]$.

Lemma 2. *Let $\Omega \subset [1,R] \times [1,R]$ be a convex region and let f be a C^1 function on Ω. For any subinterval $\mathbf{I} \subset [0,1]$ one has*

$$\sum_{\substack{(a,b)\in\Omega \bigcap \mathbb{Z}^2_{vis}, \\ \bar{b}\in\mathbf{I}_a}} f(a,b) = \frac{6|\mathbf{I}|}{\pi^2} \iint_\Omega f(x,y)\,\mathrm{d}x\mathrm{d}y + \mathcal{F}_{R,\Omega,f,\mathbf{I}},$$

where

$$\begin{aligned} \mathcal{F}_{R,\Omega,f,\mathbf{I}} \ll_\delta \quad & m_f ||f||_{\infty,\Omega} R^{3/2+\delta} + ||f||_{\infty,\Omega} R \log R \\ & + ||Df||_{\infty,\Omega} Area(\Omega) \log R, \end{aligned}$$

for any $\delta > 0$, here $\bar{b}$ denotes the multiplicative inverse of $b \pmod a$, i.e., $1 \le \bar{b} \le a - 1, b\bar{b} \equiv 1 \pmod a$, m_f is an upper bound for the number of intervals of monotonicity of each of the functions $y \mapsto f(x,y)$.

This is Lemma 8 in [2], where Weil type estimates ([37], [14]) for certain weighted incomplete Kloosterman sums play a crucial role in the proof.

3. Preliminary results

A concept that plays an important role in questions on the local distribution of Farey points is that of the index of a Farey fraction. In the language of visible points, the index is intrinsically related with the position of consecutive visible points in terms of their distance to the origin and the angle between the corresponding rays from the origin to these points, and in this way it naturally appears in some applications to questions originating in mathematical physics such as billiards and periodic Lorentz gas ([4], [5], [6]). Recall that for $1 < i < N(Q)$, the index of the fraction γ_i in $\mathscr{F}_Q$ is defined by

$$\nu_Q(\gamma_i) = \left[\frac{Q + q_{i-1}}{q_i}\right] = \frac{q_{i-1} + q_{i+1}}{q_i}.$$

The index also plays an important role in the proof of Theorem 1. In what follows it is essential to have good control over the index of each element in a chain of consecutive visible points, and this is our main strategic step in the process of proving the following two lemmas.

Lemma 3. *Fix a positive integer h and a subinterval $\mathbf{I} = [\alpha, \beta] \subset [0,1]$. Let*

$$S_{Q,h} = \#\left\{ \frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I} : s(a_i, q_i) \ge s(a_{i+1}, q_{i+1}) \ge \cdots \ge s(a_{i+h}, q_{i+h}) \right\}.$$

Then for any $\epsilon > 0$,

$$S_{Q,h} = \frac{6|\mathbf{I}|Q^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) + O_\epsilon\left(Q^{2-\frac{1}{h+1}+\epsilon}\right),$$

where the vector $\underline{0} = (0, \ldots, 0) \in \{0,1\}^h$ and $\mathscr{T}_{\underline{0}}$ is defined as in (1).

Proof. Let $\frac{a}{q} < \frac{a'}{q'}$ be consecutive Farey fractions. We know that $a'q - aq' = 1$. By Lemma 1 one has

$$s(a', q') - s(a, q) = -\frac{1}{4} + \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right).$$

Hence $s(a, q) \geq s(a', q')$, if and only if

$$-\frac{1}{4} + \frac{1}{12}\left(\frac{q}{q'} + \frac{q'}{q} + \frac{1}{qq'}\right) \leq 0,$$

since $a, q, a', q' \in \mathbb{N}$, this is equivalent to $q^2 + q'^2 - 3qq' < 0$, and therefore

$$\lambda_1 = \frac{3-\sqrt{5}}{2} < \frac{q}{q'} < \lambda_2 = \frac{3+\sqrt{5}}{2}.$$

We have

$$S_{Q,h} = \#\left\{\frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I} : \begin{array}{c} \lambda_1 < \frac{q_i}{q_{i+1}} < \lambda_2 \\ \vdots \\ \lambda_1 < \frac{q_{i+h-1}}{q_{i+h}} < \lambda_2 \end{array}\right\}.$$

For a large positive number $L < Q$ which will be chosen later, denote

$$S_{Q,h,L} = \#\left\{\frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I} : \begin{array}{cc} \lambda_1 < \frac{q_i}{q_{i+1}} < \lambda_2, & \\ \lambda_1 < \frac{q_{i+1}}{q_{i+2}} < \lambda_2, & q_{i+1} > \frac{Q}{L}, \\ \vdots & \vdots \\ \lambda_1 < \frac{q_{i+h-1}}{q_{i+h}} < \lambda_2, & q_{i+h-1} > \frac{Q}{L}, \end{array}\right\},$$

we have the inequalities

$$\begin{aligned} S_{Q,h,L} \leq S_{Q,h} &\leq S_{Q,h,L} + \#\left(\bigcup_{j=1}^{h-1}\left\{\frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I} : q_{i+j} \leq \frac{Q}{L}\right\}\right) \\ &\leq S_{Q,h,L} + (h-1) \cdot \sum_{q \leq \frac{Q}{L}} q \\ &\leq S_{Q,h,L} + (h-1)\frac{Q^2}{L^2}, \end{aligned}$$

hence

$$S_{Q,h} = S_{Q,h,L} + O\left(\frac{Q^2}{L^2}\right). \tag{2}$$

For $\frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I}$ with $q_{i+1} > Q/L, \ldots, q_{i+h-1} > Q/L$, the index satisfies

$$\begin{array}{rcccl} v_Q\left(\frac{a_{i+1}}{q_{i+1}}\right) & = & \frac{q_i+q_{i+2}}{q_{i+1}} & < & \frac{2Q}{Q/L} = 2L, \\ & & \vdots & & \\ v_Q\left(\frac{a_{i+h-1}}{q_{i+h-1}}\right) & = & \frac{q_{i+h-2}+q_{i+h}}{q_{i+h-1}} & < & \frac{2Q}{Q/L} = 2L, \end{array}$$

so that we may write $S_{Q,h,L}$ as

$$S_{Q,h,L} = \sum_{1 \le k_1, \ldots, k_{h-1} < 2L} \sum_{\substack{\frac{a_i}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I}, \\ q_{i+1} > \frac{Q}{L},\ v_Q\left(\frac{a_{i+1}}{q_{i+1}}\right) = k_1, \\ \vdots \\ q_{i+h-1} > \frac{Q}{L},\ v_Q\left(\frac{a_{i+h-1}}{q_{i+h-1}}\right) = k_{h-1},}} \sum_{\substack{\lambda_1 < \frac{q_i}{q_{i+1}} < \lambda_2, \\ \vdots \\ \lambda_1 < \frac{q_{i+h-1}}{q_{i+h}} < \lambda_2.}} 1.$$

First of all for consecutive Farey fractions $\frac{a_i}{q_i} < \frac{a_{i+1}}{q_{i+1}}$ one has $a_{i+1}q_i - a_iq_{i+1} = 1$, hence $a_i \equiv -\bar{q}_{i+1} \pmod{q_i}$, where the integer $\bar{x}(1 \le \bar{x} \le q)$ denotes the multiplicative inverse of $x \pmod q$ for any integer x with $\gcd(x, q) = 1$. Since $1 < a_i < q_i$, one has $a_i = q_i - \bar{q}_{i+1}$ and

$$\frac{a_i}{q_i} = 1 - \frac{\bar{q}_{i+1}}{q_i} \in \mathscr{F}_Q \bigcap \mathbf{I} \Longleftrightarrow \bar{q}_{i+1} \in \mathbf{I}_{q_i} = [(1-\beta)q_i, (1-\alpha)q_i],$$

where $\mathbf{I} = [\alpha, \beta] \subset [0, 1]$. Next for any positive integers $h, k_1, k_2, \ldots, k_{h-1}$, let

$$\mathscr{T}_{k_1,\ldots,k_{h-1}} = \bigcap_{j=1}^{h-1} T^{-j+1}\mathscr{T}_{k_j}.$$

(When $h = 1$, this is just $\mathscr{T}$.) Then for any $(x, y) \in \mathscr{T}_{k_1,\ldots,k_{h-1}}$, we have

$$L_0(x, y) = x, \quad L_1(x, y) = y,$$

and recursively,

$$L_{i+1}(x, y) = k_iL_i(x, y) - L_{i-1}(x, y), \qquad 1 \le i \le h - 1.$$

Therefore there exist real numbers ω_i, υ_i depending only on $k_1, \ldots, k_{h-1}$ such that

$$L_i(x, y) = \omega_i x + \upsilon_i y, \qquad 0 \le i \le h.$$

The set $\mathscr{T}_{k_1,\ldots,k_{h-1}} \subset \mathscr{T}$ is obtained by intersecting finitely many half planes, and so it is a finite union of convex polygons. Denote for any $t \ge 0$,

$$\mathscr{H}_{k_1,k_2,\ldots,k_{h-1}}(t) = \left\{ (x, y) \in \mathscr{T}_{k_1,k_2,\ldots,k_{h-1}} : \begin{array}{rl} \lambda_1 < \frac{L_j(x,y)}{L_{j+1}(x,y)} < \lambda_2, & 0 \le j \le h-1, \\ L_j(x, y) > t, & 1 \le j \le h-1, \end{array} \right\}$$

Here $\mathscr{H}_{k_1,\ldots,k_{h-1}}(t)$ is also a finite union of convex polygons. We now return to the formula of $S_{Q,h,L}$. Since for any $j \ge 0$, $q_{i+j} = QL_j\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\right)$, we see that

$$\begin{gathered} v_Q\left(\frac{a_{i+1}}{q_{i+1}}\right) = k_1, v_Q\left(\frac{a_{i+2}}{q_{i+2}}\right) = k_2, \ldots, v_Q\left(\frac{a_{i+h-1}}{q_{i+h-1}}\right) = k_{i+h-1}, \\ \lambda_1 < \frac{L_0(q_i/Q, q_{i+1}/Q)}{L_1(q_i/Q, q_{i+1}/Q)} < \lambda_2, \ldots, \lambda_1 < \frac{L_{h-1}(q_i/Q, q_{i+1}/Q)}{L_h(q_i/Q, q_{i+1}/Q)} < \lambda_2, \\ q_{i+1} > \frac{Q}{L}, q_{i+2} > \frac{Q}{L}, \ldots, q_{i+h-1} > \frac{Q}{L}, \end{gathered}$$

if and only if

$$\left(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}\right) \in \mathscr{H}_{k_1,\ldots,k_{h-1}}\left(\frac{1}{L}\right).$$

Therefore

$$S_{Q,h,L} = \sum_{1\le k_1,\ldots,k_{h-1}<2L} \quad \sum_{\substack{(q_i,\,q_{i+1})\in Q\mathscr{H}_{k_1,\ldots,k_{h-1}}\left(\frac{1}{L}\right)\bigcap \mathbb{Z}^2_{pr} \\ \bar{q}_{i+1}\in[(1-\beta)q_i,(1-\alpha)q_i]}} 1.$$

Applying Lemma 2 we obtain that

$$\begin{aligned} S_{Q,h,L} &= \sum_{1\le k_1,\ldots,k_{h-1}<2L} \left(\frac{6|\mathbf{I}|Q^2}{\pi^2} \iint_{\mathscr{H}_{k_1,\ldots,k_{h-1}}\left(\frac{1}{L}\right)} 1\,\mathrm{d}x\mathrm{d}y + O_\epsilon\left(Q^{\frac{3}{2}+\epsilon}\right)\right) \\ &= \frac{6|\mathbf{I}|Q^2}{\pi^2} \iint_{\mathscr{H}\left(\frac{1}{L}\right)} 1\,\mathrm{d}x\mathrm{d}y + O_\epsilon\left(L^{h-1}Q^{\frac{3}{2}+\epsilon}\right), \end{aligned}$$

where for any $t \ge 0$,

$$\begin{aligned} \mathscr{H}(t) &= \bigcup_{k_1,\ldots,k_{h-1}} \mathscr{H}_{k_1,\ldots,k_{h-1}}(t) \\ &= \left\{ (x,y)\in\mathscr{T} : \begin{array}{ll} \lambda_1 < \frac{L_j(x,y)}{L_{j+1}(x,y)} < \lambda_2, & 0\le j\le h-1 \\ L_j(x,y) > t, & 1\le j\le h-1 \end{array} \right\}. \end{aligned}$$

Since $T : \mathscr{T} \to \mathscr{T}$ is a bijective, continuous and area-preserving map and for any integer $r \ge 0$, $T^r(x,y) = (L_r(x,y), L_{r+1}(x,y))$ for $(x,y)\in\mathscr{T}$, we see that

$$\iint_{\left\{(x,y)\in\mathscr{T}:L_r(x,y)\le\frac{1}{L}\right\}} 1\,\mathrm{d}x\mathrm{d}y = \iint_{\left\{(u,v)\in\mathscr{T}:u\le\frac{1}{L}\right\}} 1\,\mathrm{d}u\mathrm{d}v = \frac{1}{2L^2},$$

hence

$$\iint_{\mathscr{H}\left(\frac{1}{L}\right)} 1\,\mathrm{d}x\mathrm{d}y = \iint_{\mathscr{T}_{\underline{0}}} 1\,\mathrm{d}x\mathrm{d}y + O\left(\frac{1}{L^2}\right),$$

where

$$\mathscr{T}_{\underline{0}} = \left\{(x,y)\in\mathscr{T} : \lambda_1 < \frac{L_j(x,y)}{L_{j+1}(x,y)} < \lambda_2,\ \ 0\le j\le h-1\right\}.$$

Therefore

$$S_{Q,h,L} = \frac{6|\mathbf{I}|Q^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) + O\left(\frac{Q^2}{L^2}\right) + O_\epsilon\left(L^{h-1}Q^{\frac{3}{2}+\epsilon}\right).$$

Letting $\frac{Q^2}{L^2} = L^{h-1}Q^{\frac{3}{2}}$, we may choose $L = Q^{\frac{1}{2(h+1)}}$, and using (2) we obtain that for any $\epsilon > 0$,

$$S_{Q,h} = S_{Q,h,L} + O\left(\frac{Q^2}{L^2}\right) = \frac{6|\mathbf{I}|Q^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) + O_\epsilon\left(Q^{2-\frac{1}{h+1}+\epsilon}\right).$$

This completes the proof of Lemma 3. $\square$

Lemma 4. *Fix a positive integer h and a subinterval $\mathbf{I}=[\alpha,\beta]\subset[0,1]$. Let*

$$S'_{Q,h}=\#\left\{\frac{a_i}{q_i}\in\mathscr{F}_Q\bigcap\mathbf{I}: s(q_i,a_i)\le s(q_{i+1},a_{i+1})\le\cdots\le s(q_{i+h},a_{i+h})\right\}.$$

Then for any $\epsilon>0$,

$$S'_{Q,h}=\frac{6|\mathbf{I}|Q^2}{\pi^2}A(\mathscr{T}_{\underline{0}})+O_\epsilon\left(Q^{2-\frac{1}{h+1}+\epsilon}\right).$$

Proof. For consecutive Farey fractions $\frac{a}{q}<\frac{a'}{q'}$, $a'q-aq'=1$. Using Lemma 1 we have

$$s(q,a)-s(q',a') \quad = \quad -\frac{1}{4}+\frac{1}{12}\left(\frac{a}{a'}+\frac{a'}{a}+\frac{1}{aa'}\right).$$

Therefore

$$s(q,a)\le s(q',a')\Longleftrightarrow\lambda_1<\frac{a}{a'}<\lambda_2.$$

Since $\frac{a}{a'}=\frac{q}{q'}-\frac{1}{a'q'}$, we have $\lambda_1-\frac{1}{a'q'}<\frac{q}{q'}<\lambda_2-\frac{1}{a'q'}$. We may write $S'_{Q,h}$ as

$$S'_{Q,h}=\#\left\{\frac{a_i}{q_i}\in\mathscr{F}_Q\bigcap\mathbf{I}:\begin{array}{c}\lambda_1-\frac{1}{a_{i+1}q_{i+1}}<\frac{q_i}{q_{i+1}}<\lambda_2-\frac{1}{a_{i+1}q_{i+1}},\\ \vdots\\ \lambda_1-\frac{1}{a_{i+h}q_{i+h}}<\frac{q_{i+h-1}}{q_{i+h}}<\lambda_2-\frac{1}{a_{i+h}q_{i+h}}.\end{array}\right\}.$$

For a large positive number $L<Q$ which will be chosen later, denote

$$S'_{Q,h,L,1}=\#\left\{\frac{a_i}{q_i}\in\mathscr{F}_Q\bigcap\mathbf{I}:\begin{array}{cc}\lambda_1-\frac{L}{Q}<\frac{q_i}{q_{i+1}}<\lambda_2, & q_{i+1}>\frac{Q}{L}\\ \vdots & \vdots\\ \lambda_1-\frac{L}{Q}<\frac{q_{i+h-1}}{q_{i+h}}<\lambda_2, & q_{i+h}>\frac{Q}{L},\end{array}\right\},$$

and

$$S'_{Q,h,L,2}=\#\left\{\frac{a_i}{q_i}\in\mathscr{F}_Q\bigcap\mathbf{I}:\begin{array}{cc}\lambda_1<\frac{q_i}{q_{i+1}}<\lambda_2-\frac{L}{Q}, & q_{i+1}>\frac{Q}{L}\\ \vdots & \vdots\\ \lambda_1<\frac{q_{i+h-1}}{q_{i+h}}<\lambda_2-\frac{L}{Q}, & q_{i+h}>\frac{Q}{L},\end{array}\right\},$$

we have the inequalities

$$S'_{Q,h,L,2}\le S'_{Q,h}\le S'_{Q,h,L,1}. \tag{3}$$

For $S'_{Q,h,L,1}$, similar computation shows that for any $\epsilon>0$,

$$S'_{Q,h,L,1} \quad = \quad \frac{6|\mathbf{I}|Q^2}{\pi^2}A\left(\mathscr{H}_{1,\frac{L}{Q}}\right)+O\left(\frac{Q^2}{L^2}\right)+O_\epsilon\left(L^{h-1}Q^{\frac{3}{2}+\epsilon}\right),$$

where where for any $t\ge0$,

$$\mathscr{H}_{1,t} \quad = \quad \left\{(x,y)\in\mathscr{T}:\lambda_1-t<\frac{L_j(x,y)}{L_{j+1}(x,y)}<\lambda_2,\ 0\le j\le h-1\right\}.$$

Since

$$\begin{aligned}\mathscr{T}_{\underline{0}}-\mathscr{H}_{1,\frac{L}{Q}} \quad &= \quad \left\{(x,y)\in\mathscr{T}:\lambda_1-\frac{L}{Q}<\frac{L_j(x,y)}{L_{j+1}(x,y)}\le\lambda_1,\ 0\le j\le h-1\right\}\\ &\subset \quad \left\{(x,y)\in\mathscr{T}:\lambda_1-\frac{L}{Q}<\frac{x}{y}\le\lambda_1\right\},\end{aligned}$$

we see that

$$A\left(\mathscr{H}_{1,\frac{L}{Q}}\right) = A\left(\mathscr{T}_{\underline{0}}\right) + O\left(\frac{L}{Q}\right).$$

Therefore

$$S'_{Q,h,L,1} = \frac{6|\mathbf{I}|Q^2}{\pi^2} A\left(\mathscr{T}_{\underline{0}}\right) + O(QL) + O\left(\frac{Q^2}{L^2}\right) + O_\epsilon\left(L^{h-1}Q^{\frac{3}{2}+\epsilon}\right).$$

Choosing $L = Q^{\frac{1}{2(h+1)}}$, we get that for any $\epsilon > 0$,

$$S'_{Q,h,L,1} = \frac{6|\mathbf{I}|Q^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) + O_\epsilon\left(Q^{2-\frac{1}{h+1}+\epsilon}\right).$$

Similarly the same asymptotic formula holds true for $S'_{Q,h,L,2}$. Hence the inequalities (3) give the asymptotic formula for $S'_{Q,h}$. This completes the proof of Lemma 4. □

4. Proof of theorem 1

Proof. We now have all the necessary ingredients to prove Theorem 1. Let us fix a region $\Omega \in \mathscr{M}$. It is known that (see [3])

$$\#\left(\mathcal{A}_\Omega(X)\right) \asymp \frac{6A(\Omega)}{\pi^2} X^2.$$

Without loss of generality, take $\underline{v} = \underline{0}$ and let us study the asymptotic behavior of $S_{\Omega,\underline{0}}(X) = \#\left\{P \in \mathcal{A}_\Omega(X) : \underline{v}_h(P) = \underline{0}\right\}$ as $X \to \infty$. Denote

$$\Omega_i = \Omega \bigcap \left\{(r,\theta) : \frac{(i-1)\pi}{4} \le \theta < \frac{i\pi}{4}, \quad 1 \le i \le 8\right\},$$

and

$$S_{\Omega_i,\underline{0}}(X) = \#\left\{P \in \mathcal{A}_{\Omega_i}(X) : \underline{v}(P) = \underline{0}\right\}.$$

Then

$$S_{\Omega,\underline{0}}(X) = \sum_{i=1}^{8} S_{\Omega_i,\underline{0}}(X) + O(1). \tag{4}$$

We consider $S_{\Omega_2,\underline{0}}(X)$ first. Assume that in polar coordinates (r,θ),

$$\Omega_2 = \{(r,\theta) : r \le f(\theta), \theta_1 \le \theta \le \theta_2\},$$

where f is a bounded non-negative continuously differentiable function and $\frac{\pi}{4} \le \theta_1 < \theta_2 \le \frac{\pi}{2}$. Fix a large integer $L > 0$, denote

$$\alpha = \frac{\theta_2 - \theta_1}{L}, \quad \alpha_i = \theta_1 + i \cdot \alpha, \ \ 0 \le i \le L.$$

Suppose the rays $\{(r,\theta) : \theta = \alpha_i\}$, $0 \le i \le L$ intercept the boundary of Ω_2 at points $A_0, A_1, \ldots, A_L$ counter-clockwise. At each point A_i, $0 \le i \le L-1$, draw a horizontal line which intercepts the ray $\{(r,\theta) : \theta = \alpha_{i+1}\}$ at the point A'_i. We see that

$$\begin{aligned} A_i &= \left(f(\alpha_i)\cos(\alpha_i), f(\alpha_i)\sin(\alpha_i)\right), \quad 0 \le i \le L-1, \\ A'_i &= \left(f(\alpha_i)\cos(\alpha_i) \cdot \frac{\tan(\alpha_i)}{\tan(\alpha_{i+1})}, f(\alpha_i)\sin(\alpha_i)\right), \quad 0 \le i \le L-1. \end{aligned}$$

Let $\Omega_{2,i}$ be the i-th region of Ω_2 inside the rays $\overrightarrow{OA_i}, \overrightarrow{OA_{i+1}}$ and $\triangle_i$ be the triangle OA_iA_i', one has

$$A\left(\Omega_{2,i}\right) - A\left(\triangle_i\right) \ll_f \frac{1}{L^2},$$

and thus

$$\begin{aligned} \#\left(\left\{P \in \mathcal{A}_{\Omega_{2,i}}(X)\right\}\right) &\asymp \frac{6A(\Omega_{2,i})X^2}{\pi^2} \\ &= \frac{6A(\triangle_i)X^2}{\pi^2} + O_f\left(\frac{X^2}{L^2}\right). \end{aligned}$$

Therefore

$$S_{\Omega_{2,i},\underline{0}}(X) \asymp S_{\triangle_i,\underline{0}}(X) + O_f\left(\frac{X^2}{L^2}\right). \tag{5}$$

Fix i and for the triangle $\triangle_i$, let $A_i' = (x_i', y_i)$, $A_i = (x_i, y_i)$, $X_i = Xy_i$ and $\mathbf{I}_i = \left[\frac{x_i'}{y_i}, \frac{x_i}{y_i}\right] \subset [0,1]$. Suppose the visible points of $X\triangle_i$ are $P_j = (a_j, b_j)$, $1 \le j \le N = N(\triangle_i, X)$, where the rays $\overrightarrow{OP_N}, \overrightarrow{OP_{N-1}}, \ldots, \overrightarrow{OP_1}$ are distributed counterclockwise around the origin. We see that $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_N}{b_N}$ are consecutive Farey fractions of order X_i inside the interval $\mathbf{I}_i$. By Lemma 3, for any $\epsilon > 0$ and $X \to \infty$,

$$\begin{aligned} S_{\triangle_i,\underline{0}}(X_i) &= \#\left\{P \in \mathcal{A}_{\triangle_i}(X) : \underline{v}(P) = \underline{0}\right\} \\ &= \#\left\{(a_j,b_j) \in \mathcal{A}_{\triangle_i}(X) : s(a_{j+h}, b_{j+h}) \le \cdots \le s(a_j,b_j)\right\} \\ &= \#\left\{\frac{a_j}{b_j} \in \mathscr{F}_{X_i} \bigcap \mathbf{I}_i : s(a_j,b_j) \ge \cdots \ge s(a_{j+h}, b_{j+h})\right\} \\ &= \frac{6|\mathbf{I}_i|X_i^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) + O_\epsilon\left(X_i^{2-\frac{1}{h+1}+\epsilon}\right) \\ &= \frac{6|\mathbf{I}_i|X^2}{\pi^2} A(\mathscr{T}_{\underline{0}}) y_i^2 + O_{f,\epsilon}\left(X^{2-\frac{1}{h+1}+\epsilon}\right). \end{aligned}$$

By (5) and the above we have

$$\begin{aligned} S_{\Omega_2,\underline{0}}(X) &= \sum_{i=0}^{L} S_{\Omega_{2,i},\underline{0}}(X) + O(L) \\ &\asymp \sum_{i=0}^{L}\left(S_{\triangle_i,\underline{0}}(X) + O_f\left(\frac{X^2}{L^2}\right)\right) + O(L) \\ &= \frac{6A(\mathscr{T}_{\underline{0}})X^2}{\pi^2}\sum_{i=0}^{L}|\mathbf{I}_i|y_i^2 + O_{f,\epsilon}\left(LX^{2-\frac{1}{h+1}+\epsilon}\right) + O_f\left(\frac{X^2}{L}\right) + O(L). \end{aligned}$$

One sees that

$$\begin{aligned}\sum_{i=0}^{L}|\mathbf{I}_i|y_i^2 &= \sum_{i=0}^{L}\left(\cot(\alpha_i)-\cot(\alpha_{i+1})\right)f(\alpha_i)^2\sin(\alpha_i)^2\\ &= \sum_{i=0}^{L}\frac{1}{\sin(\xi_i)^2}f(\alpha_i)^2\sin(\alpha_i)^2 \quad (\exists\ \xi_i\in[\alpha_i,\alpha_{i+1}])\\ &= \int_{\theta_1}^{\theta_2}f^2(\theta)\,\mathrm{d}\theta+O_f\left(\frac{1}{L}\right)=2A(\Omega_2)+O_f\left(\frac{1}{L}\right).\end{aligned}$$

Therefore

$$S_{\Omega_2,\underline{0}}(X) \asymp \frac{12A(\mathscr{T}_{\underline{0}})A(\Omega_2)X^2}{\pi^2}+O_\epsilon\left(X^{2-\frac{1}{h+1}+\epsilon}\right)+O_f\left(\frac{X^2}{L}\right)+O(L).$$

Choosing $0<\epsilon<\frac{1}{h+1}$, we may let $X\to\infty$ and then $L\to\infty$ to obtain that

$$\lim_{X\to\infty}\frac{S_{\Omega_2,\underline{0}}(X)}{\#\left(\mathcal{A}_\Omega(X)\right)}=2\cdot\frac{A(\mathscr{T}_{\underline{0}})A(\Omega_2)}{A(\Omega)}. \tag{6}$$

Since continuous functions can be approximated by C^∞ functions uniformly inside a closed interval, by a standard approximation procedure we see that (6) also holds true if f is only continuous.

We treat Ω_1 similarly with a slight difference. We will sketch the proof. Fix a large integer $L>0$ and let $\alpha_i=\frac{\pi}{4L}$, the rays $\{(r,\theta):\theta=\alpha_i\}$ intercept Ω_1 at points A_i's. At each point A_{i+1}, draw a vertical line which intercept the ray $\{(r,\theta):\theta=\alpha_i\}$ at the point A_i'. We use the triangle $\triangle_{OA_iA_i'}$ to estimate the region of Ω_1 inside the rays $\{(r,\theta):\theta=\alpha_i\}$ and $\{(r,\theta):\theta=\alpha_{i+1}\}$. Following exactly the same argument and applying Lemma 4, we can obtain that

$$\lim_{X\to\infty}\frac{S_{\Omega_1,\underline{0}}(X)}{\#\left(\mathcal{A}_\Omega(X)\right)}=2\cdot\frac{A(\mathscr{T}_{\underline{0}})A(\Omega_1)}{A(\Omega)}.$$

Next for $\Omega_i,\ 3\le i\le 8$, notice that

$$s(-a,b)=-s(a,b),\quad s(a,-b)=s(a,b),$$

the computation of asymptotic formulas in these regions can be reduced to that in regions Ω_1 and Ω_2, and we have

$$\begin{aligned}\lim_{X\to\infty}\frac{S_{\Omega_i,\underline{0}}(X)}{\#\left(\mathcal{A}_\Omega(X)\right)} &= 2\cdot\frac{A(\mathscr{T}_{\underline{0}})A(\Omega_i)}{A(\Omega)},\ 1\le i\le 4,\\ \lim_{X\to\infty}\frac{S_{\Omega_i,\underline{0}}(X)}{\#\left(\mathcal{A}_\Omega(X)\right)} &= 2\cdot\frac{A(\mathscr{T}_{\underline{1}})A(\Omega_i)}{A(\Omega)},\ 5\le i\le 8.\end{aligned}$$

Then finally (4) gives one that

$$\lim_{X\to\infty}\frac{S_{\Omega,\underline{0}}(X)}{\#\left(\mathcal{A}_\Omega(X)\right)} = 2\cdot\frac{A(\mathscr{T}_{\underline{0}})A(\Omega_{y\ge 0})+A(\mathscr{T}_{\underline{1}})A(\Omega_{y\le 0})}{A(\Omega)}.$$

This completes the proof of Theorem 1 for the special case $\underline{v}=\underline{0}$. The other cases can be treated in a similar fashion. $\square$

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