

ON THE SEMISTABILITY OF THE MINIMAL POSITIVE STEADY STATE FOR A NONHOMOGENEOUS SEMILINEAR CAUCHY PROBLEM

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ABSTRACT. This paper is contributed to the study of the Cauchy problem

$$\begin{cases} u_t = \Delta u + K(|x|)u^p + \mu f(|x|) & \text{in } R^n \times (0, T), \quad u(x, 0) = \varphi(x) & \text{in } R^n, \\ u(x, 0) = \varphi(x) & \text{in } R^n, \end{cases}$$

with non-negative initial function $\varphi \not\equiv 0$. We will study the asymptotic behavior and the semistability of the minimal positive steady state. In addition, we will prove that all slow decay positive steady states are stable and weakly asymptotically stable in some weighted L^∞ norms.

1. Introduction

In this paper, we will consider the asymptotic behavior and the stability of the positive radial solutions of the following equation

$$(1.1) \quad \Delta u + K(|x|)u^p + \mu f(|x|) = 0,$$

which are positive steady states of the following Cauchy problem:

$$(1.2) \quad \begin{cases} u_t = \Delta u + K(|x|)u^p + \mu f(|x|) & \text{in } R^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } R^n, \end{cases}$$

where $p > 1, x \in R^n, n \geq 3, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the n -dimensional Laplacian, $T > 0, \mu$ is some positive constant, $0 \leq f \in C^1(R^n \setminus \{0\})$, $K(x)$ is a given locally Hölder continuous function in $R^n \setminus \{0\}$, and $\varphi \not\equiv 0$ is a bounded non-negative continuous function in R^n , the unique solution of (1.2) is denote by $u(x, t, \varphi)$.

For the physical reasons, we consider the positive radial solutions of (1.1), when $K(x) = K(r), f(x) = f(r)$, where $r = |x|$. Then the equation (1.1) reduces to

$$(1.3) \quad u'' + \frac{n-1}{r}u' + K(x)u^p + \mu f(r) = 0 \quad r > 0.$$

For the same reasons, the regular solutions that have finite limits at $r = 0$, are particularly interesting, which lead us to consider the initial value problem

$$(1.4) \quad \begin{cases} u'' + \frac{n-1}{r}u' + K(x)u^p + \mu f(|x|) = 0, \\ u(0) = \alpha > 0, \end{cases}$$

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and we use $u_a = u(r, \alpha)$ to denote the solution of (1.4).

The hypotheses of $K(x)$ are often divided into two cases: the fast decay case and the slow decay case. In this paper, we will focus on the slow decay case, i.e. $K(r) \geq cr^l$, for some $l > -2$ and r large. First, let us introduce a collection of hypotheses on $K(x)$ and f :

(K.1) $K(x) = k_\infty|x|^l + O(|x|^{-d})$ at $|x| \rightarrow \infty$ for some constants $k_\infty > 0$ and $d > n - \lambda_2 - m(p + 1)$, where λ_2 is defined by (1.7) below.

(K.2) $K(x) = O(|x|^\tau)$ at $|x| = 0$ for some $\tau > -2$.

(K.3) $K(r)$ is locally Lipschitz continuous and $\frac{d}{dr}(r^{-l}K(r)) \leq 0$ for a.e. $r > 0$.

(f.1) $f(x) = O(|x|^{\tau_1})$ at $|x| \rightarrow 0$ for some $\tau_1 > -2$.

(f.2) $f(x) = O(|x|^{-q})$ near $|x| = \infty$, for some $q > n - m - \lambda_2$.

Also, we introduce the following notations, which will be used throughout this paper:

$$\begin{aligned}
 m &\equiv \frac{l+2}{p-1}, & b_0 &\equiv n-2-2m, \\
 L &\equiv [m(n-2-m)]^{\frac{1}{p-1}}, & c_0 &\equiv (p-1)L^{p-1}, \\
 (1.5) \quad p_c &= \begin{cases} \frac{(n-2)^2-2(l+2)(n+l)+2(l+2)\sqrt{(n+l)^2-(n-2)^2}}{(n-2)(n-10-4l)} & n > 10+4l, \\ \infty & 3 \leq n \leq 10+4l. \end{cases}
 \end{aligned}$$

Note that when $l = 0$ we have

$$p_c = \begin{cases} \frac{(n-2)^2-4n+4\sqrt{n^2-(n-2)^2}}{(n-2)(n-10)} & n > 10, \\ \infty & 3 \leq n \leq 10, \end{cases}$$

which was first introduced in [15]. Also note that we have $m > 0$ and $b_0 > 0$ when $p > \frac{n+2l+2}{n-2}$ and $l > -2$.

Consider the equation

$$(1.6) \quad \lambda^2 + b_0\lambda + c_0 = 0,$$

here b_0 and c_0 are as in (1.5). When $p > p_c$, (1.6) has two negative roots $-\lambda_2 < -\lambda_1 < 0$ and $b_0 > \lambda_2$,

$$\begin{aligned}
 (1.7) \quad \lambda_1 &= \lambda_1(n, p, l) = \frac{(n-2-2m) - \sqrt{(n-2-2m)^2 - 4(l+2)(n-2-m)}}{2}; \\
 \lambda_2 &= \lambda_2(n, p, l) = \frac{(n-2-2m) + \sqrt{(n-2-2m)^2 - 4(l+2)(n-2-m)}}{2}.
 \end{aligned}$$

While when $p = p_c$, (1.6) has two equal negative roots $-\lambda_2 = -\lambda_1 = -\frac{b_0}{2} < 0$.

There are many results about the existence and nonexistence of the positive solutions for problem (1.4). For the homogeneous case (i.e. $f \equiv 0$) Ni and Yotsutani showed that (1.4) has one solution $u(r)$ for every $\alpha > 0$ in [23]; Gui in [11] and Liu, Li, Deng, in [21] obtained some existence results. For the nonhomogeneous case, when $K(x) \equiv 1$, Bernard obtained the existence result for $0 \leq f \leq \frac{p-1}{[p(1+|x|^2)]^{p/p-1}}L^p$ in [6]; Bae and Ni obtained the nonexistence result (see Theorem D below) and the

infinite multiplicity result (see [2, Theorem 2]). For the general case, Bae, Chang and Pakh obtained the infinitely many positive solutions for problem (1.4) (see Theorem C below).

In order to state the results concerning the asymptotic behavior and the stability of the positive radial solutions, we need to clarify a few terms. A positive solution $u(r)$ of (1.3) in $(0, \infty)$ is said to have slow decay if

$$u(r) = Ar^{-\frac{2+l}{p-1}} + o(r^{-\frac{2+l}{p-1}}) \quad \text{as } r \rightarrow +\infty,$$

for some positive constant A. On the other hand, $u(r)$ is said to have fast decay if

$$u(r) = O(r^{2-n}) \quad \text{as } r \rightarrow +\infty.$$

And a solution $u(r)$ is said to be a regular solution of (1.3) if it is finite up to $r = 0$. We call $u(r)$ a radial singular ground state if instead $u(r) \rightarrow +\infty$ as $r \rightarrow 0^+$. Definitions of some weighted L^∞ norms are given as follows (adopted from [12,13]): for $\lambda > 0, \mu_1 > 0$, let

$$(1.8) \quad \|\psi\|_\lambda = \sup_{x \in R^n} |(1 + |x|)^\lambda \psi(x)|,$$

and

$$(1.9) \quad \|\|\psi\|\|_{\mu_1} = \sup_{x \in R^n} \left| \frac{(1 + |x|)^{m+\lambda_1}}{\log(2 + |x|)^{\mu_1}} \psi(x) \right|,$$

where m, λ_1 are defined in (1.5) and (1.7) respectively and ψ is a continuous function in R^n .

We say that a steady state u_α of (1.2) is stable with respect to some norm $\|\cdot\|_\lambda$ if for every $\varepsilon > 0$ there exist $\delta > 0$ such that, for $\|\varphi - u_\alpha\|_\lambda < \delta$, we have $\|u(\cdot, t; \varphi) - u_\alpha\|_\lambda < \varepsilon$ for all $t > 0$; in addition u_α is said to be weakly asymptotically stable with respect to norm $\|\cdot\|_\lambda$ if u_α is stable with respect to norm $\|\cdot\|_\lambda$ and there exists $\delta > 0$ such that, for $\|\varphi - u_\alpha\|_\lambda < \delta$, we have $\|u(\cdot, t; \varphi) - u_\alpha\|_{\lambda'} \rightarrow 0$ as $t \rightarrow \infty$ for all $\lambda' < \lambda$. Similarly we can define the stability with respect to the norm $\|\|\cdot\|\|_{\mu_1}$, and we say that the u_α is weak asymptotic stability with respect to norm $\|\|\cdot\|\|_{\mu_1}$, if the u_α is stable with respect to the norm $\|\|\cdot\|\|_{\mu_1}$ and there exists $\delta > 0$ such that, for $\|\|\varphi - u_\alpha\|\|_{\mu_1} < \delta$, we have $\|\|u(\cdot, t; \varphi) - u_\alpha\|\|_{\mu'_1} \rightarrow 0$ as $t \rightarrow \infty$ for all $\mu'_1 > \mu_1$.

The main result of Deng, Li and Yang in [8] can be stated in the following theorem.

Theorem A^[8]. Suppose that $K(r)$ satisfies (K.1) – (K.3), f satisfies (f.1) and (f.2), let $A = \{\alpha > 0, u(r, \alpha)$ is a positive solution of (1.4) for all $r > 0\}$ and $S = \{\alpha > 0, u(r, \alpha)$ is a positive solution of (1.4) for all $r > 0$ and is of slow decay}. Define $\alpha_* = \alpha(K, \mu) \equiv \inf\{\alpha \in A\}$, $\alpha_{**} = \inf\{\alpha \in S\}$, then $0 < \alpha_* \leq \alpha_{**}$ and

(i) if $p \geq p_c$, then there exists $\mu_* > 0$ such that for every $\mu \in [0, \mu_*)$, $\alpha_{**} < \infty$, and $A = [\alpha_*, \infty), S = (\alpha_{**}, \infty)$ and $u_\alpha(r)$ and $u_\beta(r)$ can not intersect each other for any $\alpha_* \leq \alpha < \beta$, i.e. $0 < u_\alpha < u_\beta$;

(ii) if $\frac{n+2+2l}{n-2} < p < p_c$ and u_α, u_β are slow decay solutions of (1.4), then they will intersect infinity many times.

Remark 1.1. The equation (1.4) has the minimum positive solution u_{α_*} , i.e. if $u(0) < \alpha_*$, then $u(r)$ has some finite zero.

Remark 1.2. The solution of (1.4) can have three decay cases:

- (i) the solution u is slow decay, i.e. $u(r) \sim r^{-m}$;
- (ii) the solution u is fast decay, i.e. $u(r) \sim r^{2-n}$;
- (iii) the solution u 's decay rate may be between the slow decay rate and fast decay rate.

Remark 1.3. From Theorem A we know that the minimum solution u_{α_*} is fast decay or it's decay may be between the slow decay rate and fast decay rate. We will prove $u_{\alpha_*} \sim r^{2-n}$ in a certain case in this paper.

Theorem B^[8]. Suppose that $p > p_c$, K satisfies (K.1) – (K.3), f satisfies (f.1) and (f.2). Then any slow decay positive steady state u_α of (1.2) is:

- (i) stable with respect to the norm $\|\cdot\|_{m+\lambda_1}$;
- (ii) weakly asymptotically stable with respect to the norm $\|\cdot\|_{m+\lambda_2}$.

Theorem C^[4]. Let $p > p_c$, assume that $K(x)$ satisfies (K.1), (K.2), f satisfies (f.1), (f.2) and (f.3) : $-(1 + |x|^{mp})f(x) \leq \min_{|z|=|x|} K(z)$. Then there exists $\mu_* > 0$ such that for every $\mu \in [0, \mu_*)$, Eq (1.4) possesses infinitely many positive entire solutions with asymptotic behavior $\frac{L}{k_\infty^{\frac{p-1}{p-1}}}|x|^{-m}$ at ∞ .

Some of the early results for $K \equiv 1$ are as follow:

Theorem D^[5]. (i) Let $p > p_c$. Suppose that near ∞

$$\max(\pm f(x), 0) \leq |x|^{-q_\pm},$$

where $q_+ > n - \lambda_2$ and $q_- > n - \lambda_2 - m$. Then, there exists $\mu_* > 0$ such that for every $\mu \in (0, \mu_*)$, equation (1.1) possesses infinitely many solutions with the asymptotic behavior $L|x|^{-m}$ at ∞ .

(ii) Let $p = p_c$. Then, the conclusions in (i) holds if we assume in addition that either f has a compact support in R^n or f dose not change sign in R^n .

The main purpose of this paper is to study the asymptotic behavior and the semistability of the minimal positive steady solution of equation (1.4). In addition, motivated by the work of Gui, Ni and Wang's results [12, 13] and Deng, Li, Yang's results [8], we will prove the stability of slow decay steady states in some weighted L^∞ norms. We will also show that the slow decay steady states are unstable, if the topology is too fine or too coarse. Our main results are as follow:

Theorem 1. Let $u = u(r)$ be the solution of (1.4), with that $K(r)$ satisfies (K.1), (K.2), f satisfies (f.1), (f.2). If u is positive and $u(r) = o(r^{-m})$ at $r = \infty$,

then we have, at $r = \infty$

$$u(r) = \begin{cases} O(r^{2-n}) & \text{if } q > n, & O(r^{2-n} \log r) \\ O(r^{2-n} \log r) & \text{if } q = n, & O(r^{2-q}) \\ O(r^{2-q}) & \text{if } m + 2 < q < n. \end{cases}$$

Remark 1.4. (i) For homogenous case, Li and Ni in [20] and Li in [19] systematically investigated the asymptotic behavior of the positive steady states.

(ii) From this Theorem, we know that the minimum positive steady state solution is of fast decay (i.e. $u_{\alpha_*} \sim r^{2-n}$ at ∞) if the decay of f is fast enough.

Theorem 2. Suppose that $K(r)$ satisfies (K.1 – K.3), f satisfies (f.1) and (f.2) and $p \geq p_c$, $q > n$, and $\alpha_* = \alpha_{**}$. Then, the minimum steady state u_{α_*} of (1.2) is semistable with respect to the norms $\|\cdot\|_{\mu_1}$ (i.e. if $u_{\alpha_*} \leq \varphi$), if $m < \mu_1 < n - 2$. In this case u_{α_*} is also weakly asymptotically semistable respect to the norms $\|\cdot\|_{\mu_1}$.

The following theorem is a extension of homogeneous equations to nonhomogeneous.

Theorem 3. Suppose that $p_c > p \geq \frac{n+2+2l}{n-2}$ $\alpha_* = \alpha_{**}$ and $K(r)$ satisfies (K.1 – K.3), f satisfies (f.1) and (f.2). In addition, assume that $A = [\alpha_*, \infty)$, $S = (\alpha_*, \infty)$. Then the following conclusions hold:

(i) If $0 < \varphi(x) \leq u_\alpha$ and $\varphi(x) \not\equiv u_\alpha$, for some $\alpha > \alpha_*$. then $\lim_{t \rightarrow \infty} u(x, t, \varphi) \rightarrow u_{\alpha_*}$.

(ii) If $\varphi(x) \geq u_\alpha$ and $\varphi(x) \not\equiv u_\alpha$, for some $\alpha > \alpha_*$. Then the solution $u(x, t, \varphi)$ must blow up in some finite time.

Theorem 4. Suppose that $K(r)$ satisfies (K.1) – (K.3) f satisfies (f.1), (f.2). In addition, $\alpha_* = \alpha_{**}$ and $\gamma > \lambda_2$ such that $\frac{L}{k_\infty^{\frac{p-1}{p-1}}}(r^{-l}K(r) - k_\infty) + f(r)r^{2+m} = O(\frac{1}{r^\gamma})$ at $r = \infty$. Then we have:

(1) Suppose that $p = p_c$ then any slow decay positive steady state u_α of (1.2) is weakly asymptotically stable with respect to the norm $\|\cdot\|_{\nu_1}$, when $0 < \nu_1 < 1$; unstable when $\nu_1 > 1$.

(2) Suppose that $p > p_c$, then any slow decay positive steady state u_α of (1.2) is weakly asymptotically stable with respect to the norm $\|\cdot\|_\lambda$, when $m + \lambda_1 < \lambda < m + \lambda_2$, unstable when $0 < \lambda < m + \lambda_1$.

Remark 1.5. It should be mentioned that the Theorem 2, the Theorem 3 and Theorem 4 are inspired by the work of Gui, Ni and Wang [12,13].

Remark 1.6. The following were proved by Deng, Li and Yang in [8]: u_α is stable when $\lambda = m + \lambda_1$ and weakly asymptotically stable when $\lambda = m + \lambda_2$.

For the stability and instability of the positive radial steady states with $f = 0$, it seems that the first general result is given by Fujita [9]. It is showed there that for $1 < p < \frac{n+2}{n}$, $u(x, t; \varphi)$ blows up in finite time for any $\varphi \geq 0, \varphi \not\equiv 0$. Thus the trivial steady state $u_0 \equiv 0$ is unstable in any topology for $1 < p < \frac{n+2}{n}$ (the same is true when $p = \frac{n+2}{n}$, as was proved by Hayakawa [14] and later by Kobayashi, Siaro, and Tanaka [16]). In the case of $p > \frac{n+2}{n}$, for $K(x) \equiv 1$ and $f(x) \equiv 0$, for the global existence of $u(x, t, \varphi)$, the condition given by Fujita is that φ is bounded by $\varepsilon e^{-|x|^2}$ for some small ε ; Weissler [25] studied the problem in L^p -space and the condition there on φ can be interpreted as to that φ is bounded by $\varepsilon(1 + |x|)^{-\gamma}$ for some constant $\gamma > \frac{2}{p-1}$ and ε small enough; Lee and Ni in [17] gave a sharp condition that φ has decay rate of $C|x|^{-\frac{2}{p-1}}$ at ∞ , where C is a positive constant; in 1992, Gui, Ni and Wang [12] prove that every positive radial steady state solution is unstable in any reasonable sense if $p < p_c$; and is stable in some weighted L^∞ if $p \geq p_c$. Further systematic study of the stability of positive steady state is given by Gui, Ni and Wang in [13]. Bae in [1] uses the two weights, $(\log r)^{\frac{p}{p-1}}$ and $r^{n-2}(\log r)^{-\frac{p}{p-1}}$, to show the stability of the steady state in case $l = 2$ recently. For the nonhomogeneous case, the stability of positive steady state of slow decay is obtained by Deng, Li, Yang in [8] (see Theorem B).

This paper is organized as follows. We introduce some Preliminaries in Section 2. The asymptotic of the minimum solution of equation (1.4) (i.e. Theorem 1) is given in Section 3 and the proofs of the Theorem 2 and Theorem 3 are given in Section 4. In Section 5, we give the proof of Theorem 4.

2. Preliminaries

Definition 2.1. A function u is said to be a super-solution of equation

$$\Delta u + f(x, u) = 0$$

in an open set $\Omega \subset R^n$ if $\Delta u + f(x, u) \leq 0$ in Ω ; and u is said to be a sub-solution if $\Delta u + f(x, u) \geq 0$ in Ω .

Adopting the definition by Wang [24], we have:

Definition 2.2. A function u is a continuous weak super-solution of

$$\begin{cases} u_t = \Delta u + f(x, t, u) & \text{in } R^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } R^n, \end{cases}$$

if

(i) u is continuous on $\Omega_T = R^n \times [0, T)$ and $u(\cdot, 0) \geq \varphi$;

(ii) u satisfies

$$(2.1) \quad \int_{R^n} u(x, t)\eta(x, t)dx|_0^{T'} \geq \int_0^{T'} \int_{R^n} [u(x, s)(\Delta\eta + \eta_t) + \eta(x, t)f(x, t, u(x, t))]dxdt$$

for all $T' \in [0, T)$ and $0 \leq \eta(x, t) \in C^{2,1}(R^n \times [0, T'])$ with $\text{supp}(\eta(\cdot, t))$ being compact in R^n for $t \in [0, T']$. Similarly, a continuous weak sub-solution is defined by reversing the inequalities in (i) and (2.1).

Lemma 2.1. Suppose that u is a positive solution of (1.3). Let $r = e^t, t \in (-\infty, +\infty)$ and $v(t) = r^j u(r)$, then v satisfies

$$v'' + (n - 2 - 2j)v' - j(n - 2 - j)v + K(e^t)e^{(j+2-pj)v^p} + \mu f(e^t)e^{(j+2)t} = 0$$

Let $j = m$, then we have that

$$(2.2) \quad v'' + b_0v' - L^{p-1}v + k(t)v^p + \mu f(e^t)e^{(m+2)t} = 0,$$

where $k(t) = e^{-lt}K(e^t)$, and m, b_0 , and L are as in (1.5).

This lemma can be proved by straight forward calculations, thus we omit it here.

Now we quote some results on the asymptotic behavior of solutions to the (1.1) (see [8]).

Proposition 2.1. (i) If $\gamma > \lambda_2$, $\frac{L}{k_\infty^{\frac{1}{p-1}}}(r^{-l}K(r) - k_\infty) + f(r)r^{2+m} = O(\frac{1}{r^\gamma})$ at $r = \infty$, and u is a solution of (1.4), which is slow decay, then we have

$$(2.3) \quad u(r) = \begin{cases} \frac{L}{k_\infty^{\frac{1}{p-1}}r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{b_1}{r^{m+\lambda_2}} + \dots + O(\frac{1}{r^{n-2+\varepsilon}}) \\ \quad \text{if } \lambda_2 \neq \Lambda\lambda_1 \\ \frac{L}{k_\infty^{\frac{1}{p-1}}r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{c_1 \log r}{r^{m+\Lambda\lambda_1}} + \frac{b_1}{r^{m+\lambda_2}} + \dots + O(\frac{1}{r^{n-2+\varepsilon}}) \\ \quad \text{if } \lambda_2 = \Lambda\lambda_1 \end{cases}$$

for some positive integer $\Lambda > 1$, where a_i, b_j and c_1 are similar to (3.18) of [7].

Proposition 2.2. Suppose that K satisfies (K.1) and (K.3) in (R, ∞) for some large R , and f satisfies (f.1). Then

(i) if \bar{u} and \underline{u} are bounded continuous weak super-and sub-solutions of (1.2), respectively, and $\bar{u} \geq \underline{u}$ on $R^n \times (0, T)$, then (1.2) has a unique solution u satisfies $\bar{u} \geq u(x, t; \varphi) \geq \underline{u}$ and $u \in C^{2,1}((R^n \setminus 0) \times (0, T))$ if $-2 < l < 0$, $u \in C^{2,1}(R^n \times (0, T))$ if $l > 0$;

(ii) If $\varphi(x)$ is a bounded continuous weak super-solution (sub-solution) but not a solution of (1.1) in R^n , then the solution of (1.2) is strictly decreasing (increasing, respectively) in $t > 0$ as long as it exists;

(iii) If φ is radial and radially decreasing, so is u in x -variable.

All the results of Proposition 2.2 can be proved by the techniques used in [24] with replacing $|x|^l u^p$ by $K(x)u^p + \mu f(x)$. For example, part (i) is similar to Lemma 1.2 of [24] if $l > 0$, Theorem 2.4 (i) of [24] if $-2 < l < 0$; part (ii) can be proved by the same argument as in Theorem 2.4 (ii) of [24] if $-2 < l < 0$, or Lemma 2.6 (ii) of [24] and the strong maximum principle if $l \geq 0$; part (iii) can be proved similarly by Theorem 2.3 of [24] if $-2 < l < 0$, Lemma 2.6 of [24] if $l \geq 0$.

The following results are well-known, so its proof is omitted here (see [24]).

Proposition 2.3 (i) Suppose $\bar{u}_1(|x|)$ and $\bar{u}_2(|x|)$ are super-solutions to (1.1) in $B_{R_1} := \{|x| < R_1\}$ and $B_{R_2}^c := \{|x| \geq R_2\}$, respectively. Assume that $R_1 > R_2$ and $\bar{u}_1(R_1) > \bar{u}_2(R_1)$, $\bar{u}_1(R_2) < \bar{u}_2(R_2)$. Let $R = \min\{r \in (R_2, R_1) \mid \bar{u}_1(r) \geq \bar{u}_2(r)\}$, and

$$(2.4) \quad \bar{u}(|x|) = \begin{cases} \bar{u}_1(|x|) & 0 \leq |x| \leq R, \\ \bar{u}_2(|x|) & |x| > R. \end{cases}$$

Then $\bar{u}(|x|)$ is a continuous weak super-solution to (1.1) in R^n .

(ii) Suppose $\underline{u}_1(|x|)$ and $\underline{u}_2(|x|)$ are sub-solutions to (1.1) in B_{R_1} and $B_{R_2}^c$, respectively. Assume that $R_1 > R_2$ and $\underline{u}_1(R_1) < \underline{u}_2(R_1)$, $\underline{u}_1(R_2) > \underline{u}_2(R_2)$. Let $R = \min\{r \in (R_2, R_1) \mid \underline{u}_1(r) \leq \underline{u}_2(r)\}$, and

$$(2.5) \quad \underline{u}(|x|) = \begin{cases} \underline{u}_1(|x|) & 0 \leq |x| \leq R, \\ \underline{u}_2(|x|) & |x| > R. \end{cases}$$

Then $\underline{u}(|x|)$ is a continuous weak sub-solution to (1.1) in R^n .

3. Proof of Theorem 1

In order to prove Theorem 1, we now give the following lemma, which is inspired by the work of Y-Li and W.-M. Ni [20].

Lemma 3.1. Suppose that $K(r) = O(r^l)$ at ∞ for some $l > -2$, f satisfies (f.1), (f.2), and $q > m + 2$ and u is a solution of (1.1) which is positive in $(0, \infty)$ with $u(r) = o(r^{-m})$ at ∞ , then $u(r) = O(r^{-m-\delta})$ at $r = \infty$ for some $\delta > 0$.

Proof. Set $v(r) = r^m u(r)$ for $r > 0$, then $v(r) \rightarrow 0$ as $r \rightarrow \infty$ and v satisfies the following equation

$$\Delta v - \frac{2m}{r} v' - \frac{m(n-2-m)}{r^2} v + K(r)r^{-l} \frac{v^p}{r^2} + \mu r^m f = 0. \tag{3.1}$$

Since $v(r) \rightarrow 0$ as $r \rightarrow \infty$, we have for any $\varepsilon > 0$

$$\Delta v - \frac{2m}{r} v' - \frac{m(n-2-m)}{r^2} v + \mu r^m f + m\varepsilon \frac{v}{r^2} \geq 0 \quad \text{at } \infty. \tag{3.2}$$

Defining:

$$L_\varepsilon v \equiv \Delta v - \frac{2mv'}{r} - m(n-2-m-\varepsilon) \frac{v}{r^2} + \mu r^m f,$$

immediately by (3.1) we have

$$L_\varepsilon v - m\varepsilon \frac{v}{r^2} + K(r)r^{-l} \frac{v^p}{r^2} = 0.$$

By the (3.2), there exists an $R_\varepsilon > 0$ such that $L_\varepsilon v \geq 0$ in $R^n \setminus B_{R_\varepsilon}(0)$.

On the other hand, for $0 < \varepsilon < n - 2 - m$, let $\varphi_\varepsilon(x) = |x|^{\beta_\varepsilon}$ we have

$$L_\varepsilon \varphi_\varepsilon = [\beta(\beta - 1) + (n - 1 - 2m)\beta - m(n - 2 - m - \varepsilon)]|x|^{\beta_\varepsilon - 2} + \mu r^m f$$

in $R^n \setminus 0$. Choosing $\beta_\varepsilon < 0$ sufficiently small such that,

$$\beta_\varepsilon(\beta_\varepsilon - 1) + (n - 1 - 2m)\beta_\varepsilon - m(n - 2 - m - \varepsilon) \leq 0,$$

and

$$\frac{r^m f}{r_\varepsilon^{\beta_\varepsilon} - 2} \rightarrow 0 \quad \text{at} \quad \infty.$$

So there exists an $R'_\varepsilon > 0$ such that

$$L_\varepsilon \varphi_\varepsilon \leq 0 \quad \text{in} \quad R^n \setminus B_{R'_\varepsilon}(0).$$

Setting $R''_\varepsilon = \max\{R'_\varepsilon, R_\varepsilon\}$, $C_\varepsilon = v(R''_\varepsilon)(R''_\varepsilon)^{-\beta_\varepsilon}$, we see that

$$\begin{cases} L_\varepsilon(v - C_\varepsilon \varphi_\varepsilon) \geq 0 & \text{in} \quad R^n \setminus B_{R''_\varepsilon}(0), \quad v - C_\varepsilon \varphi_\varepsilon = 0 \\ v - C_\varepsilon \varphi_\varepsilon = 0 & \text{on} \quad \partial B_{R''_\varepsilon}(0), \quad v - C_\varepsilon \varphi_\varepsilon \rightarrow 0 \\ v - C_\varepsilon \varphi_\varepsilon \rightarrow 0 & \text{at} \quad \infty, \end{cases}$$

since $\beta_\varepsilon < 0$. Observing that the coefficient of the term v in L_ε is negative, we conclude by the maximum principle that $v - C_\varepsilon \varphi_\varepsilon \leq 0$ in $R^n \setminus B_{R''_\varepsilon}(0)$, i.e. $v(r) \leq C_\varepsilon r^{\beta_\varepsilon}$ at ∞ . This guarantees that $u(r) \leq C_\varepsilon r^{-m+\beta_\varepsilon}$ at ∞ , and our proof is completed. \square

The proof of Theorem 1. From (1.3) we have, by integration from 0 to r ,

$$u_r(r) + \frac{1}{r^{n-1}} \int_0^r (K(s)u^p + \mu f)s^{n-1} ds = 0.$$

Now integrating from r to ∞ , we obtain

$$u(r) = \int_r^\infty \frac{1}{t^{n-1}} \left[\int_0^t (K(s)u^p + \mu f)s^{n-1} ds \right] dt,$$

since $u(\infty) = 0$ by our assumption on u . Changing the order of the integrations, we have that there exists $R > 0$, for $r \geq R$,

$$\begin{aligned} u(r) &= \frac{1}{n-2} r^{2-n} \int_0^r (K(s)u^p + \mu f)s^{n-1} ds + \frac{1}{n-2} r^{2-n} \int_r^\infty (K(s)u^p + \mu f)s ds \\ &\leq C[r^{2-n} + r^{2-n} \int_R^r u^p s^{l+n-1} ds + r^{2-n} \int_R^r s^{n-1} f ds + \int_r^\infty (u^p s^{l+1} + s f) ds]. \end{aligned}$$

Let

$$u_1 = r^{2-n} \int_0^r f s^{n-1} ds + \int_r^\infty f s ds.$$

By the Lemma 3.1, we obtain that for some $\varepsilon > 0$

$$u(r) \leq C[r^{2-n} + r^{2-n} \int_R^r s^{-p(m+\varepsilon)+l+n-1} ds + \int_r^\infty s^{-p(m+\varepsilon)+l+1} ds + u_1(r)]$$

By the similar computation, we have, at $r = \infty$

$$u_1(r) \leq \begin{cases} cr^{2-n} & \text{if } q > n, \\ cr^{2-n} \log r & \text{if } q = n, \\ cr^{2-q} & \text{if } m + 2 < q < n. \end{cases}$$

So if $q > n$, we have

$$\begin{aligned} u &\leq C[r^{2-n} + r^{2-n} \int_1^r s^{-p(m+\varepsilon)+l+n-1} ds + \int_r^\infty s^{-p(m+\varepsilon)+l+1} ds] \\ &\leq \begin{cases} c[r^{2-n} + r^{-m-p\varepsilon}] & \text{if } m + p\varepsilon \neq n - 2, \\ c[r^{2-n} + r^{2-n} \log r] & \text{if } m + p\varepsilon = n - 2, \end{cases} \end{aligned}$$

since $-p(m + \varepsilon) + l + 2 = -m - p\varepsilon$. If $m + p\varepsilon > n - 2$, we have

$$u(r) \leq cr^{-(n-2)} \quad \text{at } \infty.$$

Otherwise, we repeat the arguments above and for some $\rho > 0$, we have

$$u(r) \leq \begin{cases} c[r^{2-n} + r^{-m-p^2\varepsilon}] & \text{if } m + p\varepsilon < n - 2, \\ c[r^{2-n} + r^{-[m+p(n-2-m-\rho)]}] & \text{if } m + p\varepsilon = n - 2. \end{cases}$$

Let ρ be so small that $m + p(n - 2 - m - \rho) > n - 2$, then we have, for $r > R$

$$u(r) \leq \begin{cases} c[r^{2-n} + r^{-m-p^2\varepsilon}] & \text{if } m + p\varepsilon < n - 2, \\ cr^{2-n} & \text{if } m + p\varepsilon = n - 2. \end{cases}$$

Iterating this process, we can show that in case $m + p\varepsilon < n - 2$

$$u(r) \leq c[r^{2-n} + r^{-m-p^k\varepsilon}], \quad \text{at } \infty$$

for any positive integer k . Since $p > 1$, then we have $u(r) \leq cr^{2-n}$. By the same way, for $r > R$, we have

$$u \leq \begin{cases} cr^{2-n} & \text{if } q > n, \\ cr^{2-n} \log r & \text{if } q = n, \\ cr^{2-q} & \text{if } m + 2 < q < n. \end{cases}$$

The proof is complete. □

4. The stability of the minimum steady state and proofs of Theorem 2 and Theorem 3

The Proof of Theorem 2. Because $q > n$, we have $u_{\alpha_*} = O(r^{2-n})$ at ∞ . Let $v = u_{\alpha_*} + ar^{-\nu}$, for $r > 1, 0 < a < 1$. By simple computation we obtain

$$\begin{aligned} \Delta v + K(r)v^p + \mu f &= \Delta(u_{\alpha_*} + ar^{-\nu}) + K(r)(u_{\alpha_*} + ar^{-\nu})^p + \mu f \\ &= ar^{-(\nu+2)}\nu(\nu + 2 - n) + K(r)[(u_{\alpha_*} + r^{-\nu})^p - u_{\alpha_*}^p] \quad \text{at } \infty \end{aligned}$$

So if $m < \nu < n - 2$, for any $0 < a < 1$ there exists $R_1 > 1$ independent of a such that

$$\Delta v + K(r)v^p + \mu f = ar^{-(\nu+2)}[\nu(\nu + 2 - n) + o(1)] \leq 0 \quad r > R_1.$$

For each fixed $a > 0$, we choose $\beta > \alpha_*$ sufficiently close to α such that $v(R_1) > u_\beta(R_1)$. By the asymptotic expansion of the slow decay solution u_β , we know there exists $R_2 > R_1$ such that $v(R_2) < u_\beta(R_2)$. Therefore by Proposition 2.3 (ii) we can construct a sup-solution $\bar{u}(r)$ such that $\bar{u}(r) > u_{\alpha_*}$, and

$$\bar{u}(r) = \begin{cases} u_\beta & \text{if } r \leq R'_2, \\ v(r) & \text{if } r > R'_2, \end{cases}$$

where R'_2 is the first zero of $u_\beta - v$.

Let

$$\begin{aligned} \delta_3 &= \delta_3(a, \beta) =: \inf_{r \geq 0} (\bar{u}(r) - u_{\alpha_*}(r))(1+r)^\nu > 0 \\ \varepsilon_3(a, \beta) &=: \|\bar{u}(r) - u_{\alpha_*}\|_\nu, \end{aligned}$$

then

$$\varepsilon_3(a, \beta) = \sup\{(1+r)^\nu(u_\beta - u_{\alpha_*})_{r \in (0, R'_2)}, (1+r)^\nu(\bar{u}(r) - u_{\alpha_*})_{r \in (R'_2, \infty)}\};$$

and we have that

$$\begin{aligned} \lim_{\beta \rightarrow \alpha_*} \sup_{r \in (0, 1)} \{(1+r)^\nu(u_\beta - u_{\alpha_*})\} &= 0; \\ \lim_{a \rightarrow 0, \beta \rightarrow \alpha_*} \sup_{r \in (1, R'_2)} \{(1+r)^\nu(u_\beta - u_{\alpha_*})\} &\leq \lim_{a \rightarrow 0} \sup_{r \in (1, R'_2)} \{(1+r)^\nu(v - u_{\alpha_*})\} \\ &= \lim_{a \rightarrow 0} \sup_{r \in (1, R'_2)} \{(1+r)^\nu ar^{-\nu}\} \leq \lim_{a \rightarrow 0} \sup_{r \in (1, \infty)} \{(1+r)^\nu ar^{-\nu}\} = 0; \end{aligned}$$

similarly

$$\lim_{a \rightarrow 0} \sup_{r \in (R'_2, \infty)} \{(1+r)^\nu(\bar{u}(r) - u_{\alpha_*})\} = 0.$$

So we have

$$\lim_{a \rightarrow 0, \beta \rightarrow \alpha_*} \varepsilon_3(a, \beta) = 0.$$

By Proposition 2.2, we know that the solution $u(x, t, \bar{u})$ of (1.2) is strictly decreasing in t and radially symmetric in x . And $\bar{u} > u(x, t, \bar{u}) \geq u_{\alpha_*}$ by the comparison principle. So $\lim_{t \rightarrow \infty} u(x, t, \bar{u})$ exist, denoted by u_∞ and $u_\infty \leq \bar{u}$. Furthermore $\bar{u} = o(r^{-m})$ at ∞ , therefore $u_\infty = u_{\alpha_*}$.

Choosing $\delta < \delta_3$, then for any $\varphi(x)$ such that $\|\varphi(x) - u_{\alpha_*}(x)\|_\mu < \delta$, we have $\varphi(x) < \bar{u}(x)$. and then by the comparison principle,

$$(4.1) \quad u_{\alpha_*} < u(x, t, \varphi) < u(x, t, \bar{u}) < \bar{u}.$$

So we have that $\lim_{t \rightarrow \infty} u(x, t, \varphi) = u_{\alpha_*}(x)$ uniformly for x in any ball in R^n .

For any $\varepsilon > 0$, we can find $a > 0, \beta > \alpha$ such that $\varepsilon_3(a, \beta) < \varepsilon$, and we have by (4.1) that

$$\|u(x, t, \varphi(x)) - u_{\alpha_*}(x)\|_\nu < \varepsilon,$$

which prove the semistability of the solution u_{α_*} of (1.4).

Now for every $\nu' < \nu, R > R'_2$, we have by (4.1) that

$$|(1 + |x|)^{\nu'}(u(\cdot, t; \varphi) - u_{\alpha_*})| \leq \begin{cases} a(1 + |x|)^{\nu'}|x|^{-\nu} & \text{if } |x| \geq R, \\ (1 + R)^{\nu'}\|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{L^\infty(B_R)} & \text{if } |x| < R, \end{cases}$$

$$\leq \begin{cases} aR^{\nu'-\nu} & \text{if } |x| \geq R, \\ (1 + R)^{\nu'}\|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{L^\infty(B_R)} & \text{if } |x| < R. \end{cases}$$

Now for any $\varepsilon > 0, \exists R = R(\varepsilon) > R'_2, \exists T(\varepsilon)$ when $t > T(\varepsilon)$ we have:

$$aR^{\nu'-\nu} < \varepsilon/2, (1 + R)^{\nu'}\|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{L^\infty(B_R)} < \varepsilon/2.$$

So, letting $t > T(\varepsilon)$ we get

$$\|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{\nu'} \leq \varepsilon.$$

Since ε is arbitrary small, it follows that $\lim_{t \rightarrow \infty} \|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{\nu'} = 0$. Thus we complete the proof. \square

The Proof of Theorem 3. Without loss of generality we may assume that $\varphi < u_\alpha$ in R^n , because the assumptions that $\varphi \leq u_\alpha$ and $\varphi \not\equiv u_\alpha$ together with the strong maximum principle for parabolic equations immediately imply that $u(x, t, \varphi) < u_\alpha$ for all $x \in R^n$ and $t > 0$. Thus we may replace φ by $u(\cdot, \varepsilon, \varphi)$ for some $\varepsilon > 0$ if necessary. Suppose we can build a radial bounded continuous weak super-solution ψ of (1.1) staying above $\varphi(x)$ and below $u_\alpha(x)$ i.e. $0 < \varphi(x) \leq \psi(x) \leq u_\alpha(x)$. Denote by $\bar{u}(x, t)$ the solution of (1.2) with initial value $\psi(x)$. Then by the comparison principle, we have

$$u(x, t, 0) < u(x, t, \varphi) \leq \bar{u}(x, t) \leq u_\alpha(x)$$

for $x \in R^n$ and $t > 0$; moreover, $\bar{u}(x, t)$ is radial in x and decreasing in t by Proposition 2.2. Thus $\bar{u}(x, t) \rightarrow$ some radial bounded steady state $u_{\alpha'}(x)$ of (1.2) as $t \rightarrow \infty$ uniformly for bounded x . If $u_{\alpha'}(x)$ is the slow decay solution, then by (ii) of Theorem A, $u_{\alpha'}(x)$ and $u_\alpha(x)$ intersect, which would be a contradiction. So $u_{\alpha'}(x) = u_{\alpha_*}(x)$. Besides, $u(x, t, 0)$ is radial in x and increasing in t by Proposition 2.2. So $\lim_{t \rightarrow \infty} u(x, t, 0)$ exists, and is equal to $u_{\alpha_*}(x)$. By the comparison principle, we have that

$$u_{\alpha_*}(x) \leq \limsup_{t \rightarrow \infty} u(x, t, \varphi) \leq u_{\alpha'}(x).$$

So we have

$$\lim_{t \rightarrow \infty} u(x, t) = u_{\alpha_*}.$$

Now, we derive the construction of a super-solution $\psi(x)$ as mentioned above. By (ii) of Theorem A, any two positive radial slow decay solutions of (1.1) must intersect each other if $\frac{n+2+2l}{n-2} < p < p_c$. We set $z(\alpha, \beta)$ to be the first zero of $u_\alpha - u_\beta$ where $\beta \in (\alpha_*, \alpha)$. We first observed that $u_\beta \rightarrow u_\alpha$ uniformly on compact subsets.

Claim: Fixed some $\beta_1 > \beta_2 \in (\alpha_*, \alpha)$. Then $z(\alpha, \beta) \leq z(\beta_1, \beta_2) < \infty$ as $\beta \rightarrow \alpha$.

The proof follows closely that of [12, Lemma 3.1]. For otherwise if we have $z(\alpha, \beta) > z(\beta_1, \beta_2)$. Then we would have $u_\alpha(x) > u_\beta > u_{\beta_1} > u_{\beta_2}$ on $[0, z(\beta_1, \beta_2)]$ as $\beta \rightarrow \alpha$. Let $V = u_\alpha(x) - u_\beta$, and $v = u_{\beta_1} - u_{\beta_2}$, then

$$\begin{aligned} \Delta V + pK(|x|)W^{p-1}V &= 0, \\ \Delta v + pK(|x|)w^{p-1}v &= 0, \end{aligned}$$

where W is the mean value between $u_\alpha(x)$ and u_β , w is that between u_{β_1}, u_{β_2} so that $W > w$ on $[0, z(\beta_1, \beta_2)]$. Then we would have in $B_{z(\beta_1, \beta_2)}(0)$

$$\int_{B_{z(\beta_1, \beta_2)}(0)} v\Delta V - V\Delta v + pK(|x|)(W^{p-1} - w^{p-1})Vv = 0,$$

or

$$\int_{\partial B_{z(\beta_1, \beta_2)}(0)} -V\frac{\partial v}{\partial \eta} + \int_{B_{z(\beta_1, \beta_2)}(0)} pK(|x|)(W^{p-1} - w^{p-1})Vv = 0,$$

which is a contradiction, so $z(\alpha, \beta) \leq z(\beta_1, \beta_2) < \infty$ as $\beta \rightarrow \alpha$.

Thus there exists $\max\{\alpha_*, \frac{\alpha}{2}\} < \beta' < \alpha$ such that $\varphi < u_{\beta'}$ in $[0, z(\alpha, \beta')]$, setting:

$$\psi(x) = \begin{cases} u_{\beta'}(r) & \text{if } r \leq z(\alpha, \beta'), \\ u_\alpha(r) & \text{if } r > z(\alpha, \beta'). \end{cases}$$

We see that $\varphi(x) \leq \psi(x) \leq u_\alpha(x)$ and $\psi(x)$ is continuous, it is standard to verify that $\psi(x)$ is a continuous weak super-solution of (1.1).

Part (ii) of Theorem 3 may be handled in a similar fashion. As (i), we may assume, without loss of generality that $\varphi > u_\alpha$. Since $u_\beta \rightarrow u_\alpha$ uniformly in $[0, z(\alpha, 2\alpha)]$ as $\beta \rightarrow \alpha$ and $z(\alpha, \beta) < \infty$ as $\alpha \rightarrow \beta$, there exists $\frac{3}{2}\alpha > \tilde{\beta} > \alpha$ such that $\varphi > u_{\tilde{\beta}}$ in $[0, z(\alpha, \tilde{\beta})]$, thus setting:

$$\tilde{\psi}(x) = \begin{cases} u_{\tilde{\beta}}(r) & \text{if } r \leq z(\alpha, \tilde{\beta}), \\ u_\alpha(r) & \text{if } r > z(\alpha, \tilde{\beta}). \end{cases}$$

We have $\varphi(x) > \tilde{\psi}(x) \geq u_\alpha(x)$ for all $x \in R^n$. It is standard to verify that $\tilde{\psi}(x)$ is a continuous weak sub-solution of (1.1). The conclusion follows similarly from [24, Theorem 3.10], i.e. $u(x, t, \varphi)$ must blow up in finite time. \square

5 The stability and the weakly asymptotically stability of the slow decay steady states and proof of Theorem 4

The Proof of Theorem 4. We first consider the case $p = p_c$. We need to construct various super-solutions and sub-solutions to (1.1). For any given slow steady state $u_\alpha(r)$, we consider

$$v(r) = u_\alpha(r) + a(\log r)^{\nu_1} / r^{(m+\lambda_1)} \quad r > 1,$$

where a, ν_1 are constants and $\nu_1 > 0$.

We compute

$$\begin{aligned} \Delta v + K(r)v^p + \mu f &= a\nu_1(\nu_1 - 1)(\log r)^{(\nu_1-2)}r^{-(m+2+\lambda_1)} \\ &+ a[(n-1)\nu_1 - \nu_1(2m+2\lambda_1+1)](\log r)^{(\nu_1-1)}r^{-(m+2+\lambda_1)} \\ &+ a[(m+\lambda_1)(m+\lambda_1+1) - (m+\lambda_1)(n-1)](\log r)^{\nu_1}r^{-(m+2+\lambda_1)} \\ &+ K[(u_\alpha + a(\log r)^{\nu_1}r^{-(m+\lambda_1)})^p - u_\alpha^p]. \end{aligned}$$

Note that when $p = p_c$, we have $\lambda_1 = \lambda_2, 2(m + \lambda_1) = n - 2$ and

$$(m + \lambda_1)(m + \lambda_1 - n + 2) = -pL^{p-1}.$$

Then, by (2.3), we deduce that

$$\begin{aligned} \Delta v + K(r)v^p + \mu f &= av_1(\nu_1 - 1)(\log r)^{(\nu_1-2)}r^{-(m+2+\lambda_1)} \\ &+ av_1[n - 2 - 2(m + \lambda_1)](\log r)^{\nu_1-1}r^{-(m+2+\lambda_1)} \\ &+ a[(m + \lambda_1)(m + \lambda_1 + 2 - n) + pL^{p-1}](\log r)^{\nu_1}r^{-(m+2+\lambda_1)} \\ &+ o(r^{-(m+2+\lambda_1)}(\log r)^{\nu_1-2}) \\ &= a(\log)^{(\nu_1-2)}r^{-(m+2+\lambda_1)}[\nu_1(\nu_1 - 1) + o(1)] \quad \text{at } r = \infty. \end{aligned}$$

For any $\nu_1 > 1$, there exists $R_1 > 1$ such that $\Delta v + K(r)v^p + \mu f > 0$ in $|x| > R_1$ for any $0 < a < 1$. On the other hand, for any $u_\beta(r)$ with $\beta > \alpha$, it is known from Theorem A (i) that $u_\beta(r) > u_\alpha(r), r \geq 0$. Therefore we can fix $\beta > \alpha$ and choose $a > 0$ small enough such that $v(R_1) < u_\beta(R_1)$. By the asymptotic expansion (2.3), we know that there exists $R_2 > R_1$ such that $v(R_2) > u_\beta(R_2)$. Therefore by Proposition 2.3 (ii) we can construct a sub-solution $u, r \geq 0$ such that $u > u_\alpha, r \geq 0$ and

$$u(r) = \begin{cases} u_\beta & \text{if } r \leq R'_2, \\ v(r) & \text{if } r > R'_2. \end{cases}$$

where R'_2 is the first zero of $u_\beta - v$.

So we can choose $u(r)$ such that $u(r)$ is decreasing in $r \geq 0$ and that $\|u(r) - u_\alpha(r)\|_{\nu_1}$ is as small as one wishes, by choosing $\beta - \alpha$ and a sufficiently small. We claim that the solution $u(x, t, u)$ of (1.2) either blow up in finite time or converges to a singular solution of (1.3) as $t \rightarrow +\infty$. If not, from Proposition 2.2, we know that $u(x, t, u)$ is strictly increasing in t , radially symmetric in x and decreasing in $|x|$. Then $u_\infty(|x|) = \lim_{t \rightarrow \infty} u(x, t, u)$ be a regular solution of (1.1). It is easy to check that $u_\infty(|x|)$ is a distributional solution of (1.1) in R^n . So it must have expansion (2.3) at infinity. However, at infinity we have $u_\infty(|x|) \geq v(r) \geq u_\alpha(r) + a(\log r)^{\nu_1}/r^{(m+\lambda_1)}, \nu_1 > 1$, this contradicts (2.3). This proves that $u_\alpha(r)$ is unstable in $\|\cdot\|_{\nu_1}$ when $\nu_1 > 1$. (The instability is also manifested in the following way: If we choose $-1 < a < 0$, we can also construct similarly a super-solution $\bar{u}(r)$ such that $\bar{u}(r) < u_\alpha(r)$ and $u_\alpha(r) - \bar{u}(r) = a(\log r)^{\nu_1}r^{(m+\lambda_1)}$ for $r \geq R_2$. It can be shown that $u_\infty(|x|) = \lim_{t \rightarrow \infty} u(x, t, \bar{u})$, and $u_\infty(|x|)$ is the minimum steady solution).

If $0 < \nu_1 < 1$, for any $0 < a < 1$ there exists $R_1 > 1$ independent of a such that

$$\Delta v + K(r)v^p + \mu f \leq 0, \quad r > R_1 > 1.$$

For any u_β with $\beta > \alpha$, we have $u_\beta > u_\alpha$. For each fixed a , we choose $\beta > \alpha$ sufficiently close to α such that $v(R_1) > u_\beta(R_1)$. Note $a_1 < 0$ (the coefficient of $r^{-(m+\lambda_1)}$) and $\nu_1 < 1$. There exists $R_2 > R_1$ such that $u_\beta(R_2) > v(R_2)$. Therefore we can construct a super-solution $\bar{u}(r) > u_\alpha(r)$ and $\bar{u}(r) - u_\alpha(r) = a(\log r)^{\nu_1}r^{-(m+\lambda_1)}, r \geq R_2$. Let

$$\delta_1 = \delta_1(a, \beta) =: \inf_{r \geq 0} (\bar{u}(r) - u_\alpha(r))(\log(2+r))^{-\nu_1}(1+r)^{m+\lambda_1}$$

$$\varepsilon_1 = \varepsilon_1(a, \beta) := \|\bar{u}(r) - u_\alpha(r)\|_{\nu_1}.$$

Then $0 < \delta_1 < \varepsilon_1$, and as the proof of Theorem 2 we have

$$\lim_{a \rightarrow 0, \beta \rightarrow \alpha} \varepsilon_1(a, \beta) = 0.$$

By Proposition 2.2, we know that the solution $u(x, t, \bar{u})$ of (1.2) is strictly decreasing in t and radially symmetric in x , and $u(x, t, \bar{u}) > u_\alpha$. Let $u_\infty = \lim_{t \rightarrow \infty} u(x, t, \bar{u})$. It is easy to see that u_∞ is a solution of (1.1) in R^n . Then u_∞ has expansion (2.5) at $r = \infty$. Furthermore the coefficient a_1 (i.e. the coefficient of $r^{-(m+\lambda_1)}$) is the same for u_∞ and u_α because $\nu_1 < 1$. Therefore $u_\infty(|x|) = u_\alpha(r)$.

Similarly, by choosing $-1 < a < 0$ and $\beta < \alpha$ sufficiently close to α , we can construct a sub-solution $\underline{u}(r) < u_\alpha(r)$ such that

$$0 < \delta_2 = \delta(a, \beta) := \inf_{r \geq 0} (u_\alpha(r) - \underline{u}(r))(\log(2+r))^{-\nu_1}(1+r)^{m+\lambda_1},$$

$$0 < \varepsilon_2 = \varepsilon_2(a, \beta) := \|\|u_\alpha(r) - \underline{u}(r)\|\|_\mu,$$

and as the proof of Theorem 2 we have

$$\lim_{a \rightarrow 0, \beta \rightarrow \alpha} \varepsilon_2(a, \beta) = 0.$$

Moreover, $\lim_{t \rightarrow \infty} u(x, t, \underline{u}) = u_\alpha(|x|)$ uniformly for x in any ball in R^n .

For any $\varepsilon > 0$, we can find $a > 0, a' < 0, \beta > \alpha, \beta' < \alpha$ such that $\varepsilon_1(\alpha, \beta) < \varepsilon, \varepsilon_2(\alpha', \beta') < \varepsilon$. Choose $\delta = \min\{\delta_1(a, \beta), \delta_2(a', \beta')\}$. Then for any $\varphi(x)$ such that $\|\|\varphi(x) - u_\alpha(x)\|\|_{\nu_1} < \delta$, we have $\underline{u}(x) \leq \varphi(x) \leq \bar{u}(x)$ and then by the comparison principle, we have

$$\|\|u(x, t, \varphi(x)) - u_\alpha(x)\|\|_{\nu_1} < \varepsilon.$$

To show that u_α is weakly asymptotic stable with respect to the norm $\|\|\cdot\|\|_{\nu_1}$. We need to show that there exists $\delta > 0$, if $\|\|\varphi - u_\alpha\|\|_{\nu_1} < \delta$, then

$$\lim_{t \rightarrow \infty} \|\|u(\cdot, t, \varphi) - u_\alpha\|\|_{\nu'_1} \rightarrow 0$$

for every $\nu'_1 > \nu_1$.

By Proposition 2.2 we have

$$u < u(\cdot, t; u) < u(\cdot, t; \varphi) < u(\cdot, t; \bar{u}) < \bar{u};$$

and

$$\lim_{t \rightarrow \infty} u(\cdot, t; \bar{u}) = u_\alpha = \lim_{t \rightarrow \infty} u(\cdot, t; u) \quad \text{in } R^n.$$

Now for every $\nu'_1 > \nu_1, R > R'_2$, we have

$$\begin{aligned} & |(\log(2+r))^{-\nu'_1}(1+r)^{m+\lambda_1}(u(\cdot, t; \varphi) - u_\alpha)| \\ & \leq \begin{cases} 2a \frac{(\log r)^{\nu_1}(1+r)^{m+\lambda_1}}{(\log(2+r))^{\nu'_1} r^{m+\lambda_1}} \\ \text{if } r \geq R, \\ (\log(2+r))^{-\nu'_1}(1+r)^{m+\lambda_1} \|u(\cdot, t; \varphi) - u_\alpha\|_{L^\infty(B_R)} \\ \text{if } r < R, \end{cases} \end{aligned}$$

where B_R is a ball of radius R centered at 0.

As the proof of Theorem 2, we have

$$\lim_{t \rightarrow \infty} \|u(\cdot, t, \varphi) - u_{\alpha_*}\|_{\nu'_1} = 0.$$

For the case $p > p_c$ and $\lambda < m + \lambda_2$, we can argue similarly for the stability or instability of u_α in a range of weighted norm $\|\cdot\|_\lambda$ as follows.

Let $v(r) = u_\alpha + ar^{-\lambda}$ $r > 1$. Then we have

$$\Delta v + Kv^p + \mu f = aK[\lambda(\lambda + 2 - n) + pL^{p-1} + o(1)]r^{-(\lambda+2)} \quad \text{at } r = \infty.$$

We observe that

$$\lambda(\lambda + 2 - n) + pL^{p-1} \begin{cases} > 0 & \text{if } \lambda < m + \lambda_1, \\ < 0 & \text{if } m + \lambda_1 < \lambda < m + \lambda_2, \end{cases}$$

and the proof is the similar to that of the Theorem 2, we omit it here. \square

References

- [1] S. Bae, *Positive entire solutions of semilinear elliptic equations with quadratically vanishing coefficient*, J. Differential Equations, **237** (2007) 159-197.
- [2] ———, *Positive entire solutions of inhomogeneous semilinear elliptic equations with supercritical exponent*, Discrete Contin. Dyn. Syst. (2005) 50-59.
- [3] ———, *Separation structure of positive radial solutions of a semilinear elliptic equations in R^n* , J. Differential Equations, **194** (2003) 460-499.
- [4] S. Bae, T.-K. Chang, and D.-H. Pank, *Infinite multiplicity of positive entire solutions for a semilinear elliptic equation*, J. Differential Equations, **181** (2002), 367-387.
- [5] S. Bae and W.-M. Ni, *Existence and infinite multiplicity for an inhomogeneous semilinear elliptic equation on R^n* , Math. Ann. **320** (2001) 191-210.
- [6] G. Bernard, *An inhomogeneous semilinear equation in entire space*, J. Differential Equations, **125** (1996) 184-214.
- [7] Y.-B. Deng, Y. Li, and Y. Liu, *On the stability of the positive radial steady states for a semilinear Cauchy problem*, Nonlinear Anal. **54** (2003), 291-318.
- [8] Y.-B. Deng, Y. Li, and F. Yang, *On the stability of the positive steady states for a nonhomogeneous semilinear Cauchy problem*, J. Differential Equations, **228** (2006), no. 2, 507-529.
- [9] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* , J. Fac. Sci. Univ. Tokyo, **13** (1966), 109-124.
- [10] C.-F. Gui, *Positive entire solutions of equation $\Delta u + F(x, u) = 0$* , J. Differential Equations, **99** (1992), 245-280.
- [11] ———, *On positive entire solutions of the elliptic equation $\Delta u + K(x)u^p = 0$ and its applications to Riemannian geometry*, Proc. Roy. Soc. Edinburgh Sect. A, **126** (1996), 225-237.
- [12] C.-F. Gui, W.-M. Ni, and X.-F. Wang, *On the stability and instability of positive steady state of a semilinear heat equation in R^n* , Comm. Pure Appl. Math. **45** (1992), 1153-1181.
- [13] ———, *Further study on a nonlinear heat equation*, J. Differential Equations, **169** (2001), 588-613.
- [14] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic equations*, Proc. Japan Acad. **49** (1973), 503-525.
- [15] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Ration. Mech. Anal. **49** (1972) 241-269.
- [16] K. Kobayashi, T. siaro, and H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan, **29** (1977), no. 3, 407-424.
- [17] T.-Y. Lee and W.-M. Ni, *Global existence, Large time behavior and life span of solutions of semilinear parabolic Cauchy problems*, Trans. Amer. Math. Soc. **333** (1992), 365-371.

- [18] Y. Li, *Remarks on a semilinear elliptic equations on R^n* , J. Differential Equations, **74** (1988), 34-49.
- [19] ———, *Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in R^n* , J. Differential Equations, **95**, 1992, (304-330).
- [20] Y. Li, and W.-M. Ni, *On conformal scalar curvature equations in R^n* , Duke Math. J. **57** (1988), 895-924.
- [21] Y. Liu, Y. Li, and Y.-B. Deng, *Separation Property of solutions for a semilinear elliptic equation*, J. Differential Equations, **163** (2000), 381-406.
- [22] W.-M. Ni, *On the elliptic equation $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$, Its generalizations and applications in geometry*, Indian Univ. Math. J. **31** (1982), 493-529.
- [23] W.-M. Ni and S. Yotsutani, *Semilinear elliptic equations of Matukuma type and relate topics*, Japan J. Appl. Math. **5** (1988), 1-32.
- [24] X.-F. Wang, *On Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc. **337** (1993), 549-590.
- [25] F. Weissler, *Existence and nonexistence of global solutuion for semilinear heat equation*, Israel J. Math. **38** (1981), 29-40.
- [26] E. Yanagida and S. Yotsutani, *Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in R^n* , Arch. Rational Mech. Anal. **124** (1993), 239-259.

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