Math. Res. Lett. **15** (2008), *no.* 5, 881–892

THE STRUCTURE OF SALLY MODULES OF RANK ONE

Shiro Goto, Koji Nishida, and Kazuho Ozeki

Abstract. A complete structure theorem of Sally modules of $\mathfrak{m}$-primary ideals I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ satisfying the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ is given, where $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$ denote the first two Hilbert coefficients of I.

1. Introduction

This paper aims to give a structure theorem of Sally modules of rank one.

Let A be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. Then we have integers $\{e_i = \mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

holds true for all $n \gg 0$. Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \ \subseteq \ A[t]$$

denote, respectively, the Rees algebras of I and Q, where t stands for an indeterminate over A. We put

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \text{and} \quad G = \mathrm{G}(I) := R'/t^{-1}R' \ \cong \ \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Let $B = T/\mathfrak{m}T$, which is the polynomial ring with d indeterminates over the field k. Following W. V. Vasconcelos [11], we then define

$$\mathrm{S}_Q(I) = IR/IT$$

and call it the Sally module of I with respect to Q. We notice that the Sally module $S = \mathrm{S}_Q(I)$ is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

The Sally module S was introduced by W. V. Vasconcelos [11], where he gave an elegant review, in terms of his *Sally* module, of the works [8, 9, 10] of J. Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded ring G and the Hilbert coefficients e_i's of I.

As is well-known, we have the inequality ([5])

$$e_1 \ge e_0 - \ell_A(A/I)$$

Received by the editors October 5, 2007.

2000 *Mathematics Subject Classification.* 13H10, 13B22, 13H15.

Key words and phrases. Cohen-Macaulay local ring, Buchsbaum ring, associated graded ring, Rees algebra, Sally module, Hilbert coefficient.

and C. Huneke [4] showed that $e_1 = e_0 - \ell_A(A/I)$ if and only if $I^2 = QI$. When this is the case, both the graded rings G and $\mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1 = e_0 - \ell_A(A/I)$ enjoy very nice properties. The reader may consult with the recent work of Wang [13], which establishes the ubiquity of ideals I with $I^2 = QI$.

J. Sally [10] firstly investigated the second border, that is the ideals I satisfying the equality

$$e_1 = e_0 - \ell_A(A/I) + 1$$

and gave several very important results. Among them, one can find the following characterization of ideals I with $e_1 = e_0 - \ell_A(A/I) + 1$ and $e_2 \neq 0$, where $B(-1)$ stands for the graded B-module whose grading is given by $[B(-1)]_n = B_{n-1}$ for all $n \in \mathbb{Z}$. The reader may also consult with [1] and [12] for further ingenious use of Sally modules.

Theorem 1.1 (Sally [10] , Vasconcelos [11]). *The following three conditions are equivalent to each other.*

(1) $S \cong B(-1)$ *as graded T-modules.*
(2) $e_1 = e_0 - \ell_A(A/I) + 1$ *and if* $d \geq 2$, $e_2 \neq 0$.
(3) $I^3 = QI^2$ *and* $\ell_A(I^2/QI) = 1$.

When this is the case, the following assertions hold true.

(i) $e_2 = 1$, *if* $d \geq 2$.
(ii) $e_i = 0$ *for all* $3 \leq i \leq d$.
(iii) $\operatorname{depth} G \geq d - 1$.

This beautiful theorem says, however, nothing about the case where $e_2 = 0$. It seems natural to ask what happens, when $e_2 = 0$, on the ideals I which satisfy the equality $e_1 = e_0 - \ell_A(A/I) + 1$. This long standing question has motivated the recent research [2], where the authors gave several partial answers to the question. The present research is a continuation of [2, 10, 11] and aims at a simultaneous understanding of the structure of Sally modules of ideals I which satisfy the equality $e_1 = e_0 - \ell_A(A/I) + 1$.

Let us now state our own result. The main result of this paper is the following Theorem 1.2, which contains Theorem 1.1 of Sally–Vasconcelos as the case where $c = 1$. Our contribution in Theorem 1.2 is the implication (1) $\Rightarrow$ (3), the proof of which is based on the new result that the equality $I^3 = QI^2$ holds true if $e_1 = e_0 - \ell_A(A/I) + 1$ (cf. Theorem 3.1).

Theorem 1.2. *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.
(2) $\mathfrak{m}S = (0)$ *and* $\operatorname{rank}_B S = 1$.
(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded T-modules for some* $0 < c \leq d$, *where* $\{X_i\}_{1 \leq i \leq c}$ *are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$, *and the following assertions hold true.*

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.
(ii) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.

(iii) *Suppose* $c < d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$$

for all $n \geq 0$. *Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose* $c = d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$$

for all $n \geq 1$. *Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Thus Theorem 1.2 settles a long standing problem, although the structure of ideals I with $e_1 = e_0 - \ell_A(A/I) + 2$ or the structure of Sally modules S with $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 2$ remains unknown.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1.2 in Section 3. In Section 2 we will pick up from the paper [2] some auxiliary results on Sally modules, all of which are known, but let us note them for the sake of the reader's convenience. In Section 4 we shall discuss two consequences of Theorem 1.2. The results are more or less known by [2, 10, 11]. However, thanks to Theorem 1.2, not only the statements of the results but also the proofs are substantially simplified, so that we would like to note the improved statements, and would like to indicate a brief proof of Theorem 1.1 as well. In Section 5 we will construct one example in order to see the ubiquity of ideals I which satisfy condition (3) in Theorem 1.2. We will show that, for given integers $0 < c \leq d$, there exists an $\mathfrak{m}$-primary ideal I in a certain Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that

$$d = \dim A, \quad e_1 = e_0 - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. We assume that the field $k = A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It], T = A[Qt], R' = A[It, t^{-1}]$, and $G = R'/t^{-1}R'$. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n]$$

denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = e_i$ for all $0 \leq i \leq d$ (cf. [6]). We denote by $\mu_A(*)$ the number of generators.

2. Auxiliary results

In this section let us firstly summarize some known results on Sally modules, which we need throughout this paper. See [2] and [11] for the detailed proofs.

The first two results are basic facts on Sally modules developed by Vasconcelos [11].

Lemma 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$.
(2) *The homogeneous components* $\{S_n\}_{n\in\mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \le 0, \\ I^{n+1}/IQ^n & \text{if } n \ge 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.
(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$, *where* $M = \mathfrak{m}T + T_+$ *is the graded maximal ideal in* T. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* T/M*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q+1$, *where* $\mathrm{r}_Q(I)$ *stands for the reduction number of* I *with respect to* Q.
(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [2, Lemma 2.1]. $\square$

Proposition 2.2. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$. *Hence* $\dim_T S = d$, *if* $S \neq (0)$.
(2) $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.
(3) *We have* $e_1 = e_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *Hence* $e_1 = e_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.
(4) *Suppose that* $S \neq (0)$. *Let* $s = \mathrm{depth}_T\, S$. *Then* $\mathrm{depth}\, G = s - 1$ *if* $s < d$. S *is a Cohen-Macaulay* T*-module if and only if* $\mathrm{depth}\, G \geq d - 1$.

Proof. See [2, Proposition 2.2]. $\square$

Combining Lemma 2.1 (3) and Proposition 2.2, we readily get the following results of Northcott [5] and Huneke [4].

Corollary 2.3 ([4, 5])**.** *We have* $e_1 \geq e_0 - \ell_A(A/I)$. *The equality* $e_1 = e_0 - \ell_A(A/I)$ *holds true if and only if* $I^2 = QI$. *When this is the case,* $e_i = 0$ *for all* $2 \leq i \leq d$.

The following result is one of the keys for our proof of Theorem 1.2.

Theorem 2.4 ([2])**.** *The following conditions are equivalent.*

(1) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.
(2) $S \cong \mathfrak{a}$ *as graded* T*-modules for some non-zero graded ideal* $\mathfrak{a}$ *of* B.

Proof. See [2, Theorem 2.4]. $\square$

The following result is also due to [2], which will enable us to reduce the proof of Theorem 1.2 to the proof of the fact that $I^3 = QI^2$ if $e_1 = e_0 - \ell_A(A/I) + 1$.

Proposition 2.5 ([2])**.** *Suppose* $e_1 = e_0 - \ell_A(A/I) + 1$ *and* $I^3 = QI^2$. *Let* $c = \ell_A(I^2/QI)$. *Then the following assertions hold true.*

(1) $0 < c \leq d$ *and* $\mu_B(S) = c$.
(2) $\mathrm{depth}\, G \geq d - c$ *and* $\mathrm{depth}_B\, S = d - c + 1$.
(3) $\mathrm{depth}\, G = d - c$, *if* $c \geq 2$.

(4) *Suppose $c < d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ for all $n \geq 0$. Hence*
$$e_i = \begin{cases} 0 & \text{if } i \neq c+1 \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$
for $2 \leq i \leq d$.

(5) *Suppose $c = d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ for all $n \geq 1$. Hence $e_i = 0$ for $2 \leq i \leq d$.*

Proof. See [2, Corollary 2.5]. □

The following result might be known. However, since we can find no good references, let us include a brief proof.

Proposition 2.6. *Let $Q \subseteq I \subseteq J$ be ideals in a commutative ring A. Assume that $J = I + (h)$ for some $h \in A$. Then $I^3 = QI^2$, if $J^2 = QJ$.*

Proof. Since $hI \subseteq J^2 = QJ = QI + Qh$, for each $i \in I$ there exist $j \in QI$ and $q \in Q$ such that $hi = j + qh$. Hence $h(i-q) = j \in QI$. On the other hand, we have $(i-q)I^2 \subseteq (i-q)J^2 = (i-q)(QI + Qh) = (i-q)QI + jQ \subseteq QI^2$, because $i - q \in I$ and $j \in QI$. Thus $(i-q)I^2 \subseteq QI^2$, so that we have $iI^2 \subseteq QI^2$ for all $i \in I$. Hence $I^3 = QI^2$. □

3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. See Proposition 2.2 (3) for the equivalence of conditions (1) and (2) in Theorem 1.2. The implication (3) ⇒ (2) is clear. So, we must show the implication (1) ⇒ (3) together with the last assertions in Theorem 1.2. Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then, thanks to Theorem 2.4, we get an isomorphism
$$\varphi : S \to \mathfrak{a}$$
of graded B-modules, where $\mathfrak{a}$ is a graded ideal of B. Notice that once we are able to show $I^3 = QI^2$, the last assertions of Theorem 1.2 readily follow from Proposition 2.5. On the other hand, since $\mathfrak{a} \cong S = BS_1$ (cf. Lemma 2.1 (5)), the ideal $\mathfrak{a}$ of B is generated by linearly independent linear forms $\{X_i\}_{1 \leq i \leq c}$ $(0 < c \leq d)$ of B and so, the implication (1) ⇒ (3) in Theorem 1.2 follows. We have $c = \ell_A(I^2/QI)$, because $\mathfrak{a}_1 \cong S_1 = I^2/QI$ (cf. Lemma 2.1 (2)). Thus our Theorem 1.2 has been proven modulo the following theorem, which follows also, in the case where $d \leq 2$, from a result of M. Rossi [7, Corollary 1.5].

Theorem 3.1. *Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then $I^3 = QI^2$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then S is B-free of rank one (recall that the B-module S is torsionfree; cf. Proposition 2.2 (1)) and so, since $S_1 \neq (0)$ (cf. Lemma 2.1 (3)), $S \cong B(-1)$ as graded B-modules. Thus $I^3 = QI^2$ by Lemma 2.1 (5).

Let us assume that $d \geq 2$ and that our assertion holds true for $d-1$. Since the field $k = A/\mathfrak{m}$ is infinite, without loss of generality we may assume that a_1 is a superficial element of I. Let
$$\overline{A} = A/(a_1), \quad \overline{I} = I/(a_1), \text{ and } \quad \overline{Q} = Q/(a_1).$$

We then have $\mathrm{e}_i(\overline{I}) = e_i$ for all $0 \le i \le d-1$, whence

$$\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1.$$

Therefore the hypothesis of induction on d yields $\overline{I}^3 = \overline{Q}\,\overline{I}^2$. Hence, because the element a_1t is a nonzerodivisor on G if $\operatorname{depth} G > 0$, we have $I^3 = QI^2$ in that case.

Assume that $\operatorname{depth} G = 0$. Then, thanks to Sally's technique ([10], [3, Lemma 2.2]), we also have $\operatorname{depth} \mathrm{G}(\overline{I}) = 0$. Hence $\ell_{\overline{A}}(\overline{I}^2/\overline{Q}\,\overline{I}) = d-1$ by Proposition 2.5 (2), because $\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1$. Consequently, $\ell_A(S_1) = \ell_A(I^2/QI) \ge d-1$, because $\overline{I}^2/\overline{Q}\,\overline{I}$ is a homomorphic image of I^2/QI. Let us take an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a}$ is a graded ideal of B. Then, since

$$\ell_A(\mathfrak{a}_1) = \ell_A(S_1) \ge d-1,$$

the ideal $\mathfrak{a}$ contains $d-1$ linearly independent linear forms, say $X_1, X_2, \cdots, X_{d-1}$ of B, which we enlarge to a basis $X_1, \cdots, X_{d-1}, X_d$ of B_1. Hence

$$B = k[X_1, X_2, \cdots, X_d],$$

so that the ideal $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B$ in the polynomial ring

$$B/(X_1, X_2, \cdots, X_{d-1})B = k[X_d]$$

is principal. If $\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1})B$, then $I^3 = QI^2$ by Lemma 2.1 (5), since $S = BS_1$. However, because $\ell_A(I^2/QI) = \ell_A(\mathfrak{a}_1) = d-1$, we have $\operatorname{depth} G \ge 1$ by Proposition 2.5 (2), which is impossible. Therefore $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B \ne (0)$, so that we have

$$\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1}, X_d^{\alpha})B$$

for some $\alpha \ge 1$. Notice that $\alpha = 1$ or $\alpha = 2$ by Lemma 2.1 (4). We must show that $\alpha = 1$.

Assume that $\alpha = 2$. Let us write, for each $1 \le i \le d$, $X_i = \overline{b_it}$ with $b_i \in Q$, where $\overline{b_it}$ denotes the image of $b_it \in T$ in $B = T/\mathfrak{m}T$. Then $\mathfrak{a} = (\overline{b_1t}, \overline{b_2t}, \cdots, \overline{b_{d-1}t}, \overline{(b_dt)^2})$. Notice that

$$Q = (b_1, b_2, \cdots, b_d),$$

because $\{X_i\}_{1 \le i \le d}$ is a k-basis of B_1. We now choose elements $f_i \in S_1$ for $1 \le i \le d-1$ and $f_d \in S_2$ so that $\varphi(f_i) = X_i$ for $1 \le i \le d-1$ and $\varphi(f_d) = X_d^2$. Let $z_i \in I^2$ for $1 \le i \le d-1$ and $z_d \in I^3$ such that $\{f_i\}_{1 \le i \le d-1}$ and f_d are, respectively, the images of $\{z_it\}_{1 \le i \le d-1}$ and z_dt^2 in S. We now consider the relations $X_if_1 = X_1f_i$ in S for $1 \le i \le d-1$ and $X_d^2f_1 = X_1f_d$, that is

$$b_iz_1 - b_1z_i \in Q^2I$$

for $1 \le i \le d-1$ and

$$b_d^2z_1 - b_1z_d \in Q^3I.$$

Notice that

$$Q^3 = b_1Q^2 + (b_2, b_3, \cdots, b_{d-1})^2{\cdot}(b_2, b_3, \cdots, b_d) + b_d^2Q$$

and write

$$b_d^2z_1 - b_1z_d = b_1\tau_1 + \tau_2 + b_d^2\tau_3$$

with $\tau_1 \in Q^2I$, $\tau_2 \in (b_2, b_3, \cdots, b_{d-1})^2 \cdot (b_2, b_3, \cdots, b_d)I$, and $\tau_3 \in QI$. Then

$$b_d^2(z_1 - \tau_3) = b_1(\tau_1 + z_d) + \tau_2 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2.$$

Hence $z_1 - \tau_3 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2$, because the sequence $b_1, b_2, \cdots, b_d$ is A-regular. Let $z_1 - \tau_3 = b_1h + h'$ with $h \in A$ and $h' \in (b_2, b_3, \cdots, b_{d-1})^2$. Then since

$$b_1[b_d^2h - (\tau_1 + z_d)] = \tau_2 - b_d^2h' \in (b_2, b_3, \cdots, b_d)^3,$$

we have $b_d^2h - (\tau_1 + z_d) \in (b_2, b_3, \cdots, b_d)^3$, whence $b_d^2h \in I^3$.

We need the following.

Claim. *$h \notin I$ but $h \in \tilde{I}$. Hence $\tilde{I} \neq I$.*

Proof. If $h \in I$, then $b_1h \in QI$, so that $z_1 = b_1h + h' + \tau_3 \in QI$, whence $f_1 = 0$ in S (cf. Lemma 2.1 (2)), which is impossible. Let $1 \leq i \leq d-1$. Then

$$b_iz_1 - b_1z_i = b_i(b_1h + h' + \tau_3) - b_1z_i = b_1(b_ih - z_i) + b_i(h' + \tau_3) \in Q^2I.$$

Therefore, because $b_i(h' + \tau_3) \in Q^2I$, we get

$$b_1(b_ih - z_i) \in (b_1) \cap Q^2I.$$

Notice that

$$\begin{aligned}
(b_1) \cap Q^2I &= (b_1) \cap [b_1QI + (b_2, b_3, \cdots, b_d)^2I] \\
&= b_1QI + [(b_1) \cap (b_2, b_3, \cdots, b_d)^2I] \\
&= b_1QI + b_1(b_2, b_3, \cdots, b_d)^2 \\
&= b_1QI
\end{aligned}$$

and we have $b_ih - z_i \in QI$, whence $b_ih \in I^2$ for $1 \leq i \leq d-1$. Consequently $b_i^2h \in I^3$ for all $1 \leq i \leq d$, so that $h \in \tilde{I}$, whence $\tilde{I} \neq I$. $\square$

Because $\ell_A(\tilde{I}/I) \geq 1$, we have

$$\begin{aligned}
e_1 &= e_0 - \ell_A(A/I) + 1 \\
&= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) + [1 - \ell_A(\tilde{I}/I)] \\
&\leq \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\
&\leq \mathrm{e}_1(\tilde{I}) \\
&= e_1,
\end{aligned}$$

where $\mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \leq \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 2.3). Hence $\ell_A(\tilde{I}/I) = 1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that

$$\tilde{I} = I + (h) \quad \text{and} \quad \tilde{I}^2 = Q\tilde{I}$$

by Corollary 2.3 (recall that Q is a reduction of $\tilde{I}$ also). We then have, thanks to Proposition 2.6, that $I^3 = QI^2$, which is a required contradiction. This completes the proof of Theorem 1.2 and that of Theorem 3.1 as well. $\square$

4. Consequences

In this section let us review three results of [2, 10, 11] in order to see how our Theorem 1.2 works to prove or improve them. Let us begin with Theorem 1.1.

Proof of Theorem 1.1. Notice that condition (1) (resp. (2)) in Theorem 1.1 is equivalent to condition (3) (resp. (1)) in Theorem 1.2 with $c = 1$. The implication (1) $\Rightarrow$ (3) and the last assertions of Theorem 1.1 are contained in Theorem 1.2. Suppose condition (3) of Theorem 1.1 is satisfied. Then $S = TS_1$ since $I^3 = QI^2$, whence $\mathfrak{m}S = (0)$ and $\mu_B(S) = 1$ because $\ell_A(S_1) = 1$ (recall that $S_1 = I^2/QI$ and $\ell_A(I^2/QI) = 1$). Thus condition (2) in Theorem 1.2 is satisfied. $\square$

The following result is the main result of [2], which is exactly the case $c = 2$ of Theorem 1.2. We would like to refer the reader to [2] for the proof, which can be substantially simplified by Theorem 1.2.

Corollary 4.1 ([2, Theorem 1.2]). *Suppose that $d \geq 2$. Then the following four conditions are equivalent to each other.*

(1) $\mathfrak{m}S = (0)$, $\operatorname{rank}_B S = 1$, *and* $\mu_B(S) = 2$.
(2) *There exists an exact sequence*

$$0 \to B(-2) \to B(-1) \oplus B(-1) \to S \to 0$$

of graded T-modules.
(3) $e_1 = e_0 - \ell_A(A/I) + 1$, $e_2 = 0$, *and* $\operatorname{depth} G \geq d - 2$.
(4) $I^3 = QI^2$, $\ell_A(I^2/QI) = 2$, $\mathfrak{m}I^2 \subseteq QI$, *and* $\ell_A(I^3/Q^2I) < 2d$.

When this is the case, the following assertions hold true

(i) $\operatorname{depth} G = d - 2$.
(ii) $e_3 = -1$, *if* $d \geq 3$.
(iii) $e_i = 0$ *for all* $4 \leq i \leq d$.
(iv) $\ell_A(I^3/Q^2I) = 2d - 1$.

Later we need the following result in Section 5, which is due to [2] and is exactly the case $c = d$ of Theorem 1.2. Here we have deleted from the original statement the superfluous condition that $I^3 = QI^2$ in conditions (2) and (3) (cf. Proposition 2.6 also). We refer the reader to [2] for the proof.

Corollary 4.2 ([2, Corollary 2.6]). *Suppose that $d \geq 2$. Then the following three conditions are equivalent to each other.*

(1) $S \cong B_+$ *as graded T-modules.*
(2) $e_1 = e_0 - \ell_A(A/I) + 1$ *and* $e_i = 0$ *for all* $2 \leq i \leq d$.
(3) $\ell_A(\tilde{I}/I) = 1$ *and* $\tilde{I}^2 = Q\tilde{I}$.

When this is the case, the graded rings G, R, and R' are all Buchsbaum rings with Buchsbaum invariant

$$\mathbb{I}(G) = \mathbb{I}(R) = \mathbb{I}(R') = d.$$

We have learned the following example from Rossi.

Example 4.3. Let A be a 3-dimensional regular local ring and let x, y, z be a regular system of parameters. We put

$$I = (x^2 - y^2, x^2 - z^2, xy, yz, zx) \quad \text{and} \quad Q = (x^2 - y^2, x^2 - z^2, yz).$$

Then $\tilde{I} = \mathfrak{m}^2 = I + (z^2)$ and $\ell_A(\tilde{I}/I) = 1$. Since $\mathfrak{m}^4 = Q\mathfrak{m}^2$, the ideal I satisfies condition (3) in Corollary 4.2, so that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ and $\mathrm{e}_2(I) = \mathrm{e}_3(I) = 0$. The graded rings $G = \mathrm{G}(I)$, $R = \mathrm{R}(I)$, and $R' = \mathrm{R}'(I)$ are all Buchsbaum rings with $\mathbb{I}(G) = \mathbb{I}(R) = \mathbb{I}(R') = 3$.

5. An example

In this section we construct one example which satisfies condition (3) in Theorem 1.2. Our goal is the following.

Theorem 5.1. *Let $0 < c \leq d$ be integers. Then there exists an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

To construct necessary examples we may assume that $c = d$. In fact, suppose that $0 < c < d$ and assume that we have already chosen an $\mathfrak{m}_0$-primary ideal I_0 in a certain Cohen-Macaulay local ring $(A_0, \mathfrak{m}_0)$ such that $c = \dim A_0$, $\mathrm{e}_1(I_0) = \mathrm{e}_0(I_0) - \ell_{A_0}(A_0/I_0) + 1$, and $c = \ell_{A_0}(I_0^2/Q_0 I_0)$ with $Q_0 = (a_1, a_2, \cdots, a_c)A_0$ a reduction of I_0. Let $n = d - c$ and let $A = A_0[[X_1, X_2, \cdots, X_n]]$ be the formal power series ring. We put $I = I_0 A + (X_1, X_2, \cdots, X_n)A$ and $Q = Q_0 A + (X_1, X_2, \cdots, X_n)A$. Then A is a Cohen-Macaulay local ring with $\dim A = \dim A_0 + n = d$ and the maximal ideal $\mathfrak{m} = \mathfrak{m}_0 A + (X_1, X_2, \cdots, X_n)A$. The ideal Q is a reduction of I and because $X_1, X_2, \cdots, X_n$ forms a super regular sequence in A with respect to I (recall that $\mathrm{G}(I) = \mathrm{G}(I_0)[Y_1, Y_2, \cdots, Y_n]$ is the polynomial ring, where Y_i's are the initial forms of X_i's), we have $\mathrm{e}_i(I) = \mathrm{e}_i(I_0)$ $(i = 0, 1)$ and $I^2/QI \cong I_0^2/Q_0 I_0$, whence $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ and $\ell_A(I^2/QI) = c$. This observation allows us to concentrate our attention on the case where $c = d$.

Let m, $d > 0$ be integers. Let

$$U = k[\{X_j\}_{1 \leq j \leq m}, Y, \{V_i\}_{1 \leq i \leq d}, \{Z_i\}_{1 \leq i \leq d}]$$

be the polynomial ring with $m + 2d + 1$ indeterminates over an infinite field k and let

$$\begin{aligned} \mathfrak{a} \;=\; & [(X_j \mid 1 \leq j \leq m) + (Y)] \cdot [(X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)] \\ & + (V_i V_j \mid 1 \leq i, j \leq d,\ i \neq j) + (V_i^2 - Z_i Y \mid 1 \leq i \leq d). \end{aligned}$$

We put $C = U/\mathfrak{a}$ and denote the images of X_j, Y, V_i, and Z_i in C by x_j, y, v_i, and a_i, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{a}} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$. Let $M = C_+ := (x_j \mid 1 \leq j \leq m) + (y) + (v_i \mid 1 \leq i \leq d) + (a_i \mid 1 \leq i \leq d)$ be the graded maximal ideal in C. Let Λ be a subset of $\{1, 2, \cdots, m\}$. We put

$$J = (a_i \mid 1 \leq i \leq d) + (x_\alpha \mid \alpha \in \Lambda) + (v_i \mid 1 \leq i \leq d) \quad \text{and} \quad \mathfrak{q} = (a_i \mid 1 \leq i \leq d).$$

Then $M^2 = \mathfrak{q}M$, $J^2 = \mathfrak{q}J + \mathfrak{q}y$, and $J^3 = \mathfrak{q}J^2$, whence $\mathfrak{q}$ is a reduction of both M and J, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $A = C_M$, $I = JA$, and $Q = \mathfrak{q}A$. We are now interested in the Hilbert coefficients e_i's of the ideal I as well as the structure of the associated graded ring and the Sally module of I. Let us maintain the same notation as in the previous sections. We then have the following, which shows that the ideal I is a required example.

Theorem 5.2. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with $\dim A = d$.*
(2) *$S \cong B_+$ as graded T-modules, whence $\ell_A(I^2/QI) = d$.*
(3) *$\mathrm{e}_0(I) = m + d + 2$ and $\mathrm{e}_1(I) = \sharp\Lambda + d + 1$.*
(4) *$\mathrm{e}_i(I) = 0$ for all $2 \leq i \leq d$.*
(5) *G is a Buchsbaum ring with $\operatorname{depth} G = 0$ and $\mathbb{I}(G) = d$.*

We divide the proof of Theorem 5.2 into a few steps. Let us begin with the following.

Proposition 5.3. *Let $\mathfrak{p} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$ in U. Then $\ell_{C_\mathfrak{p}}(C_\mathfrak{p}) = m + d + 2$.*

Proof. Let $\widetilde{U} = U[\{\frac{1}{Z_i}\}_{1 \leq i \leq d}]$ and put $\widetilde{k} = k[\{Z_i\}_{1 \leq i \leq d}, \{\frac{1}{Z_i}\}_{1 \leq i \leq d}]$ in $\tilde{U}$. Let $X_j' = \frac{X_j}{Z_1}$ $(1 \leq j \leq m)$, $V_i' = \frac{V_i}{Z_1}$ $(1 \leq i \leq d)$, and $Y' = \frac{Y}{Z_1}$. Then $\{X_j'\}_{1 \leq j \leq m}$, Y', and $\{V_i'\}_{1 \leq i \leq d}$ are algebraically independent over $\widetilde{k}$,

$$\widetilde{U} = \widetilde{k}[\{X_j'\}_{1 \leq j \leq m}, Y', \{V_i'\}_{1 \leq i \leq d}], \quad \text{and}$$

$$\begin{aligned} \mathfrak{a}\widetilde{U} &= [(X_j' \mid 1 \leq j \leq m) + (Y')]{\cdot}[(X_j' \mid 1 \leq j \leq m) + (Y') + (V_i' \mid 1 \leq i \leq d)] \\ &+ (V_i'V_j' \mid 1 \leq i, j \leq d,\, i \neq j) + (\frac{Z_1}{Z_i}{V_i'}^2 - Y' \mid 1 \leq i \leq d). \end{aligned}$$

Let $W = \widetilde{k}[\{X_j'\}_{1 \leq j \leq m}, \{V_i'\}_{1 \leq i \leq d}]$ in $\widetilde{U}$ and

$$\begin{aligned} \mathfrak{b} &= [(X_j' \mid 1 \leq j \leq m) + ({V_1'}^2)]{\cdot}[(X_j' \mid 1 \leq j \leq m) + (V_i' \mid 1 \leq i \leq d)] \\ &+ (V_i'V_j' \mid 1 \leq i, j \leq d,\, i \neq j) + (\frac{Z_1}{Z_i}{V_i'}^2 - {V_1'}^2 \mid 2 \leq i \leq d) \end{aligned}$$

in W. Then, substituting Y' with ${V_1'}^2$ in $\tilde{U}$, we get the isomorphism

$$\widetilde{U}/\mathfrak{a}\widetilde{U} \cong \overline{U} := W/\mathfrak{b}$$

of $\tilde{k}$-algebras, under which the prime ideal $\mathfrak{p}\widetilde{U}/\mathfrak{a}\widetilde{U}$ corresponds to the prime ideal $P/\mathfrak{b}$ of $\overline{U}$, where $P = W_+ := (X_j' \mid 1 \leq j \leq m) + (V_i' \mid 1 \leq i \leq d)$. Then, since $\mathfrak{b} + ({V_1'}^2) = P^2$ and

$$\ell_{W_P}([\mathfrak{b} + ({V_1'}^2)]W_P/\mathfrak{b}W_P) = 1,$$

we get

$$\begin{aligned} \ell_{\overline{U}_P}(\overline{U}_P) &= \ell_{W_P}(W_P/P^2W_P) + \ell_{W_P}([\mathfrak{b} + ({V_1'}^2)]W_P/\mathfrak{b}W_P) \\ &= (m + d + 1) + 1 \\ &= m + d + 2. \end{aligned}$$

Thus $\ell_{C_\mathfrak{p}}(C_\mathfrak{p}) = \ell_{\overline{U}_P}(\overline{U}_P) = m + d + 2$. $\square$

We have by the associative formula of multiplicity that

$$\mathrm{e}_0(\mathfrak{q}) = \ell_{C_\mathfrak{p}}(C_\mathfrak{p}) \cdot \mathrm{e}_0^{C/\mathfrak{p}C}([\mathfrak{q} + \mathfrak{p}C]/\mathfrak{p}C) = m + d + 2,$$

because $\mathfrak{p} = \sqrt{\mathfrak{a}}$ and $C/\mathfrak{p}C = U/\mathfrak{p} = k[Z_i \mid 1 \le i \le d]$. On the other hand, we have $\ell_C(C/\mathfrak{q}) = m + d + 2$, since

$$C/\mathfrak{q} = k[\{X_j\}_{1 \le j \le m}, Y, \{V_i\}_{1 \le i \le d}]/\mathfrak{c}^2$$

where

$$\mathfrak{c} = (X_j \mid 1 \le j \le m) + (Y) + (V_i \mid 1 \le i \le d).$$

Thus $\mathrm{e}_0(\mathfrak{q}) = \ell_C(C/\mathfrak{q})$, so that C is a Cohen-Macaulay ring and $\mathrm{e}_0(\mathfrak{q}) = m + d + 2$.

Proposition 5.4. *$\ell_C(\tilde{J}/J) = 1$ and $\tilde{J}^2 = \mathfrak{q}\tilde{J}$.*

Proof. Let $K = J + (y)$. Then $\ell_C(K/J) = 1$ and $K^2 = \mathfrak{q}K = J^2$. Hence $\tilde{K} = K$ because $K^2 = \mathfrak{q}K$, while we have $\tilde{K} = \tilde{J}$ because $K^2 = J^2$. Thus the assertions follow. $\square$

We are now in a position to finish the proof of Theorem 5.2.

Proof of Theorem 5.2. Since $\tilde{I} = \tilde{J}A$, by Proposition 5.4 we get $\ell_A(\tilde{I}/I) = 1$ and $\tilde{I}^2 = Q\tilde{I}$. Hence by Corollary 4.2 $S \cong B_+$ as graded T-modules, so that

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$$

and $\mathrm{e}_i(I) = 0$ for all $2 \le i \le d$. We have $\mathrm{e}_1(I) = \sharp\Lambda + d + 1$, because $\ell_A(A/I) = m - \sharp\Lambda + 2$ and $\mathrm{e}_0(I) = \mathrm{e}_0(\mathfrak{q}) = m + d + 2$. The ring $G = \mathrm{G}(I)$ is a Buchsbaum ring with $\operatorname{depth} G = 0$ and $\mathbb{I}(G) = d$ by Corollary 4.2, which completes the proof of Theorem 5.2. $\square$

References

[1] A. Corso, C. Polini, and M. Vaz Pinto, *Sally modules and associated graded rings*, Comm. Algebra **26** (1998) 2689–2708.

[2] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*. Michigan Math. J. (to appear).

[3] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. **56** (1997) 64–76.

[4] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987) 293–318.

[5] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. **35** (1960) 209–214.

[6] L. J. Ratliff, Jr. and D. E. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J. **27** (1978) 929–934.

[7] M. E. Rossi, *A bound on the reduction number of a primary ideal*, Proc. Amer. Math. Soc. **128** (2000) 1325–1332.

[8] J. D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. Algebra **56** (1979) 168–183.

[9] ———, *Tangent cones at Gorenstein singularities*, Composito Math. **40** (1980) 167–175.

[10] ———, *Hilbert coefficients and reduction number 2*, J. Alg. Geo. and Sing. **1** (1992) 325–333.

[11] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics **159** (1994) 410–422.

[12] M. Vaz Pinto, *Hilbert functions and Sally modules*, J. Algebra **192** (1977) 504–523.

[13] H.-J. Wang, *Links of symbolic powers of prime ideals*, Math. Z. **256** (2007) 749–756.

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: goto@math.meiji.ac.jp

Department of Mathematics and Informatics, Graduate School of Science and Technology, Chiba University, 1-33 Yatoi-cho, Inage-ku, Chiba-shi, 263 Japan
E-mail address: nishida@math.s.chiba-u.ac.jp

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
E-mail address: kozeki@math.meiji.ac.jp