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ASYMPTOTIC VANISHING CONDITIONS WHICH FORCE REGULARITY IN LOCAL RINGS OF PRIME CHARACTERISTIC

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Dedicated to Mel Hochster on the occasion of his 65th birthday

Abstract. Let $(R, \mathfrak{m}, k)$ be a local (Noetherian) ring of positive prime characteristic p and dimension d. Let $G_\bullet$ be a minimal resolution of the residue field k, and for each $i \geq 0$, let $\mathfrak{t}_i(R) = \lim_{e\to\infty} \lambda(H_i(F^e(G_\bullet)))/p^{ed}$. We show that if $\mathfrak{t}_i(R) = 0$ for some $i > 0$, then R is a regular local ring. Using the same method, we are also able to show that if R is an excellent local domain and $\mathrm{Tor}_i^R(k, R^+) = 0$ for some $i > 0$, then R is regular (where R^+ is the absolute integral closure of R). Both of the two results were previously known only for $i = 1$ or 2 via completely different methods.

1. Introduction

Throughout this paper, we assume $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring of characteristic $p > 0$ and dimension d. The Frobenius endomorphism $f_R : R \to R$ is defined by $f_R(r) = r^p$ for $r \in R$. Each iteration f_R^e defines a new R-module structure on R, denoted ${}^{f^e}\!R$, for which $a \cdot b = a^{p^e} b$. For any R-module M, $F_R^e(M)$ stands for $M \otimes_R {}^{f^e}\!R$, the R-module structure of which is given by base change along the Frobenius endomorphism. When M is a cyclic module R/I, it is easy to show that $F^e(R/I) \cong R/I^{[p^e]}$, where $I^{[p^e]}$ denotes the ideal generated by the p^e-th power of the generators of I.

In the sequel, $\lambda(-)$ denotes the length function. q usually denotes a varying power p^e.

In [6], the second author introduced the following higher Tor counterparts for the Hilbert-Kunz multiplicity

$$\mathfrak{t}_i(R) = \lim_{q\to\infty} \lambda(\mathrm{Tor}_i(k, {}^{f^e}\!R))/q^d \tag{1.1}$$

and showed that R is regular if and only if $\mathfrak{t}_i(R) = 0$ for $i = 1$ or $i = 2$. In another paper [2], (which extends results obtained in [10]), the first author proved that an excellent local domain R is regular if and only if $\mathrm{Tor}_1^R(k, R^+) = 0$, where R^+ is the absolute integral closure of R (i.e, the integral closure of R in the algebraic closure of its field of fraction). It is not difficult to see that this is also equivalent to

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$\mathrm{Tor}_2^R(k, R^+) = 0$. In fact, since R^+ is a big Cohen-Macaulay algebra [5], by Proposition 2.5 of [10], the condition $\mathrm{Tor}_2^R(k, R^+) = 0$ forces R to be Cohen-Macaulay. Therefore $\mathrm{Tor}_1^R(R/(\mathbf{x}), R^+) = 0$ for any system of parameters $\mathbf{x}$ for R. $R/(\mathbf{x})$ has a filtration by k. Tensoring the short exact sequences in the filtration by R^+, from the resulting long exact sequences and the condition $\mathrm{Tor}_2^R(k, R^+) = 0$, one gets $\mathrm{Tor}_1^R(k, R^+) = 0$. The methods used in [6] and [2] are completely different. However, neither of them works for the case $i \geq 3$ (unless R is assumed to be Cohen-Macaulay).

The main result of this paper, Theorem 3.1 below, states that over an equidimensional complete local ring $(R, \mathfrak{m}, k)$ of prime characteristic, if a finitely generated free resolution of k has *stably phantom homology* at the i-th spot for some $i > 0$, then R is regular. As we observe in Proposition 2.6, under mild conditions, a complex of finitely generated free modules with finite length homology has stably phantom homology at some spot if and only if a similar asymptotic length as defined in (1.1) vanishes at the same spot. Consequently, Theorem 3.1 allows us to simultaneously extend the main results in [6] and [2] to all $i > 0$. Specifically, we have

Corollary 3.2 Let R be a local ring of characteristic p. If $\mathfrak{t}_i(R) = 0$ for some $i > 0$ then R is regular.

Corollary 3.5 Let $(R, \mathfrak{m}, k)$ be an excellent local domain. If $\mathrm{Tor}_i^R(k, R^+) = 0$ for some $i > 0$ then R is regular.

As a further consequence of Corollary 3.2, we give an extended version (in the prime characteristic case) of a criterion of regular local rings due to Bridgeland and Iyengar [3].

2. Asymptotic vanishing and stably phantom homology

We recall some basic definitions in tight closure theory. The reader should refer to [4] for details. Let R^o denote the complement of the union of the minimal primes of R. Let $N \subseteq M$ be two R-modules. We write $N_M^{[q]}$ for $\ker(F^e(M) \to F^e(M/N))$. We say that $x \in M$ is in the *tight closure* of N, N^*, if there exist $c \in R^o$ and an integer q_0 such that for all $q \geq q_0$, $cx^q \in N_M^{[q]}$.

We say that $c \in R^o$ is a q_0-*weak test element* (or simply a weak test element) if for every finitely generated module M and submodule N, $x \in M$ is in N^* if and only if $cx^q \in N_M^{[q]}$ for all $q \geq q_0$. If this holds with $q_0 = 1$,we call c a *test element*.

Let $(G_\bullet, \partial_\bullet)$ be a complex over R. We say that $G_\bullet$ has *phantom homology* at the ith spot (or simply, $G_\bullet$ is phantom at the ith spot), if $\ker \partial_i \subseteq (\mathrm{im}\, \partial_{i+1})^*_{G_i}$. We say that $G_\bullet$ has *stably phantom homology* at the ith spot (or simply, $G_\bullet$ is stably phantom at the ith spot) if $F^e(G_\bullet)$ has phantom homology at the ith spot for all $e \geq 0$.

We say a complex $G_\bullet$ is a *left complex* if $G_i = 0$ for $i < 0$. A left complex $G_\bullet$ of projective modules is call a *phantom resolution* of M if $H_0(G_\bullet) \cong M$ and $G_\bullet$ has phantom homology for all $i > 0$. If such a complex $G_\bullet$ is finite, we say $G_\bullet$ is a *finite phantom resolution* of M.

By a *resolution* of a module M, we always mean a resolution of M by finitely generated free modules. It is easy to check that for a given module M, whether a resolution of M has stably phantom homology at the ith spot is independent of the choice of the resolution. We will use this fact many times without explicitly mentioning it.

The following two lemmas are easy consequences of the above definitions, we leave them for the reader to verify.

Lemma 2.1. *Let $G_\bullet$ be a complex of finitely generated R-modules.*

(1) *If for some $d \in R^o$ (d need not be a test element), $dH_i(F^e_R(G_\bullet)) = 0$ for all $e \geq 0$, then $G_\bullet$ has stably phantom homology at the ith spot.*
(2) *Suppose R admits a test element c. If $G_\bullet$ has phantom homology at the ith spot for some i, then $cH_i(G_\bullet) = 0$. In particular, if $G_\bullet$ has stably phantom homology at the ith spot for some i, then $cH_i(F^e(G_\bullet)) = 0$ for all $e \geq 0$.*

Lemma 2.2. *Suppose R admits a test element. Let $\alpha : F_\bullet \to G_\bullet$ be a chain map between two complexes $F_\bullet$ and $G_\bullet$ over R. If both $F_\bullet$ and the mapping cone of α have stably phantom homology at the ith spot, then so does $G_\bullet$.*

We will make ample use of one of the main results of Seibert's paper [11]:

Proposition 2.3 ([11], Propostion 1)**.** *Let R be a local ring of characteristic p. Let $G_\bullet$ be a left complex of finitely generated free R-modules, and N a finitely generated R module such that $G_\bullet \otimes_R N$ has homology of finite length. Let $t = \dim N$. Then for each $i \geq 0$ there exists $c_i \in \mathbb{R}$ such that*

$$\lambda\left((H_i(F^e_R(G_\bullet) \otimes_R N)\right) = c_i q^t + O(q^{t-1}).$$

We need a version of [4], Theorem 8.17 for equidimensional rings that are not reduced.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a complete equidimensional local ring of dimension d. Suppose that $N \subseteq W \subseteq M$ are finitely generated modules such that W/N has finite length. Then $W \subseteq N^*_M$ if and only if $\lim_{q\to\infty} \lambda(W^{[q]}_M / N^{[q]}_M)/q^d = 0$.*

Proof. The implication $\Rightarrow$ follows from [4], Theorem 8.17(a).

Let $J = \sqrt{0}$ be the nilradical, and let $\overline{R} = R/J$. By mapping a free module onto M and taking preimages, there is no loss of generality in assuming that M is free. Clearly, for each q, $\lambda\left(\dfrac{W^{[q]}_M + JM}{N^{[q]}_M + JM}\right) \leq \lambda\left(\dfrac{W^{[q]}_M}{N^{[q]}_M}\right)$, so by [4], Theorem8.17(b), $W + JM$ is in the tight closure of $N + JM$ computed over $\overline{R}$. This implies that $W \subseteq N^*_M$ when computed over R (see [4], Proposition 8.5(j)). This shows $\Leftarrow$. $\square$

Definition 2.5. Let R be a local ring of dimension d. Let I be the intersection of the primary components of (0) which have dimension d. Then $R^{eq} := R/I$.

Proposition 2.6. *Let R be a local ring of dimension d, and $G_\bullet$ a left complex of finitely generated free modules with finite length homology. Then for $i \geq 0$*

$$\lim_{q\to\infty} \frac{\lambda\left(H_i(F^e_R(G_\bullet))\right)}{q^d} = \lim_{q\to\infty} \frac{\lambda\left(H_i(F^e_{R^{eq}}(G_\bullet \otimes_R R^{eq}))\right)}{q^d}.$$

In the case that either is 0 (in which case both are), $G_\bullet \otimes_R R^{eq}$ has stably phantom homology at the ith spot (over R^{eq}). The converse is also true when R^{eq} admits a test element c that is regular on R^{eq} (e.g, R is excellent, reduced and unmixed).

Proof. Let I be as in the definition of R^{eq}. Let J be the intersection of primary components of (0) of dimension strictly less than d, so $(0) = I \cap J$ and $\dim(R/J) < d$. There is then an exact sequence

$$0 \to R \to R^{eq} \oplus R/J \to R/(I+J) \to 0 \tag{2.1}$$

(and we note that $\dim(R/(I+J)) < d$).

By Proposition 2.3, for all j,

$$\lim_{q\to\infty} \frac{\lambda\left(H_j\left(F^e(G_\bullet)\otimes_R R/J\right)\right)}{q^d} = \lim_{q\to\infty} \frac{\lambda\left(H_j\left(F^e(G_\bullet)\otimes_R R/(I+J)\right)\right)}{q^d} = 0. \tag{2.2}$$

If we tensor the short exact sequence 2.1 with $F^e(G_\bullet)$, we get a short exact sequence of complexes. The resulting long exact sequence in homology and the observations in equation(2.2) give the desired equation.

Suppose now that the given limit is 0. Without loss of generality, we assume $R = R^{eq}$. Let ∂_i denote the map from G_i to G_{i-1} in the complex $G_\bullet$. It is easy to check that $(\ker \partial_i)^{[q]}_{G_i} \subseteq \ker(\partial_i^{[q]})$ and $(\operatorname{im} \partial_{i+1})^{[q]}_{G_i} = \operatorname{im}(\partial_{i+1}^{[q]})$. So Lemma 2.4 shows that $G_\bullet$ has stably phantom homology at the ith spot over R^{eq}.

Conversely, assume R is equidimensional with a test element c. Suppose $G_\bullet$ has stably phantom homology at the ith spot. Then $cH_i(F^e_R(G_\bullet)) = 0$ for all $e \geq 0$. Since c is regular on R, one has an embedding $H_i(F^e_R(G_\bullet)) \otimes R/cR \hookrightarrow H_i(F^e_{R/cR}(G_\bullet \otimes (R/cR)))$, i.e., $H_i(F^e_R(G_\bullet)) \hookrightarrow H_i(F^e_{R/cR}(G_\bullet \otimes (R/cR)))$ for all $e \geq 0$. Thus

$$\lim_{q\to\infty} \frac{\lambda\left(H_i(F^e_R(G_\bullet))\right)}{q^d} \leq \lim_{q\to\infty} \frac{\lambda\left(H_i(F^e_{R/cR}(G_\bullet \otimes (R/cR)))\right)}{q^d} = 0 \tag{2.3}$$

The equality on the right hand side follows from Proposition 2.3 again. □

3. The main theorem

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be an excellent local ring of characteristic p. Let $G_\bullet$ be a resolution of k. If for some $i > 0$ the complex $G_\bullet \otimes_R R^{eq}$ is stably phantom at the ith spot, then R is regular.*

Proof. Write $(0) = I \cap J$ as in Proposition 2.6. Let $c \in R$ be an element which is a q_0-weak test element in R^{eq}, and let $c_1 \in J$ but not in any minimal prime of I. Set $d = cc_1$. We claim that if M is any module of finite length, $(P_\bullet, \beta_\bullet)$ is a resolution for M, and $w \in \ker(\beta_i^{[q]})$, then $dw^{q_0} \in \operatorname{im}(\beta_{i+1}^{[qq_0]})$. We can induce on the length of the module M, with $\lambda(M) = 1$ known. If $\lambda(M) > 1$ take a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ with $1 \leq \lambda(M_1) < \lambda(M)$. Let $G_\bullet$ be a resolution of M_1. Then there is a chain map $G_\bullet \to P_\bullet$, and the mapping cone $T_\bullet$ is a resolution of M_2. By the induction hypothesis and the observation above, $G_\bullet \otimes_R R^{eq}$ and $T_\bullet \otimes_R R^{eq}$ are stably phantom over R^{eq} at the ith spot. Then Lemma 2.2 shows $P_\bullet \otimes_R R^{eq}$ is stably phantom over R^{eq} at the ith spot. Thus, if $w \in \ker(\beta_i^{[q]})$, then $cw^{q_0} \in \operatorname{im}(\beta_{i+1}^{[qq_0]}) + IP_i$, and $c_1 I = 0$, so $dw^{q_0} = c_1 c w^{q_0} \in \operatorname{im}(\beta_{i+1}^{[qq_0]})$.

We next wish to show that the claim above is true for any finitely generated module M. We can induce on $\dim M$, with the case $\dim M = 0$ done. Assume that $\dim M > 0$. The same mapping cone argument as above applied to $0 \to H^0_{\mathfrak{m}}(M) \to M \to M/H^0_{\mathfrak{m}}(M) \to 0$ shows that we may assume that $\operatorname{depth}_{\mathfrak{m}} M > 0$. Let $x \in \mathfrak{m}$

be a nonzerodivisor on M (so $\dim(M/xM) < \dim M$). The mapping cone argument for the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ gives an exact sequence $H_i(F^e(P_\bullet)) \xrightarrow{x^q} H_i(F^e(P_\bullet)) \to H_i(F^e(T_\bullet))$. Let $w \in \ker(\beta_i^{[q]})$. Its image in $F^e(T_\bullet)$ gives a homology element, which by the induction argument, vanishes when raised to the q_0 power and multiplied by d. Equivalently, $dw^{q_0} \in \operatorname{im}(\beta_{i+1}^{[qq_0]}) + x^{qq_0}P_i$. However, the argument still applies for any power of x, so by the Krull intersection theorem, $dw^{q_0} \in \operatorname{im}(\beta_{i+1}^{[qq_0]})$, as desired.

A consequence of the above argument is that if M is any finitely generated R-module, and $(P_\bullet, \beta_\bullet)$ is a resolution of M, then $P_\bullet \otimes_R R^{eq}$ is stably phantom at the ith spot. But for any $j \geq 0$, the homology at the $i+j$th spot is the ith homology of the jth syzygy of M, and therefore, $P_\bullet \otimes_R R^{eq}$ is stably phantom at all spots above i as well. By Theorem 2.1.7 of [1], $(\operatorname{syz}_{i-1} M)/I(\operatorname{syz}_{i-1} M)$ has a finite phantom resolution over R^{eq}, and therefore, if we take $P_\bullet$ to be minimal (so $P_\bullet \otimes R^{eq}$ is minimal), we see that $P_\bullet$ is bounded on the left. But this means $\operatorname{pd}_R(\operatorname{syz}_{i-1} M) < \infty$, and hence $\operatorname{pd}_R M < \infty$. Hence R is regular. □

Corollary 3.2. *Let R be a local ring of characteristic p. If $\mathfrak{t}_i(R) = 0$ for some $i > 0$ then R is a regular local ring.*

Proof. The hypothesis is stable under completion, and R is regular if and only if $\widehat{R}$ is, so we may assume that R is complete.

By Proposition 2.6, $G_\bullet \otimes R^{eq}$ has stably phantom homology at the ith spot. Thus, by Proposition 3.1, R is regular. □

Remark 3.3. Regarding Theorem 3.1, one should not expect more generally that if a resolution of an arbitrary module M has stably phantom homology at the ith spot for some $i > 0$, then $\operatorname{pd} M < \infty$. Although this is the case when M is of finite length over a local complete intersection [8], it is not true in general, even over Gorenstein rings. The reader should refer to [7] for an explicit example.

As an immediate consequence of 3.1, we have the following improved version of a theorem due to Bridgeland and Iyengar in the characteristic p case,

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional local ring in characteristic $p > 0$. Assume $C_\bullet$ is a complex of free R-modules with $C_i = 0$ for $i \notin [0, d]$, the R-module $H_0(C_\bullet)$ is finitely generated, and $\lambda(H_i(C_\bullet)) < \infty$ for $i > 0$. If k or any syzygy of k is a direct summand of $H_0(C_\bullet)$, then R is regular.*

Proof. The proof is the same as that of Theorem 3.1 in [6] □

Since the original theorem of Bridgeland and Iyengar holds true in the equicharacteristic case, we expect Theorem 3.4 also holds in equicharacteristic case. But we do not have a proof.

As another corollary of Theorem 3.1, we may also characterize regularity for excellent local domains via vanishing of higher Tor's of the residue field with R^+.

Corollary 3.5. *Let $(R, \mathfrak{m}, k)$ be an excellent domain. If $\operatorname{Tor}_i^R(k, R^+) = 0$ for some $i > 0$ then R is regular.*

Proof. Let $(G_\bullet, \alpha_\bullet)$ be a minimal free resolution of k. It suffices to show that $G_\bullet$ is stably phantom at the ith spot, so that Theorem 3.1 applies.

Suppose $w \in \ker(\alpha_i^{[q]})$. Then $w^{1/q} \in \ker(\alpha_i \otimes_R R^+) = \operatorname{im}(\alpha_{i+1} \otimes_R R^+)$ (the second equality holds since $\operatorname{Tor}_i^R(k, R^+) = 0$). This can be expressed using only finitely many elements of R^+, so there is a module finite extension $S \supseteq R$ such that $w \in \operatorname{im}(\alpha_{i+1}^{[q]} \otimes_R S) \cap G_i \subseteq \operatorname{im}(\alpha_{i+1}^{[q]})^*_{G_i}$, as desired. □

Remark 3.6. By exactly the same argument, one can also show that for a reduced, excellent and equidimensional local ring R, if $\operatorname{Tor}_i^R(k, R^\infty) = 0$ for some $i > 0$, then R is regular, where R^∞ denotes $\cup_q R^{1/q}$.

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