THE JACOBIAN IDEAL OF A HYPERPLANE ARRANGEMENT

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ABSTRACT. The Jacobian ideal of a hyperplane arrangement is an ideal in the polynomial ring whose generators are the partial derivatives of the arrangements defining polynomial. In this article, we prove that an arrangement can be reconstructed from its Jacobian ideal.

1. Introduction

Let $V \cong \mathbb{C}^\ell$ and choose coordinates for V^* such that we can identify the symmetric algebra $S = S(V^*)$ with the polynomial ring $\mathbb{C}[z_1, \ldots, z_\ell]$. A hyperplane in V is a codimension one affine space in V. A hyperplane arrangement in V is a finite collection of hyperplanes denoted by $\mathcal{A}$. When all the hyperplanes of an arrangement contain the origin we say the arrangement is central. For most of this note we assume the arrangement is central. In this case we can 'projectivize' all the hyperplanes and view the arrangement as an arrangement of hyperplanes in $\mathbb{CP}^{\ell-1}$. Further, we say a central arrangement $\mathcal{A}$ is essential if $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

For each $H \in \mathcal{A}$ choose a linear polynomial $\alpha_H \in S$ such that $H = \ker \alpha_H$. Let $Q = \prod_{H \in \mathcal{A}} \alpha_H$ denote the defining polynomial of the arrangement $\mathcal{A}$. Then the main character of this note is the homogeneous ideal in S defined by

$$J(Q) := \left(\frac{\partial Q}{\partial z_1}, \ldots, \frac{\partial Q}{\partial z_\ell} \right).$$

We call this ideal the Jacobian ideal and sometimes denote it by $J(\mathcal{A})$. The Jacobian ideal determines a closed subscheme $\operatorname{Proj} S/J(Q)$ of the projective space $\mathbb{CP}^{\ell-1}$, which we call the Jacobian scheme.

The purpose of this paper is to prove the following result, which simply put, states that the Jacobian scheme contains all the information of the arrangement. We say two hyperplane arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *identical* when $Q_1 = cQ_2$ for some $c \in \mathbb{C}^*$ where Q_1 and Q_2 are the defining polynomials respectively.

Theorem 1.1. *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two central and essential arrangements in dimension $\ell \geq 3$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are identical if and only if the Jacobian schemes $\operatorname{Proj} S/J(\mathcal{A}_1)$ and $\operatorname{Proj} S/J(\mathcal{A}_2)$ are equal as closed subschemes of $\mathbb{CP}^{\ell-1}$.*

The proof of Theorem 1.1 is inspired by a Torelli-type theorem of Dolgachev and Kapranov [2, 1]. In [2], Dolgachev and Kapranov prove that the module $D(\mathcal{A})$ of

derivations of a generic arrangement $\mathcal{A}$ contains all the information of the arrangement. More precisely, they consider the set of jumping lines of the torsion free (actually locally free when $\mathcal{A}$ is a generic arrangement [5, 8]) sheaf $\widetilde{D}(\mathcal{A})$ on the projective space. From the set of jumping lines, the arrangement $\mathcal{A}$ can be recovered. Then in [1], by considering a certain subsheaf of $\widetilde{D}(\mathcal{A})$, Dolgachev extended these results to a wider class of arrangements.

Instead of jumping lines, we consider the subscheme obtained as the intersection $\overline{K} \cap \operatorname{Proj} S/J(\mathcal{A}) \subset \mathbb{CP}^{\ell-1}$, for a given hyperplane $K \subset V$. In particular, we focus on the $(\ell-3)$-dimensional components of $\overline{K} \cap \operatorname{Proj} S/J(\mathcal{A})$ and compute the $(\ell-3)$-dimensional degree (i.e. the coefficient of the $\ell-3$ term in the Hilbert polynomial). Then we can prove that $K \in \mathcal{A}$ precisely when this degree is maximized. We also note that the reduced Jacobian scheme $\operatorname{Proj} S/\sqrt{J(Q)}$ does not contain all the information of $\mathcal{A}$ (see Remark 4.2).

Another closely related result is found in [3, Prop. 1.1]. Let $f \in S_d$ be a homogeneous polynomial of degree d. Then Donagi proved that the Jacobian ideal $J(f)$ recovers f up to PGL-action. Our main result in this paper strengthens this assertion for hyperplane arrangements, namely, the saturated Jacobian ideal $\operatorname{Sat}(J(Q))$ recovers the defining equation Q up to constant multiple.

2. Minimal Components of $J(Q)$

In this section, we will study the minimal primary components of the Jacobian ideal of the arrangement $\mathcal{A}$.

Throughout this paper we use the following notation. Let $L(\mathcal{A})$ be the intersection lattice of $\mathcal{A}$ which is the set of all intersections of elements from $\mathcal{A}$ with the order being reverse inclusion. Moreover, let $L_k(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \operatorname{codim}(X) = k\}$. For $X \in L(\mathcal{A})$ let $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ and $L(\mathcal{A})_X = \{Y \in L(\mathcal{A}) \mid X \subsetneq Y\}$. Then we define the Möbius function μ on $L(\mathcal{A})$ by setting $\mu(V) = 1$ and the recursive formula:

$$\mu(X) = - \sum_{Y \in L(\mathcal{A})_X} \mu(Y).$$

We assume the dimension $\ell \geq 3$. For given an intersection $X \in L(\mathcal{A})$, put

$$Q_X \quad = \quad \prod_{H \in \mathcal{A}_X} \alpha_H,$$

$$\overline{Q}_X \quad = \quad \frac{Q}{Q_X} = \prod_{X \not\subseteq H} \alpha_H.$$

Obviously $Q = Q_X \overline{Q}_X$. Let us denote $I(X) := \sum_{H \in \mathcal{A}_X} S\alpha_H$ the prime ideal representing X.

Since the Jacobian ideal $J(Q)$ determines the singular loci of the union $\bigcup_{H \in \mathcal{A}} H$ of hyperplanes, we have

$$\sqrt{J(Q)} = \bigcap_{X \in L_2(\mathcal{A})} I(X).$$

This implies that the set of minimal associated primes of $J(Q)$ is $\{I(X) \mid X \in L_2(\mathcal{A})\}$. The localization technique enables us to obtain the corresponding minimal primary components as follows.

Lemma 2.1. *The set of minimal components of the Jacobian ideal $J(Q)$ is equal to $\{J(Q_X) \mid X \in L_2(\mathcal{A})\}$.*

Remark 2.2. Generally, the Jacobian ideal $J(Q)$ has a lot of embedded primes. If $\mathcal{A}$ is a free arrangement, then $S/J(Q)$ is known to be Cohen-Macaulay [7]. In this case, $J(Q)$ has no embedded primes. Thus we have the primary decomposition $J(Q) = \bigcap_{X \in L_2(\mathcal{A})} J(Q(X))$, [7].

Remark 2.3. The degree of the ideal $J(Q_X)$ is

$$\deg J(Q_X) = \mu(X)^2 = (|\mathcal{A}_X| - 1)^2.$$

Hence the degree of the Jacobian ideal $J(Q)$ is $\sum_{X \in L_2(\mathcal{A})} \mu(X)^2$. For details see [6, Theorem 2.5].

3. $\operatorname{Proj} S/J(Q)$ **intersected with a hyperplane**

Fix a hyperplane $K = \{\beta = 0\}$ that is not necessarily in $\mathcal{A}$. In this section, we consider the codimension two components of $\overline{K} \cap \operatorname{Proj} S/J(Q) = \operatorname{Proj} S/(J(Q) + (\beta))$ in $\mathbb{CP}^{\ell-1}$. In particular, we compute its degree in terms of the Möbius function.

The essential part of the computation is the following 2-dimensional case.

Lemma 3.1. *Let $Q(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \ldots + a_n z_2^n \in \mathbb{C}[z_1, z_2]$ be a non-zero degree n homogeneous polynomial of two variables. Suppose $\{Q = 0\}$ defines a distinct n lines. Then $J(Q) + (z_2) = (z_1^{n-1}, z_2)$ and*

$$\dim \mathbb{C}[z_1, z_2]/(J(Q) + (z_2)) = n - 1.$$

Recall that if $I \subset S$ is a homogeneous ideal and assume $\dim \operatorname{Proj} S/I \leq m$, then the Hilbert polynomial is of the form

$$\operatorname{HP}(S/I, d) = \frac{a_m}{m!} d^m + \frac{a_{m-1}}{(m-1)!} d^{m-1} + \cdots.$$

Let us denote the coefficient a_m by $\deg_{(m)} \operatorname{Proj} S/I$, which depends only on the m-dimensional components of the closed subscheme $\operatorname{Proj} S/I \subset \mathbb{CP}^{\ell-1}$. By definition, if $\dim \operatorname{Proj} S/I < m$, then $\deg_{(m)} \operatorname{Proj} S/I = 0$.

Lemma 3.2. *For any arrangement $\mathcal{A}$ with defining polynomial Q and any hyperplane $K = \{\beta = 0\}$, we have*

$$(3.1) \qquad \deg_{(\ell-3)} \operatorname{Proj} S/(J(Q) + (\beta)) = \sum_{X \in L_2, X \subset K} \mu(X).$$

Proof. First note that every $(\ell - 3)$-dimensional component of $\operatorname{Proj} S/(J(Q) + (\beta))$ is of the form $\operatorname{Proj} S/(J(Q_X) + (\beta)) \subset \mathbb{CP}^{\ell-1}$ such that $X \in L_2(\mathcal{A})$ and $X \subset K$. Then the lemma is immediate from Lemma 3.1. $\qquad\square$

We denote the right hand side of (3.1) by $\mu_{\mathcal{A}}(K) := \sum_{X \in L_2, X \subset K} \mu(X)$.

4. Reconstruction of the arrangement by the Jacobian scheme

In this section we prove that the Jacobian ideal of a hyperplane arrangement and its saturation contain all the information from the arrangement. Hence, we prove Theorem 1.1. Let $\mathcal{A}$ be a central arrangement.

Lemma 4.1. *If the hyperplane K is in $\mathcal{A}$, then $\mu_{\mathcal{A}}(K) = |\mathcal{A}| - 1$. If K is not in $\mathcal{A}$, $\ell \geq 3$ and $\mathcal{A}$ is essential, then $\mu_{\mathcal{A}}(K) < |\mathcal{A}| - 1$.*

Proof. The first statement follows easily from the definition (3.1). Suppose that K is not in $\mathcal{A}$. Put the set $L_2(\mathcal{A})^K = \{X \in L_2(\mathcal{A}) \mid X \subset K\}$. If $L_2(\mathcal{A})^K$ is empty, there is nothing to prove. If $L_2(\mathcal{A})^K = \{X\}$ consists of one element, then there exists $H \in \mathcal{A}$ such that $X \nsubseteq H$ since $\mathcal{A}$ is essential. Hence $|\mathcal{A}_X| \leq |\mathcal{A}| - 1$. We also obtain $\mu_{\mathcal{A}}(K) = |\mathcal{A}_X| - 1 < |\mathcal{A}| - 1$. Finally suppose $L_2(\mathcal{A})^K = \{X_1, X_2, \ldots, X_p\}$ with $p \geq 2$. Then from the assumption, we have $\mathcal{A}_{X_i} \cap \mathcal{A}_{X_j} = \emptyset$ for $1 \leq i < j \leq p$. Thus we have $\mu_{\mathcal{A}}(K) = \sum_{i=1}^{p} |\mathcal{A}_{X_i}| - p < |\mathcal{A}| - 1$. $\square$

Now, we can prove Theorem 1.1. Let $\mathcal{A}$ be an essential hyperplane arrangement with $\ell \geq 3$. Let $K = \{\beta = 0\}$ be a hyperplane and $\overline{K} \subset \mathbb{CP}^{\ell-1}$ the projectivization. Then the scheme theoretic intersection with $\operatorname{Proj} S/J(Q)$ is obtained by

$$\overline{K} \cap \operatorname{Proj} S/J(Q) = \operatorname{Proj} S/(J(Q) + (\beta)).$$

From Lemma 4.1, $\deg_{(\ell-3)} \overline{K} \cap \operatorname{Proj} S/J(Q)$ is not greater than $|\mathcal{A}| - 1$ and maximized precisely when $K \in \mathcal{A}$. This reconstructs $\mathcal{A}$ from $\operatorname{Proj} S/J(Q)$. $\square$

Example 4.2. It may be worth noting that from the reduced Jacobian scheme $\operatorname{Proj} S/\sqrt{J(Q)}$, we can not reconstruct $\mathcal{A}$. Suppose $\mathcal{A}_1$ is defined by $Q_1 = z_1 z_2 z_3 (z_1 + z_2 - z_3)$ and $\mathcal{A}_2$ is defined by $Q_2 = Q_1 \times (z_1 - z_3)$. Recall in general $\operatorname{Proj} S/\sqrt{J(Q)}$ is the reduced scheme structure on the singular locus, which is the union of codimension two intersections $X \in L_2(\mathcal{A})$. Then the radical of Jacobian ideals are equal, more precisely,

$$\sqrt{J(Q_1)} = \sqrt{J(Q_2)} = (z_1, z_2) \cap (z_1, z_2 - z_3) \cap (z_2, z_1 - z_3) \cap (z_1, z_3) \cap (z_2, z_3) \cap (z_1 + z_2, z_3).$$

So, the reduced Jacobian ideal does not even record the number of hyperplanes.

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