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ON THE BASE LOCUS OF THE LINEAR SYSTEM OF GENERALIZED THETA FUNCTIONS

Christian Pauly

Abstract. Let $\mathcal{M}_r$ denote the moduli space of semi-stable rank-r vector bundles with trivial determinant over a smooth projective curve C of genus g. In this paper we study the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system of the determinant line bundle $\mathcal{L}$ over $\mathcal{M}_r$, i.e., the set of semi-stable rank-r vector bundles without theta divisor. We construct base points in $\mathcal{B}_{g+2}$ over any curve C, and base points in $\mathcal{B}_4$ over any hyperelliptic curve.

1. Introduction

Let C be a complex smooth projective curve of genus g and let $\mathcal{M}_r$ denote the coarse moduli space parametrizing semi-stable rank-r vector bundles with trivial determinant over the curve C. Let $\mathcal{L}$ be the determinant line bundle over the moduli space $\mathcal{M}_r$ and let $\Theta \subset \mathrm{Pic}^{g-1}(C)$ be the Riemann theta divisor in the degree $g-1$ component of the Picard variety of C. By [BNR] there is a canonical isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |r\Theta|$, under which the natural rational map $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$ is identified with the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|, \qquad E \mapsto \theta(E) \subset \mathrm{Pic}^{g-1}(C).$$

The underlying set of $\theta(E)$ consists of line bundles $L \in \mathrm{Pic}^{g-1}(C)$ with $h^0(C, E \otimes L) > 0$. For a general semi-stable vector bundle E, $\theta(E)$ is a divisor. If $\theta(E) = \mathrm{Pic}^{g-1}(C)$, we say that E has no theta divisor. We note that the indeterminacy locus of the theta map θ, i.e., the set of bundles E without theta divisor, coincides with the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system $|\mathcal{L}|$.

Over the past years many authors [A], [B2], [He], [Hi], [P], [R], [S] have studied the base locus $\mathcal{B}_r$ of $|\mathcal{L}|$ and their analogues for the powers $|\mathcal{L}^k|$. For a recent survey of this subject we refer to [B1].

It is natural to introduce for a curve C the integer $r(C)$ defined as the minimal rank for which there exists a semi-stable rank-$r(C)$ vector bundle with trivial determinant over C without theta divisor (see also [B1] section 6). It is known [R] that $r(C) \geq 3$ for any curve C and that $r(C) \geq 4$ for a generic curve C. Our main result shows the existence of vector bundles of low ranks without theta divisor.

Theorem 1.1. *We assume that $g \geq 2$. Then we have the following bounds.*

(1) $r(C) \leq g+2$.
(2) $r(C) \leq 4$, *if C is hyperelliptic.*

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The first part of the theorem improves the upper bound $r(C) \leq \frac{(g+1)(g+2)}{2}$ given in [A]. The statements of the theorem are equivalent to the existence of a semi-stable rank-$(g+2)$ (resp. rank-4) vector bundle without theta divisor — see section 2.1 (resp. 2.2). The construction of these vector bundles uses ingredients which are already implicit in [Hi].

Theorem 1.1 seems to hint towards a dependence of the integer $r(C)$ on the curve C.

Notations: If E is a vector bundle over C, we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of E by $\mu(E) := \frac{\deg E}{\mathrm{rk} E}$, the canonical bundle over C by K and the degree d component of the Picard variety of C by $\mathrm{Pic}^d(C)$.

2. Proof of Theorem 1.1

2.1. Semi-stable rank-$(g+2)$ vector bundles without theta divisor. We consider a line bundle $L \in \mathrm{Pic}^{2g+1}(C)$. Then L is globally generated, $h^0(L) = g+2$ and the evaluation bundle E_L, which is defined by the exact sequence

$$0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0, \tag{1}$$

is stable (see e.g. [Bu]), with $\deg E_L = 2g+1$, $\mathrm{rk} E_L = g+1$ and $\mu(E_L) = 2 - \frac{1}{g+1}$.

A cohomology class $e \in \mathrm{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1})$ determines a rank-$(g+2)$ vector bundle $\mathcal{E}_e$ given as an extension

$$0 \longrightarrow E_L \longrightarrow \mathcal{E}_e \longrightarrow LK^{-1} \longrightarrow 0. \tag{2}$$

Proposition 2.1. *For any non-zero class e, the rank-$(g+2)$ vector bundle $\mathcal{E}_e$ is semi-stable.*

Proof. Consider a proper subbundle $A \subset \mathcal{E}_e$. If $A \subset E_L$, then $\mu(A) \leq \mu(E_L) = 2 - \frac{1}{g+1}$ by stability of E_L, so the subbundles of E_L cannot destabilize $\mathcal{E}_e$. If $A \not\subset E_L$, we introduce $S = A \cap E_L \subset E_L$ and consider the exact sequence

$$0 \longrightarrow S \longrightarrow A \longrightarrow LK^{-1}(-D) \longrightarrow 0,$$

where D is an effective divisor. If $\mathrm{rk} S = g+1$ or $S = 0$, we easily conclude that $\mu(A) < \mu(\mathcal{E}_e) = 2$. If $\mathrm{rk} S < g+1$ and $S \neq 0$, then stability of E_L gives the inequality $\mu(S) < \mu(E_L) = 2 - \frac{1}{g+1}$. We introduce the integer $\delta = 2\mathrm{rk} S - \deg S$. Then the previous inequality is equivalent to $\delta \geq 1$. Now we compute

$$\mu(A) = \frac{\deg S + \deg LK^{-1}(-D)}{\mathrm{rk} S + 1} \leq \frac{2\mathrm{rk} S - \delta + 3}{\mathrm{rk} S + 1} = 2 + \frac{1-\delta}{\mathrm{rk} S + 1} \leq 2 = \mu(\mathcal{E}_e),$$

which shows the semi-stablity of $\mathcal{E}_e$. □

We tensorize the exact sequence (1) with L and take the cohomology

$$0 \longrightarrow H^0(E_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0. \tag{3}$$

Note that $h^1(E_L^* \otimes L) = h^0(E_L \otimes KL^{-1}) = 0$ by stability of E_L. The second map μ is the multiplication map and factorizes through $\mathrm{Sym}^2 H^0(L)$, i.e.,

$$\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L) = \ker \mu.$$

By Serre duality a cohomology class $e \in \mathrm{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1}) = H^0(E_L^* \otimes L)^*$ can be viewed as a hyperplane $H_e \subset H^0(E_L^* \otimes L)$. Then we have the following

Proposition 2.2. *If $\Lambda^2 H^0(L) \subset H_e$, then the vector bundle $\mathcal{E}_e$ satisfies*

$$h^0(\mathcal{E}_e \otimes \lambda) > 0, \qquad \forall \lambda \in \mathrm{Pic}^{g-3}(C).$$

Proof. We tensorize the exact sequence (2) with $\lambda \in \mathrm{Pic}^{g-3}(C)$ and take the cohomology

$$0 \longrightarrow H^0(E_L \otimes \lambda) \longrightarrow H^0(\mathcal{E}_e \otimes \lambda) \longrightarrow H^0(LK^{-1}\lambda) \xrightarrow{\cup e} H^1(E_L \otimes \lambda) \longrightarrow \cdots$$

Since $\deg LK^{-1}\lambda = g$, we can write $LK^{-1}\lambda = \mathcal{O}_C(D)$ for some effective divisor D. It is enough to show that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ holds for λ general. Hence we can assume that $h^0(LK^{-1}\lambda) = h^0(\mathcal{O}_C(D)) = 1$.

If $h^0(E_L \otimes \lambda) > 0$, we are done. So we assume $h^0(E_L \otimes \lambda) = 0$, which implies $h^1(E_L \otimes \lambda) = 1$ by Riemann-Roch. Hence we obtain that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ if and only if the cup product map

$$\cup e : H^0(\mathcal{O}_X(D)) \longrightarrow H^1(E_L \otimes \lambda) = H^0(E_L^* \otimes L(-D))^*$$

is zero. Furthermore $\cup e$ is zero if and only if $H^0(E_L^* \otimes L(-D)) \subset H_e$. Now we will show the inclusion

$$H^0(E_L^* \otimes L(-D)) \subset \Lambda^2 H^0(L). \tag{4}$$

We tensorize the exact sequence (1) with $L(-D)$ and take cohomology

$$0 \longrightarrow H^0(E_L^* \otimes L(-D)) \longrightarrow H^0(L) \otimes H^0(L(-D)) \xrightarrow{\mu} H^0(L^2(-D)) \longrightarrow \cdots$$

Since $h^0(E_L^* \otimes L(-D)) = 1$, we conclude that $h^0(L(-D)) = 2$ and $H^0(E_L^* \otimes L(-D)) = \Lambda^2 H^0(L(-D)) \subset \Lambda^2 H^0(L)$.

Finally the proposition follows: if $\Lambda^2 H^0(L) \subset H_e$, then by (4) $H^0(E_L^* \otimes L(-D)) \subset H_e$ for general D, or equivalently $h^0(\mathcal{E}_e \otimes \lambda) > 0$ for general $\lambda \in \mathrm{Pic}^{g-3}(C)$. □

We introduce the linear subspace $\Gamma \subset \mathrm{Ext}^1(LK^{-1}, E_L)$ defined by

$$\Gamma := \ker \left(\mathrm{Ext}^1(LK^{-1}, E_L) = H^0(E_L^* \otimes L)^* \longrightarrow \Lambda^2 H^0(L)^* \right),$$

which has dimension $\frac{g(g-1)}{2} > 0$. Then for any non-zero cohomology class $e \in \Gamma$ and any $\gamma \in \mathrm{Pic}^2(C)$ satisfying $\gamma^{g+2} = L^2K^{-1} = \det \mathcal{E}_e$, the rank-$(g+2)$ vector bundle

$$\mathcal{E}_e \otimes \gamma^{-1}$$

has trivial determinant, is semi-stable by Proposition 2.1 and has no theta divisor by Proposition 2.2.

2.2. Hyperelliptic curves. In this subsection we assume that C is hyperelliptic and we denote by σ the hyperelliptic involution. The construction of a semi-stable rank-4 vector bundle without theta divisor has been given in [Hi] section 6 in the case $g = 2$, but it can be carried out for any $g \geq 2$ without major modification. For the convenience of the reader, we recall the construction and refer to [Hi] for the details and the proofs.

Let $w \in C$ be a Weierstrass point. Any non-trivial extension

$$0 \longrightarrow \mathcal{O}_C(-w) \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is a stable, σ-invariant, rank-2 vector bundle with $\deg G = -1$. By [Hi] Theorem 4 a cohomology class $e \in H^1(\mathrm{Sym}^2 G)$ determines a symplectic rank-4 bundle

$$0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

Moreover it is easily seen that, for any non-zero class e, the vector bundle $\mathcal{E}_e$ is semi-stable. By [Hi] Lemma 16 the composite map

$$D_G : \mathbb{P}H^1(\mathrm{Sym}^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \qquad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D_G} : H^1(\mathrm{Sym}^2 G) \longrightarrow H^0(\mathrm{Pic}^{g-1}(C), 4\Theta).$$

The involution $i(L) = KL^{-1}$ on $\mathrm{Pic}^{g-1}(C)$ induces a linear involution on $|4\Theta|$ with eigenspaces $|4\Theta|_\pm$. Note that $4\Theta \in |4\Theta|_+$. We now observe that $\theta(\mathcal{E}) \in |4\Theta|_+$ for any symplectic rank-4 vector bundle $\mathcal{E}$ — see e.g. [B2]. Moreover we have the equality $\theta(\sigma^*\mathcal{E}) = i^*\theta(\mathcal{E})$ for any vector bundle $\mathcal{E}$. These two observations imply that the linear map $\widetilde{D_G}$ is equivariant with respect to the induced involutions σ and i. Since $\mathrm{im}\, \widetilde{D_G} \subset H^0(\mathrm{Pic}^{g-1}(C), 4\Theta)_+$, we obtain that one of the two eigenspaces $H^1(\mathrm{Sym}^2 G)_\pm$ is contained in the kernel $\ker \widetilde{D_G}$, hence give base points for the theta map. We now compute as in [Hi] using the Atiyah-Bott-fixed-point formula

$$h^1(\mathrm{Sym}^2 G)_+ = g - 1, \qquad h^1(\mathrm{Sym}^2 G)_- = 2g + 1.$$

One can work out that $H^1(\mathrm{Sym}^2 G)_+ \subset \ker \widetilde{D_G}$. Hence any $\mathcal{E}_e$ with non-zero $e \in H^1(\mathrm{Sym}^2 G)_+$ is a semi-stable rank-4 vector bundle without theta divisor.

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Département de Mathématiques, Université de Montpellier II - Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

E-mail address: `pauly@math.univ-montp2.fr`