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POSITIVE QUATERNIONIC KÄHLER MANIFOLDS AND SYMMETRY RANK: II

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ABSTRACT. Let M be a positive quaternionic Kähler manifold of dimension $4m$. If the isometry group $\mathrm{Isom}(M)$ has rank at least $\frac{m}{2}+3$, then M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$. The lower bound for the rank is optimal if m is even.

1. Introduction

A quaternionic Kähler manifold M is an oriented Riemannian $4n$-manifold, $n \geq 2$, whose holonomy group is contained in $Sp(n)Sp(1) \subset SO(4n)$. If $n = 1$ we add the condition that M is Einstein and self-dual. Equivalently, there exists a 3-dimensional subbundle S of the endmorphism bundle, $\mathrm{End}(TM, TM)$, locally generated by three anti-commuting almost complex structures $I, J, K = IJ$ so that the Levi-Civita connection preserves S. It is well-known [3] that a quaternionic Kähler manifold M is always Einstein, and is necessarily locally hyperkähler if its Ricci tensor vanishes. A quaternionic Kähler manifold M is called *positive* if it has positive scalar curvature. By [13] (for $n = 1$) and [20] (for $n \geq 2$, compare [16] [17]) a positive quaternionic Kähler manifold M has a twistor space a complex Fano manifold. Hitchin [13] proved a positive quaternionic Kähler 4-manifold M must be isometric to $\mathbb{C}P^2$ or S^4. Hitchin's work was extended by Poon-Salamon [19] to dimension 8, which proves that a positive quaternionic Kähler 8-manifold M must be isometric to $\mathbb{H}P^2$, $Gr_2(\mathbb{C}^4)$ or $G_2/SO(4)$.

This leads to the Salamon-Lebrun conjecture:

Every positive quaternionic Kähler manifold is a quaternionic symmetric space.

Very recently, the conjecture was further verified for $n = 3$ in [12], using the approach initiated in [20] [19] (compare [17]). For a positive quaternionic Kähler manifold M, Salamon [20] proved that the dimension of its isometry group is equal to the index of certain twisted Dirac operator, by the Atiyah-Singer index theorem, which is a characteristic number of M coupled with the Kraines 4-form Ω (in analog with the Kähler form), and it was applied to prove that the isometry group of M is large in lower dimensions (up to dimension 16).

By [17] a positive quaternionic Kähler $4n$-manifold M is simply connected and the second homotopy group $\pi_2(M)$ is a finite group or $\mathbb{Z}$, and M is isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$ according to $\pi_2(M) = 0$ or $\mathbb{Z}$.

An interesting question is to study positive quaternionic Kähler manifold in terms of its isometry group. This approach dates back to the work [19] for $n = 2$ [12] for

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$n = 3$ to proving the action is transitive, and [5] [18] for cohomogeneity one actions (and hence the isometry group must be very large). [4] classified positive quaternionic Kähler $4n$-manifolds with isometry rank $n+1$, using an approach on hyper-Kähler quantizations. [6] establishes a connectedness theorem and using this tool the author proved that, a positive quaternionic Kähler $4n$-manifolds of symmetry rank $\geq n-2$ must be either isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$, if $n \geq 10$.

In this paper we will combine Morse theory of the momentum map on quaternionic Kähler manifold [2] and the connectedness theorem in [6] to prove the following

Theorem 1.1. *Let M be a positive quaternionic Kähler manifold of dimension $4m$. Then the isometry group $Isom(M)$ has rank (denoted by $rank(M)$) at most $(m+1)$, and M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$ if $rank(M) \geq \frac{m}{2}+3$.*

Notice that the fixed point set of an isometric circle action on a quaternionic Kähler manifold of dimension $4m$ is either a quaternionic Kähler submanifold or a Kähler manifold. In the latter case the fixed point set has dimension at most $2m$ (the middle dimension of the manifold). Moreover, if a fixed point component is contained in $\mu^{-1}(0)$ then it must be a quaternionic Kähler submanifold, and if it is in the complement $M - \mu^{-1}(0)$ then it is Kähler (see [2]).

Theorem 1.2. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with an isometric S^1-action. Assume $m \geq 3$. If N is a fixed point component of codimension 4, then M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$.*

The idea of proving Theorem 1.2 is as follows: by the assumption we know that the fixed point component $N \subset \mu^{-1}(0)$ since $4m-4 \geq 2m+1$ (cf. [2] Remark 3.2). Furthermore, by [8] the reduction $\mu^{-1}(0)/S^1$ has dimension at most $4m-4$ (cf. [5]). We will prove in section 3 that $N = \mu^{-1}(0)$. This together with the equivariant Morse equality implies that $M = \mathbb{H}P^m$ if $b_2(M) = 0$ (cf. Lemma 4.1) and so Theorem 1.2 follows by [17].

With Theorem 1.2 in hand, the proof of Theorem 1.1 follows by induction on m and Theorems 2.1 and 2.4.

Theorem 1.1 is optimal if m is even since the rank of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4})$ is $\frac{m}{2}+2$. We conjecture that when m is odd, the lower bound for the rank in Theorem 1.1 may be improved by 1, that is

Conjecture 1.3. *Let M be a positive quaternionic Kähler manifold of dimension $8m+4$. Then M is isometric to $\mathbb{H}P^{2m+1}$ or $Gr_2(\mathbb{C}^{2m+3})$ if $rank(M) \geq m+3$.*

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2. Preliminaries

In this section we recall some results on quaternionic Kähler manifolds needed in later sections.

Let (M, g) be a quaternionic Kähler manifold of dimension $4m$. Let $F \to M$ be the principal $Sp(m)Sp(1)$-bundle over M. Locally, $F \to M$ can be lifted to a principal $Sp(m) \times Sp(1)$-bundle, i.e., the fiberwise double cover of F. Let E, H be the locally defined bundles associated to the standard complex representation of $Sp(m)$ and $Sp(1)$ respectively. The complexified cotangent bundle $T^*M_{\mathbb{C}}$ is isomorphic to $E \otimes_{\mathbb{C}} H$. The adjoint representations of $Sp(m)$ and $Sp(1)$ give two bundles S^2E and S^2H over M, respectively. Given the inclusion of the holonomy algebra $sp(m) \oplus sp(1)$ into $so(4m)$, the bundle $S^2E \oplus S^2H$ can be regarded as a subbundle of the bundle of 2-forms $\Lambda^2 T^*M_{\mathbb{C}}$. The bundle S^2H has fiber the Lie algebra $sp(1)$ and the local basis $\{I, J, K\}$ cooresponding to $i, j, k \in sp(1)$ which are almost complex structures satisfying that $IJ = -JI = K$.

The Kraines 4-form, Ω, associated to a quaternionic Kähler manifold M, is a non-degenerate closed form which is defined by

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

where ω_1, ω_2 and ω_3 are the locally defined 2-forms associated to the almost complex structures I, J and K. The form Ω is globally defined and non-degenerate, namely Ω^m is a constant non-zero multiple of the volume form. It is well-known that Ω is parallel if and only if M has holonomy in $Sp(m)Sp(1)$, if $m \geq 2$. Moreover, by [22], M has holonomy in $Sp(m)Sp(1)$ if and only if Ω is closed, provided $m \geq 3$.

A quaternionic Kähler manifold may not have a global almost complex structure, e.g., the quaternionic projective space $\mathbb{H}P^m$. If I, J, K are integrable and covariantly constant with respect the metric, the holonomy group reduces to $Sp(m)$, quaternionic Kähler manifold is hyperkähler. Wolf [23] classified quaternionic symmetric spaces of compact type, they are $\mathbb{H}P^m$, the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2})$, and the oriented real Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4})$, and exactly one quaternionic symmetric space for each compact simple Lie algebra, $G_2/SO(4)$, $F_4/Sp(3)Sp(1)$, $E_6/SU(6)Sp(1)$, $E_7/\mathrm{Spin}(12)Sp(1)$, $E_8/E_7Sp(1)$.

Theorem 2.1 ([17])**.** *(i) (Fininteness) For any $m \in \mathbb{Z}_+$, there are, modulo isometries and rescalings, only finitely many positive quaternionic Kähler manifolds of dimension $4m$.*

(ii) (Strong rigidity) Let (M, g) be a positive quaternionic Kähler manifold of dimension $4m$. Then M is simply connected and

$$\pi_2(M) = \{ \begin{array}{l} 0, \ (M,g) = \mathbb{H}P^m \\ \mathbb{Z}, \ (M,g) = Gr_2(\mathbb{C}^{m+2}) \\ \textit{finite with } 2\textit{-torsion}, \ \textit{otherwise} \end{array}$$

A submanifold N in a quaternionic Kähler manifold is called a *quaternionic submanifold* if the locally defined almost complex structures I, J, K preserve the tangent bundle of N.

Proposition 2.2 ([9]). *Any quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic and quaternionic Kählerian.*

Theorem 2.3 ([6]). *Let M be a positive quaternionic Kähler manifold of dimension $4m$. Assume $f = (f_1, f_2) : N \to M \times M$, where $N = N_1 \times N_2$ and $f_i : N_i \to M$ are quaternionic immersions of compact quaternionic Kähler manifolds of dimensions $4n_i$, $i = 1, 2$. Let Δ be the diagonal of $M \times M$. Set $n = n_1 + n_2$. Then:*
(2.3.1) If $n \geq m$, then $f^{-1}(\Delta)$ is nonempty.
(2.3.2) If $n \geq m + 1$, then $f^{-1}(\Delta)$ is connected.
(2.3.3) If f is an embedding, then for $i \leq n - m$ there is a natural isomorphism, $\pi_i(N_1, N_1 \cap N_2) \to \pi_i(M, N_2)$ and a surjection for $i = n - m + 1$.

As a direct corollary of (2.3.3) we have

Theorem 2.4 ([6]). *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If $N \subset M$ is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \to M$ is $(2n - m + 1)$-connected.*

3. Hyperkähler quotient and Quaternionic Kähler quotient

a. Hyperkähler quotient

Let M be a hyperkähler manifold having a metric g and covariantly constant complex structures I, J, K which behave algebraically like quaternions:

$$I^2 = J^2 = K^2 = -1; IJ = -JI = K$$

Let G be a compact Lie group acting on M by isometries preserving the structures I, J, K. The group G preserves the three Kähler forms $\omega_1, \omega_2, \omega_3$ corresponding to the complex structures I, J, K, so we may define moment maps μ_1, μ_2, μ_3, respectively. These may be written as a single map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

Let

$$\mu_+ = \mu_2 + i\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}$$

where $\mathfrak{g}^*$ is the dual space of the Lie algebra of G.

By [14] μ_+ is holomorphic, and so $N = \mu_+^{-1}(0)$ is a complex subvariety of M, with respect to the complex structure I. By definition, $\mu^{-1}(0) = N \cap \mu_1^{-1}(0)$. The hyperkähler quotient is the quotient space $\mu^{-1}(0)/G$, denoted by $M//G$. In particular, if $\mu^{-1}(0)$ is a manifold and the induced G-action is free, then the hyperkähler quotient $M//G$ is also a hyperKähler manifold. More generally, Dancer-Swann [5] proved that the hyperkähler quotient $M//G$ may be decomposed into the union of hyperkähler manifolds, according to the isotropy decomposition of the G-action on M. However, it is wide open if the decomposition of $M//G$ is a stratified topological space, as in the sympletic quotient case [21].

In this section we will consider the structure of this decomposition in the special case that $G = S^1$ and the action is semi-free, i.e., free outside the fixed point set.

Let us start with the standard example of isometric S^1-action on quaternionic linear space $\mathbb{H}^n$ defined by

$$\varphi_t(u) = e^{2\pi i t}u; \qquad t \in [0, 1)$$

where i is one of the quaternionic units. With global quaternionic coordinates $\{u^\alpha\}$, $\alpha = 1, \cdots, n$, the standard flat metric on $\mathbb{H}^n$ may be written as:

$$ds^2 = \sum_\alpha d\bar{u}^\alpha \otimes du^\alpha$$

where $\bar{u}^\alpha$ is the quaternionic conjugate of u^α.

The Killing vector field X of the above action is $\mathbb{H}$-valued:

$$X^\alpha(u) = iu^\alpha$$

which is triholomorphic.

Consider the $\mathbb{H}$-valued 2-form

$$\omega = \sum_\alpha d\bar{u}^\alpha \wedge du^\alpha$$

Observe that ω is purely imaginary since $\omega + \bar{\omega} = 0$. Note that $\omega = \omega_1 i + \omega_2 j + \omega_3 k$, where ω_i is as above.

It is easy to see that the moment map (cf. [7])

$$\mu^X = \sum_\alpha \bar{u}^\alpha i u^\alpha : \mathbb{H}^n \to i\mathbb{R}^3$$

Write $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$. The zero set of the holomorphic moment map $\mu_+^{-1}(0)$ is the complex algebraic variety of complex dimension $(2n - 1)$:

$$\{(a, b) \in \mathbb{C}^n \oplus j\mathbb{C}^n : \langle a, \bar{b}\rangle = \sum_\alpha a^\alpha b^\alpha = 0\}$$

In particular, if $n = 1$, then $\mu_+^{-1}(0)$ is a reducible algebraic curve with two irreducible components the standard complex lines.

The hyperkähler quotient $\mathbb{H}^n//S^1$ is an open cone over a $(4n - 5)$-dimensional manifold W:

$$W = \{(a, b) \in \mathbb{C}^n \oplus \mathbb{C}^n : |a|^2 = |b|^2 = 1, \langle a, \bar{b}\rangle = \sum_\alpha a^\alpha b^\alpha = 0\}/S^1$$

In particular, $\mathbb{H}//S^1 = \{0\}$, a single point.

Theorem 3.1. *Let M^{4m} be a hyperkähler manifold with an isometric effective S^1-action preserving the hyperkähler structure. Let μ be the hyperkähler moment map. If $Y \subset M^{S^1} \cap \mu^{-1}(0)$ is a connected fixed point component of dimension $4m - 4$, then $Y \subset M//S^1$ is a connected component.*

Before we start the proof, let us give the analog of the classical Darboux-Weinstein theorem for complex symplectic manifolds. Recall that a complex symplectic manifold W is a complex manifold with a complex value symplectic form $\omega^c = \omega_2 + i\omega_3$, where ω_2, ω_3 are two real value symplectic forms.

Lemma 3.2 (Relative equivariant Darboux theorem)**.** *Let G be a compact Lie group acting on a complex symplectic manifold W. Let ω_0^c and ω_1^c be two G-equivariant complex symplectic forms on W. Assume they coincide on a closed G-invariant complex symplectic submanifold V. Then there exists a G-invariant neighborhood U_0 of V in W and a G-equivariant map*

$$\psi : U_0 \to W$$

such that

$$\psi|_V = Id_V \text{ and } \psi^*\omega_1 = \omega_0$$

Proof. We apply the path method due to Moser. Consider the form $\omega_t^c = \omega_0^c + t(\omega_1^c - \omega_0^c)$. This is a closed complex value 2-form which is non-degenerate on V and thus on a small G-invariant tubular neighborhood U of V.

Since ω_0^c and ω_1^c are closed, and so is $\omega_0^c - \omega_1^c$, and thus we may find a complex value 1-form β on a neighborhood of V such that $d\beta = \omega_0^c - \omega_1^c$. Indeed, U is G-equivariant diffeomorphic to an open neighborhood of the zero section of a G-equivariant vector bundle on V, hence retracts on V. By applying the Poincaré lemma to $\omega_0^c - \omega_1^c$ we get the G-invariant 1-form β, that can be chosen so that $\beta_x = 0$ for all $x \in V$.

The complex value closed 2-form ω_t^c being non-degenerate, it defines a time dependent G-invariant vector field X_t such that the contraction $i_{X_t}\omega_t^c = \beta$. Note that $X_t = 0$ on V. Its flow φ_t keeps V fixed, and thus one can find a G-invariant neighborhood U_0 of V where φ_t is defined and such that $\varphi_t(U_0) \subset U$.

Therefore,

$$\frac{d}{dt}[\varphi_t^*\omega_t^c] = \varphi_t^*[\frac{d\omega_t^c}{dt} + \mathfrak{L}_{X_t}\omega_t^c] = \varphi_t^*[\omega_1^c - \omega_0^c + \omega_0^c - \omega_1^c] = 0$$

because $\mathfrak{L}_{X_t}\omega_t^c = di_{X_t}\omega_t^c + i_{X_t}d\omega_t^c = d\beta$ using the Cartan formula and the definition of φ_t. The form $\varphi_t^*\omega_t^c$ does not depend on t, and it equals ω_0^c for $t = 0$. Put $\psi = \varphi_1$ the desired result follows. □

Now we are ready to prove

Proof of Theorem 3.1. Recall that $\omega^c = \omega_2 + i\omega_3$ defines a complex symplectic structure on M. Choose an open ball V in Y with restricted complex symplectic structure. The complex normal bundle of V in M is a trivial complex vector bundle $V \times \mathbb{C}^2$. The S^1-action on the bundle is the product action of a trivial action on V and an effective complex linear action on $\mathbb{C}^2$, saying, $t \cdot (z_1, z_2) = (t^p z_1, t^q z_2)$ for some $p, q \in \mathbb{Z}$. Since the hyperkähler metric is S^1-invariant, the normal exponential map $\varphi = \exp_V : V \times \mathbb{C}^2 \to M$ defines an S^1-equivariant diffeomorphism from an S^1-invariant tubular neighborhood V_0 of $V \times \{0\}$ in $V \times \mathbb{C}^2$ to an S^1-invariant tubular neighborhood U_0 of V in M. Note that $pq = -1$ since the action is effective and preserves the hyperkähler structure.

Consider the S^1-invariant complex symplectic form $\omega_0^c = \omega^c|_V \times (dz_1 \wedge dz_2)$ on $V \times \mathbb{C}^2$. The pullback form $(\varphi^{-1})^*\omega_0^c$ and $\omega^c|_{U_0}$ are both S^1-invariant complex symplectic forms on U_0 which coincide on V. By Lemma 3.2 there exists an S^1-equivariant diffeomorphism ψ such that $\psi^*\omega^c|_{U_0} = (\varphi^{-1})^*\omega_0^c$. Therefore, $(\psi \circ \varphi)^*\omega^c|_{U_0} = \omega_0^c$.

The moment map $\mu_+|_{U_0}$ of (U_0, ω^c) with the restricted S^1-action may be identified with the moment map of (V_0, ω_0^c) with the product action. Thus,

$$\mu_+^{-1}(0) \cap U_0 \cong (V \times \{0\}) \times L \cap V_0$$

where $L \subset \mathbb{C}^2$ is the zero locus of the moment map of $(\mathbb{C}^2, dz_1 \wedge dz_2)$ with the above mentioned S^1-action. By the paragraph before Theorem 3.1 we already knew that L is the reducible curve given by $z_1 z_2 = 0$. Therefore, $\mu_+^{-1}(0) \cap U_0$ is the union of two S^1-invariant complex submanifolds, $N_1, N_2 \subset M$ (with respect to the complex structure I), and the S^1-action on N_1 (resp. N_2) has a real codimension 2 fixed point set (e.g., $N_1 \cong (V \times \{0\}) \times \{(z_1, 0) : z_1 \in \mathbb{C}\} \cap V_0)$. Obviously, both N_1 and N_2 have induced Kähler structures (w.r.t. I), and the restriction of μ_1 on N_1 (resp. N_2) equals the moment map of the restricted S^1-action on N_1 (resp. N_2) with the induced Kähler structure.

By definition, $\mu^{-1}(0) \cap U_0 = \mu_+^{-1}(0) \cap U_0 \cap \mu_1^{-1}(0) = (\mu_1|_{N_1})^{-1}(0) \cup (\mu_1|_{N_2})^{-1}(0)$. Consider the moment map μ_1 of the Kähler manifold (N_1, ω_1) with the restricted S^1-action. In a small tubular neighborhood W_1 of the fixed point component $V \subset N_1$ of codimension 2, by [10] $(\mu_1|_{N_1})^{-1}(0)$ is a conic bundle over the fixed point set V. Since $(\mu_1|_{N_1})^{-1}(0)$ has dimension $4m-4$, where $\dim(N_1) = 4m-2$, thus, $(\mu_1|_{N_1})^{-1}(0) \cap W_1 = V$. Similarly, $(\mu_1|_{N_2})^{-1}(0) \cap W_2 = V$, where W_2 is a small tubular neighborhood of V in N_2. Therefore, $\mu^{-1}(0) \cap U_0' = V$ for a possibly smaller tubular neighborhood U_0' of V in M. By definition of $M//S^1$ we conclude the desired result. □

b. Quaternionic Kähler quotient

Let M be a quaternionic Kähler manifold with non-zero scalar curvature. If G acts on M by isometries, there is a well-defined moment map, which is a section $\mu \in \Gamma(S^2 H \otimes \mathfrak{g}^*)$ solving the equation

$$\langle \nabla \mu, X \rangle = \sum_{i=1}^{3} \overline{I_i X} \otimes I_i$$

for each $X \in \mathfrak{g}$; where $\bar{X} = g(X, \cdot)$ denote the 1-form dual to X with respect to the Riemannian metric. Equivalently, the above equation may be written in the following form similar to the symplectic case

$$d\mu(X) = i_X \Omega$$

A nontrivial feature for quaternionic quotient is, the section μ is uniquely determined if the scalar curvature is nonzero. Moreover, only the preimage of the zero section of the moment map, $\mu^{-1}(0)$, is well-defined.

Theorem 3.3 ([8])**.** *Let M^{4n} be a quaternionic Kähler manifold with nonzero scalar curvature acted on isometrically by S^1. If S^1 acts freely on $\mu^{-1}(0)$ then $\mu^{-1}(0)/S^1$ is a quaternionic Kähler manifold of dimension $4(n-1)$.*

Since the proof of the Galicki-Lawson's theorem is local, so if the circle action is free on a piece of the manifold, the same result applies to the moment map on this piece.

Let $f = \|\mu\|^2$. By [2] the critical set of f is the union of the zero set $f^{-1}(0) = \mu^{-1}(0)$ and the fixed point set of the circle action. Moreover, the zero set $\mu^{-1}(0)$ is connected, and a fixed point component is either contained in $\mu^{-1}(0)$ or does not interesect with

$\mu^{-1}(0)$. Following [2] the Morse function f is called *equivariantly perfect* over $\mathbb{Q}$ if the equivariant Morse equialities hold, that is if

$$\hat{P}_t(M) = \hat{P}_t(\mu^{-1}(0)) + \sum t^{\lambda_F} \hat{P}_t(F)$$

where the sum ranges over the set of connected components outside $\mu^{-1}(0)$ of the fixed point set, λ_F is the index of F, and $\hat{P}_t$ is the equivariant Poincaré polynomial for the equivariant cohomology with coefficients in $\mathbb{Q}$.

Theorem 3.4 ([2]). *Let M^{4n} be a quaternionic Kähler manifold acted on isometrically by S^1. Then the Morse function $\|\mu\|^2$ is equivariantly perfect over $\mathbb{Q}$.*

Proposition 3.5 ([2]). *Let M^{4n} be a positive quaternionic Kähler manifold acted on isometrically by S^1. Then every connected component of the fixed point set, not contained in $\mu^{-1}(0)$, is a Kähler submanifold of $M - \mu^{-1}(0)$ of real dimension less than or equal to $2n$ whose Morse index is at least $2n$, with respect to the function f.*

For each quaternionic Kähler manifold M with non-zero scalar curvature, following [22], let $\mathfrak{u}(M)$ denote the $H^*/\{\pm 1\}$-bundle over M:

$$\mathfrak{u}(M) = F \times_{Sp(n)Sp(1)} (H^*/\{\pm 1\})$$

where F is the principal $Sp(n)Sp(1)$-bundle over M. Let $\pi : \mathfrak{u}(M) \to M$ denote the bundle projection. Obviously, if G is acts on M by isometries, G can be lifted to a G-action on $\mathfrak{u}(M)$. It is proved in [22] that, if the scalar curvature is positive, $\mathfrak{u}(M)$ has a hyperkähler structure which is preserved by the lifted G-action. Let $\hat{\mu}$ denote the moment map of the lifted G-action on $\mathfrak{u}(M)$. By [22] Lemma 4.4 $\hat{\mu} = \mu \circ \pi$.

Lemma 3.6. *Let M be a positive quaternionic Kähler manifold of dimension $4n$. Assume that S^1 acts on M effectively by isometries. Let $\mu \in \Gamma(S^2H)$ be its moment map. If $N \subset \mu^{-1}(0)$ is a fixed point component of codimension 4, then $N = \mu^{-1}(0)$.*

Proof. Let $\mathfrak{u}(M)$ be as above. By Proposition 4.2 of [5], at the fixed point $x \in N$, the isotropy representation of S^1 in $SO(3) \cong \mathrm{Aut}(\mathfrak{u}(M)_x)$ factors through a finite group and hence the representation is trivial, where $\mathrm{Aut}(\mathfrak{u}(M)_x)$ is the isomorphism group of the fiber at x preserving the quaternionic structure, and $\mathfrak{u}(M)_x = \pi^{-1}(x)$ is the fiber of the bundle at x. Therefore, $\pi^{-1}(N)$ is also a fixed point component of the lifted S^1-action on $\mathfrak{u}(M)$ of codimension 4.

By [22] Lemma 4.4 we see that $\pi^{-1}(N) \subset \hat{\mu}^{-1}(0)$, where $\hat{\mu}$ is the moment map for the lifted S^1-action on $\mathfrak{u}(M)$. Now S^1 acts on the normal slice of $\pi^{-1}(N)$ in $\mathfrak{u}(M)$ through a representation in $Sp(1)$. For dimension reasoning, this representation is faithful, otherwise, a finite order subgroup of S^1 acts trivially on the whole manifold $\mathfrak{u}(M)$ and so on M, a contradiction to the effectiveness of the action from our assumption. Therefore, S^1 acts semi-freely on a neighborhood of $\pi^{-1}(N)$ in $\mathfrak{u}(M)$. By now we may apply Theorem 3.1 to show that $\pi^{-1}(N)$ is a connected component of $\hat{\mu}^{-1}(0)$. Since the moment map $\hat{\mu}$ projects to the moment map μ, we conclude N is also a connected component of $\mu^{-1}(0)$. By [2] $\mu^{-1}(0)$ is connected, thus $N = \mu^{-1}(0)$, the desired result follows. □

4. Proof of Theorem 1.2

Theorem 1.2 follows readily from the following Lemma and Theorem 2.1, where the dimension bound $m \geq 3$ implies that the fixed point component of codimension 4 has to be contained in $\mu^{-1}(0)$, by Proposition 3.5.

Lemma 4.1. *Let M be a positive quaternionic Kähler $4n$-manifold with an isometric S^1-action where $m \geq 3$. Let μ be the moment map. Assuming $b_2(M) = 0$. If $N \subset \mu^{-1}(0)$ is a fixed point component of dimension $4m-4$ of the circle action, then M is isometric to $\mathbb{H}P^m$.*

Proof. By Lemma 3.5 $\mu^{-1}(0) = N$, therefore S^1 acts trivially on $\mu^{-1}(0)$.

By Theorem 3.4

$$\hat{P}_t(M) = \hat{P}_t(N) + \sum_F t^{\lambda_F} \hat{P}_t(F)$$

where F runs over fixed point components outside N, and λ_F the Morse index of F. By Proposition 3.5 the Morse index $\lambda_F \geq 2n$ and are all even numbers. Thus the inclusion $N \to M$ is a $(2n-1)$-equivalence.

By [2] Lemma 2.2 $\hat{P}_t(M) = P_t(M)P_t(BS^1)$. Since S^1 acts trivially on F and N, we get that $\hat{P}_t(F) = P_t(F)P_t(BS^1)$ and $\hat{P}_t(N) = P_t(N)P_t(BS^1)$. The above identity reduces to

$$P_t(M) - P_t(N) = \sum_F t^{\lambda_F} P_t(F)$$

Observe that the last two terms of the left hand side, according to increasing degree in t, is $b_2(M)t^{4m-2} + t^{4m} = t^{4m}$ by Poincaré duality.

If F is a fixed point component outside $\mu^{-1}(0)$ such that $\dim_{\mathbb{R}} F > 0$, we claim that $\dim_{\mathbb{R}} F + \lambda_F \leq 4m-4$. Otherwise, by the above identity $\dim_{\mathbb{R}} F + \lambda_F = 4m-2$ is impossible, and if the even integer $\dim_{\mathbb{R}} F + \lambda_F = 4m$, we conclude that the coefficients of t^{4m-2} of the right hand side is also non-zero, since F must be a compact Kähler manifold (by Proposition 3.4) and so $P_t(F)$ has nonzero coefficient at every even degree not larger than the dimension. A contradiction.

By the above, the identity also implies that there is no isolated fixed point outside $\mu^{-1}(0)$ with Morse index $4m-2$, and there is an isolated fixed point with Morse index $4m$.

Put all together, by Morse theory, up to homotopy equivalence,

$$M \simeq N \cup_i e^{\lambda_i} \cup e^{4m}$$

where $2m \leq \lambda_i \leq \dim_{\mathbb{R}} F + \lambda_F \leq 4m-4$, and e^i denotes cell of dimension i. Therefore $H^{4m-2}(M, N) = 0$. By duality $H_2(M-N) \cong H^{4m-2}(M, N) = 0$. Since the codimension of N is 4, it follows that $H_2(M) \cong H_2(M-N) = 0$. Therefore by Theorem 2.1 $M = \mathbb{H}P^m$. The desired result follows. □

5. Proof of Theorem 1.1

Let M be a positive quaternionic Kähler manifold of dimension $4m$. We call the rank of the isometry group $\mathrm{Isom}(M)$ the symmetry rank of M, denoted by $\mathrm{rank}(M)$. By [6] we know that $\mathrm{rank}(M) \leq m+1$.

Proof of Theorem 1.1. Let $r = \mathrm{rank}(M)$. Consider the isometric T^r-action on M. Note that the T^r-action on M must have non-empty fixed point set since the Euler characteristic $\chi(M) > 0$ by [20]. Consider the isotropy representation of T^r at a fixed point $x \in M$, which must be a representation through the local linear holonomy $Sp(m)Sp(1)$ representation at $T_xM \cong \mathbb{H}^m$. If there is a stratum (a fixed point set of an isotropy group of rank ≥ 1) of codimension 4, then it must be contained in $\mu^{-1}(0)$ if $m \geq 3$ (by [2] or Proposition 3.6). By Theorem 2.1 and Lemma 4.1 the desired result follows. Thus we can assume that at x, the isotropy representation does not have any codimension 4 linear subspace fixed by some rank 1 subgroup of T^r. Let N be a maximal dimensional submanifold of M passing through x fixed by a circle subgroup of T^r.

Case (i): If $m = 0(\mathrm{mod}\ 2)$.

By the above assumption $4m-8 \geq \mathrm{dim}N \geq 2m+4$ since $\mathrm{rank}(N) = r-1 \geq \frac{m}{2}+2$, by Lemma 2.1 of [6]. Note that N is a quaternionic Kähler manifold since $N \subset \mu^{-1}(0)$. By Theorem 2.4 we see that $\pi_2(N) \cong \pi_2(M)$. By Theorem 2.1 it suffices to prove $\pi_2(N) = 0$ or $\mathbb{Z}$. By induction we may consider T^r-action on N, and applying Lemma 4.1 once again. Finally it suffices to consider the case where a 16-dimensional quaternionic Kähler submanifold of M, M^{16}, with an effective isometric action by torus of rank ≥ 5. In this case there is a quaternionic Kähler submanifold $M^{12} \subset M^{16}$ fixed by a circle group (cf. [6]). By Lemma 4.1, Theorem 2.1 and Theorem 2.4 $M^{16} = \mathbb{H}P^4$ or $\mathrm{Gr}_2(\mathbb{C}^6)$, the desired result follows.

Case (ii): If $m = 1(\mathrm{mod}\ 2)$.

Similar to the above $\mathrm{dim}N \geq 2m+6$ for the same reasoning. By Theorem 2.4 $\pi_2(N) \cong \pi_2(M)$. The same argument by induction reduces the problem to the case of a quaternionic Kähler submanifold of dimension 20, M^{20}, with an effective isometric torus action of rank ≥ 6. Once again the argument in [6] shows that M^{20} has a quatertnionic submanifold M^{16} of rank ≥ 5. By (i) we see that $M^{16} = \mathbb{H}P^4$ or $\mathrm{Gr}_2(\mathbb{C}^6)$. By Theorem 2.1 and Theorem 2.4 again we complete the proof. □

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