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CONFORMAL EQUIVALENCE BETWEEN CERTAIN GEOMETRIES IN DIMENSION 6 AND 7

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Abstract. For G_2-manifolds the Fernández-Gray class $\mathcal{X}_1 + \mathcal{X}_4$ is shown to consist of the union of the class $\mathcal{X}_4$ of G_2-manifolds locally conformal to parallel G_2-structures and that of conformal transformations of nearly parallel or weak holonomy G_2-manifolds of type $\mathcal{X}_1$. The analogous conclusion is obtained for Gray-Hervella class $\mathcal{W}_1 + \mathcal{W}_4$ of real 6-dimensional almost Hermitian manifolds: this sort of geometry consists of locally conformally Kähler manifolds of class $\mathcal{W}_4$ and conformal transformations of nearly Kähler manifolds in class $\mathcal{W}_1$. A corollary of this is that a compact $SU(3)$-space in class $\mathcal{W}_1 + \mathcal{W}_4$ or G_2-space of the kind $\mathcal{X}_1 + \mathcal{X}_4$ has constant scalar curvature if only if it is either a standard sphere or a nearly parallel G_2 or nearly Kähler manifold, respectively. The properties of the Riemannian curvature of the spaces under consideration are also explored.

1. Introduction

Reductions of the bundle of orthonormal frames over a Riemannian manifold to a principal G-bundle may be classified by the G-invariant components of the intrinsic torsion.

This idea was originally due to Gray and collaborators [11, 18] for the special instances of G_2-manifolds and almost Hermitian manifolds. It has been further refined and explored by, for instance, Bryant [6], Farinola, Falcitelli & Salamon [10], Martín Cabrera [25, 24], Martín Cabrera, Monar & Swann [26], Chiossi & Salamon [8].

For G_2- and almost Hermitian structures alike, the intrinsic torsion has 4 irreducible components. There are thus potentially 16 torsion classes for these two kinds of geometries.

In [26], Martín Cabrera, Monar & Swann showed that apart from one instance, $\mathcal{X}_1 + \mathcal{X}_2$ in our notation, every single class of G_2-structures may be realized on a compact homogeneous space. For the one exception an easy calculation shows that any G_2 structure with torsion $X_1 + X_2$ must have either $X_1 = 0$ or $X_2 = 0$.

In section 3 we show that something similar holds for the class $\mathcal{X}_1 + \mathcal{X}_4$. Namely that the latter essentially is generated by the classes $\mathcal{X}_1$ and $\mathcal{X}_4$. It is well known that G_2-structures in this class are locally conformally equivalent to nearly parallel ones. We will show that this equivalence is only really local when the G_2-structure lies in the subclass $\mathcal{X}_4$ of locally conformally parallel structures. The structure of compact locally conformally parallel G_2-manifolds has been recently described in [21, 31]. In contrast to this, we will show that if the X_1 component is non-zero at some point, it is non-zero everywhere. This is the key point in proving that a global conformal change exists.

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Differently from the G_2 case, it was only recently pointed out by Butruille [7] that a 6-dimensional almost Hermitian manifold in the Gray-Hervella class $\mathcal{W}_1 + \mathcal{W}_4$ is locally conformal to a nearly Kähler manifold. In section 5, we present a different and simpler proof of this fact for completeness. Based on this the analogous statements to those given for G_2-structures are shown to hold for 6-dimensional almost Hermitian geometry, too. In particular, any almost Hermitian 6-manifold in the class $(\mathcal{W}_1 + \mathcal{W}_4) \setminus \mathcal{W}_4$ has trivial canonical bundle. The geometries $\mathcal{W}_4$ for $SU(3)$- and $\mathcal{X}_4$ for G_2-manifolds are both special instances of G-structures with vectorial torsion. This notion was studied in [1]. The almost Hermitian manifolds and G_2-structures studied in this paper all fit in the wider framework of G-structures with three-form torsion, see [13, 2].

The aim of this note is to establish

Theorem 1. *Let (M, g, ϕ) be a compact 7-dimensional manifold locally conformally equivalent to a nearly parallel G_2-manifold. Then (M, g, ϕ) has constant scalar curvature if and only if (M, g) is either nearly parallel or conformally equivalent to the standard 7-sphere with its unique nearly parallel G_2 structure.*

Theorem 2. *Let (M, g, J) be a compact 6-dimensional manifold locally conformally equivalent to a nearly Kähler manifold. Then (M, g, J) has constant scalar curvature if and only if (M, g) is either nearly parallel or conformally equivalent to the 6-sphere with its unique nearly Kähler structure.*

In the last section we characterize complete Einstein G_2 and $SU(3)$ manifolds in the strict class $\mathcal{X}_1 + \mathcal{X}_4$ and $\mathcal{W}_1 + \mathcal{W}_4$, respectively. The phrases 'strict class' is used here to indicate that the G-structure is not in any sub-class of the one given. So a G_2-structure strictly in class $\mathcal{X}_1$ must, in particular, have non-trivial intrinsic torsion.

The results obtained in this paper are direct consequences of the following. The G-structures under consideration are described by the existence of certain *fundamental differential forms* $\omega_1, \ldots, \omega_p$, whose exterior derivatives determine the corresponding intrinsic torsion in full. First order identities on the G-invariant components of the intrinsic torsion descend from the closure of $d\omega_i$. These equations in general have non-trivial consequences as is seen by the examples considered here.

The relations coming from the second derivatives of the forms may also be seen as consequences of the first Bianchi identity, see for instance [27, 6].

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2. A lemma

The key to obtaining the results is the observation

Lemma 3. *Let M be a connected manifold equipped with a differentiable function $\phi \not\equiv 0$ and a one-form α such that*

$$\begin{aligned} d\alpha &= 0, \\ d\phi + \phi\alpha &= 0. \end{aligned} \tag{2.1}$$

Then ϕ is nowhere zero and $\alpha = -d\log|\phi|$.

Proof. Let ϕ and α be a function and one-form as in equation (2.1). By Poincaré's Lemma we may choose a covering U_i of M and functions $f_i\colon U_i \to \mathbb{R}$ such that $\alpha|_{U_i} = df_i$. Then equation (2.1) implies that the product $\phi\exp(f_i)$ is constant over each U_i. Therefore, if $\phi(p) \neq 0$ at some point p in, say U_0, then $\phi \neq 0$ over all U_0 and therefore also on each U_j that overlaps U_0. The conclusion now follows from connectedness. □

3. The G_2 case

A G_2-manifold is a 7-dimensional manifold M equipped with a special, so-called fundamental three-form ϕ, required to satisfy the following non-degeneracy condition

$$i_X\phi \wedge i_Y\phi \wedge \phi = 6g(X,Y)\, vol(g), \tag{3.1}$$

for some Riemannian metric g and orientation on M. The notation $i_X\phi$ means interior product of the vector field X with the three-form ϕ. It is well known that the covariant derivative of the fundamental three-form is determined by the exterior derivatives of ϕ and its Hodge dual $*\phi$. Using the representation theory of G_2 on the exterior algebra one may write these differentials as

$$\begin{aligned} d\phi &= \tau_0{*\phi} + 3\tau_1 \wedge \phi + *\tau_3, \\ d{*\phi} &= 4\tau_1 \wedge *\phi + \tau_2 \wedge \phi, \end{aligned}$$

for suitable forms $\tau_p \in \Omega^p$. In terms of the G_2 invariant splittings of the exterior algebra, $\tau_0 \in \Omega^0_1$, $\tau_1 \in \Omega^1_7$, $\tau_2 \in \Omega^2_{14}$, $\tau_3 \in \Omega^3_{27}$. The notation Ω^p_d indicates the space of p-forms taking values in the d-dimensional G_2 irreducible subspace $\Lambda^p_d \subset \Lambda^p$. The one-form τ_1 is also known as the *Lee form* of the G_2 manifold. The forms $\tau_0, \tau_1, \tau_2, \tau_3$ correspond to the Fernández-Gray classes [11] as follows

$$\tau_0 \leftrightarrow \mathcal{X}_1, \quad \tau_2 \leftrightarrow \mathcal{X}_2, \quad \tau_3 \leftrightarrow \mathcal{X}_3, \quad \tau_1 \leftrightarrow \mathcal{X}_4.$$

When we speak of the intrinsic torsion τ of a G_2 structure we mean the form of mixed degree $\tau = \tau_0 + \tau_1 + \tau_2 + \tau_3$ fixed by the fundamental three-form as above.

In particular, G_2-manifolds in the class $\mathcal{X}_1$ are characterized by the conditions $\tau_1 = \tau_2 = \tau_3 = 0$ and are called nearly parallel G_2-manifolds. It is well known that these spaces are Einstein with positive scalar curvature. From this it follows that τ_0 is constant [14].

A G_2-manifold in the Fernández-Gray class $\mathcal{X}_1 + \mathcal{X}_4$ satisfies $\tau_2 = \tau_3 = 0$. The structure equations for this case reduce to

$$\begin{aligned} d\phi &= \tau_0{*\phi} + 3\tau_1 \wedge \phi, \\ d{*\phi} &= 4\tau_1 \wedge *\phi, \end{aligned}$$

from which one infers

$$\begin{aligned} d^2\phi &= (d\tau_0 + \tau_0\tau_1) \wedge *\phi + 3d\tau_1 \wedge \phi, \\ d^2{*\phi} &= 4d\tau_1 \wedge *\phi. \end{aligned} \tag{3.2}$$

The latter equation implies that the component $(d\tau_1)_7 \in \Omega^2_7$ vanishes. Using this in the first equation, one deduces that the complementary component $(d\tau_1)_{14} \in \Omega^2_{14}$ also vanishes. Thus we recover the fact (see [24]) that G_2 structures in this class are, locally, conformal to a nearly parallel structure. Observe that equations (3.2) furthermore give us $d\tau_0 + \tau_0\tau_1 = 0$. Now our Lemma 3 applies with $\phi = \tau_0$ and $\alpha = \tau_1$. The connectedness of M leads to the conclusion in the form of this

Theorem 4. *Suppose M is a 7-dimensional manifold with a G_2-structure ϕ in the Fernández-Gray class $\mathfrak{X}_1 + \mathfrak{X}_4$. Then ϕ is either of class $\mathfrak{X}_4$, in which case (M, ϕ) is locally conformal to a parallel G_2-manifold, or (M, ϕ) is conformal to a nearly parallel G_2-manifold.*

4. The 6-dimensional almost Hermitian case

An almost Hermitian manifold is a Riemannian manifold (M^{2m}, g) equipped with an orthogonal almost complex structure J. The metric and the almost complex structure then define the fundamental two-form of the almost Hermitian structure:

$$\omega(X, Y) := g(JX, Y).$$

As opposed to the G_2 case above and the case of $SU(3)$ below, the components of the intrinsic torsion of a Hermitian structure cannot all be identified with differential forms [30]. Instead, the intrinsic torsion is detected by $d\omega$ along with the Nijenhuis tensor N_J.

From now on m is taken to be at least 3.

4.1. The Nijenhuis Tensor. For an almost complex structure J the Nijenhuis tensor measures the failure of the eigenspaces of J in the complexified tangent space to be involutive. Let $\pi'(X) = \frac{1}{2}(X - iJX)$ and $\pi'' = \frac{1}{2}(X + iJX)$ be the projections to the i-eigenspace T' and the $-i$-eigenspace T'', respectively. We set

$$\begin{aligned} N_J(X, Y) :=& \pi'[\pi''X, \pi''Y] + \pi''[\pi'X, \pi'Y] \\ =& \frac{1}{4}\left([X, Y] - [JX, JY] + J[JX, Y] + J[X, JY]\right). \end{aligned}$$

Using the metric we obtain an algebraically equivalent 3-tensor

$$N_J(X, Y; Z) = g(N_J(X, Y), Z),$$

with the property $N_J(JX, Y; Z) = N_J(X, JY; Z) = N_J(X, Y; JZ)$. Equivalently,

$$N_J \in \left[\!\left[\Lambda^{2,0} \otimes \Lambda^{1,0}\right]\!\right].$$

See for instance [10] for an explanation of the notation.

The space $[\![\Lambda^{3,0}]\!]$ is a subspace of $[\![\Lambda^{2,0} \otimes \Lambda^{1,0}]\!]$ in the natural way. The projection $[\![\Lambda^{2,0} \otimes \Lambda^{1,0}]\!] \to [\![\Lambda^{3,0}]\!]$ is given simply by skew-symmetrization. Write V for the orthogonal complement of $[\![\Lambda^{3,0}]\!]$ in $[\![\Lambda^{2,0} \otimes \Lambda^{1,0}]\!]$. Then we may split the Nijenhuis tensor accordingly

$$N_J = N_J^{3,0} + N_J^V.$$

One may now deduce that

$$(d\omega)^{3,0}(X, Y, Z) = 3g(N_J^{3,0}(X, Y), JZ) = 3N_J^{3,0}(X, Y, JZ). \tag{4.1}$$

Here we use sub- and superscripts to indicate projections of form. For instance, $d\omega^{3,0}$ is the projection of $d\omega$ in Λ^3 to the subspace $[\![\Lambda^{3,0}]\!]$.

The structure equations for an almost Hermitian manifold now can be written

$$d\omega = -3J_{(1)}N_J^{3,0} + 2\sigma_1 \wedge \omega + \sigma_3, \qquad N_J = N_J^{3,0} + N_J^V. \tag{4.2}$$

The first equation here employs the conventions $(J_{(1)}\alpha)(X,Y,\dots) := -\alpha(JX,Y,\dots)$, see [25]. This action of the complex structure J on differential forms is, generally speaking, distinct from the usual action given by $(J\alpha)(X_1,\dots,X_p) := (-1)^p\alpha(JX_1,\dots,JX_p)$.

The Gray-Hervella classes of an almost Hermitian manifold are in the following correspondence with the components in (4.2)

$$N_J^{3,0} \leftrightarrow \mathcal{W}_1, \quad N_J^V \leftrightarrow \mathcal{W}_2, \quad \sigma_3 \leftrightarrow \mathcal{W}_3, \quad \sigma_1 \leftrightarrow \mathcal{W}_4.$$

Almost Hermitian manifolds in the class $\mathcal{W}_1$, called nearly Kähler manifolds, are characterized by the conditions $N_J^V = \sigma_1 = \sigma_3 = 0$ or equivalently, by demanding that the covariant derivative of the almost complex structure with respect to the Levi-Civita connection be skew-symmetric, $(\nabla^g_X J)X = 0$ [15]. For an arbitrary one-form the relation

$$(d\theta)(X,Y) - (d\theta)(JX,JY) - d(J\theta)(JX,Y) + Jd(J\theta)(JX,Y) = 4g(N_J(X,Y), J(J\theta)^\#)$$

holds. Writing $d\theta^{2,0} = \frac{1}{2}(d\theta - Jd\theta)$ for the projection of $d\theta$ to $[\![\Lambda^{2,0}]\!]$ we have

Lemma 5. *Suppose (g,J) is an almost Hermitian structure in class $\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_4$. Let θ be a one-form and write $\theta' := J\theta$. Then*

$$(d\theta)^{2,0} + J_{(1)}(d\theta')^{2,0} = \tfrac{2}{3}\theta' \lrcorner (d\omega)^{3,0}$$

4.2. SU(3)-structures. A 6-dimensional manifold with an $SU(3)$-structure comes equipped with data $(g,J,\omega,\psi_+,\psi_-)$ invariant with respect to the action of $SU(3)$. Here g is a Riemannian metric, J is an almost complex structure, ω the fundamental two-form and ψ_+ and ψ_- are three-forms such that $\Psi := \psi_+ + i\psi_-$ is a complex $(3,0)$-form. These invariant tensors are not independent, in fact the triple (ω,ψ_+,ψ_-) with $\psi_+ + i\psi_-$ decomposable and compatible with ω by means of the equations below, defines both g and J, see [20]. Clearly the triple (g,ψ_+,ψ_-) will do the same. We choose a normalization with the following relations

$$\begin{gathered} \omega(X,JY) = g(X,Y), \\ \omega\wedge\psi_+ = 0 = \omega\wedge\psi_-, \\ 3\psi_+\wedge\psi_- = 2\omega^3 = 12\,vol_g, \\ *\omega = \tfrac{1}{2}\omega^2, \quad *\psi_+ = \psi_-, \quad J\psi_+ = -\psi_-. \end{gathered} \tag{4.3}$$

4.2.1. *Torsion classes and structure equations.* Under the action of $SU(3)$, $\Lambda^{3,0} = \mathbb{C}$ and $V\otimes\mathbb{C} \cong \Lambda_0^{1,1}$. This means that $V \cong 2\,\mathfrak{su}(3)$ and $[\![\Lambda^{3,0}]\!] \cong 2\mathbb{R}$. Moreover, for an $SU(3)$-structure $(\omega,\psi_\pm)$, the components of the Nijenhuis tensor can be computed from components of $(d\omega, d\psi_\pm)$. In fact, there are algebraic correspondences (see [8])

$$\begin{aligned} N_J^{3,0} &\leftrightarrow (d\omega)^{3,0} \leftrightarrow \left((d\psi_+)^{0,0}, (d\psi_-)^{0,0}\right), \\ N_J^V &\leftrightarrow \left((d\psi_+)_0^{2,2}, (d\psi_-)_0^{2,2}\right). \end{aligned}$$

The first arrow is given by equation (4.1). The notation here means the following. A generic four-form η has a component in $\Lambda^{2,2} = \Lambda^{2,0}\wedge\Lambda^{0,2}$. In dimension 6, $\Lambda^{2,2}$ has two subspaces, the primitive part $\Lambda_0^{2,2}$ (isomorphic to $\mathfrak{su}(3)\otimes\mathbb{C}$) and the real span of $\omega^2=\omega\wedge\omega$. The respective components of η are then $\eta_0^{2,2}$ and $\eta^{0,0}$.

The $SU(3)$-structure function $\nabla^g\Psi$ is completely determined by the exterior derivatives of the three forms ω, ψ_+ and ψ_-. These may be written as

$$\begin{aligned} d\omega &= 3\left(\sigma_0^+\psi_+ - \sigma_0^-\psi_-\right) + 2\sigma_1^+\wedge\omega + \sigma_3, \\ d\psi_+ &= -2\sigma_0^-\omega^2 + 3\sigma_1^+\wedge\psi_+ - \sigma_1^-\wedge\psi_- + \sigma_2^+\wedge\omega, \\ d\psi_- &= -2\sigma_0^+\omega^2 + 3\sigma_1^+\wedge\psi_- + \sigma_1^-\wedge\psi_+ + \sigma_2^-\wedge\omega. \end{aligned} \tag{4.4}$$

Here $\sigma_p^\pm$ are p-forms and σ_3 is a three-form. They correspond roughly to the classes $\mathcal{W}_1^+$, $\mathcal{W}_1^-$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_2^+$, $\mathcal{W}_2^-$, and $\mathcal{W}_3$ of [8], respectively (see also [4]). These determine the Gray-Hervella classes of the underlying almost Hermitian structure in the obvious way.

4.2.2. *A transformation.* Set $\lambda := \sigma_0^+ + i\sigma_0^-$ and $\Lambda := |\lambda|$. In neighbourhoods with λ non-vanishing an argument $\varphi := \arg(\lambda) := \arctan\left(\frac{\sigma_0^-}{\sigma_0^+}\right)$ may be chosen. We then set

$$\begin{aligned} \tilde\omega &:= \Lambda^2\omega, \\ \tilde\psi_+ &:= \Lambda^2\left(\sigma_0^+\psi_+ - \sigma_0^-\psi_-\right), \\ \tilde\psi_- &:= \Lambda^2\left(\sigma_0^-\psi_+ + \sigma_0^+\psi_-\right). \end{aligned}$$

This gives the somewhat simpler structure equations

$$\begin{aligned} d\tilde\omega &:= 3\tilde\psi_+ + 2\tilde\sigma_1^+\wedge\omega + \tilde\sigma_3, \\ d\tilde\psi_+ &:= 3\tilde\sigma_1^+\wedge\tilde\psi_+ - \tilde\sigma_1^-\wedge\tilde\psi_- + \tilde\sigma_2^+\wedge\tilde\omega, \\ d\tilde\psi_- &:= -2\tilde\omega^2 + 3\tilde\sigma_1^+\wedge\tilde\psi_- + \tilde\sigma_1^-\wedge\tilde\psi_- + \tilde\sigma_2^-\wedge\tilde\omega. \end{aligned}$$

where

$$\begin{aligned} \tilde\sigma_1^+ &:= \sigma_1^+ + \Lambda^{-1}d\Lambda, \qquad \tilde\sigma_1^- := \sigma_1^- - d\varphi, \\ \tilde\sigma_2^+ &:= \sigma_0^+\sigma_2^+ - \sigma_0^-\sigma_2^-, \qquad \tilde\sigma_2^- := \sigma_0^-\sigma_2^+ + \sigma_0^+\sigma_2^-, \\ \tilde\sigma_3 &:= \Lambda^2\sigma_3. \end{aligned}$$

In particular, the structure equations of a nearly Kähler 6-manifold can always be put on the form [19, 29]

$$d\omega = 3\psi_+, \qquad d\psi_- = -2\omega^2.$$

It is well known that these spaces are Einstein with positive scalar curvature [16].

Remark 6. The Lee form of an almost Hermitian manifold is, up to scale, the co-differential $\delta^g\omega$ of the fundamental two-form. This means that, again up to scale, σ_1^+ is the Lee-form. A pointwise conformal change of the metric $\bar g = e^{2f}g$ acts on the forms $\omega, \psi_\pm$ by $\bar\omega = e^{2f}\omega$, $\bar\psi_\pm = e^{3f}\psi_\pm$. Therefore, the conformally changed torsion component σ_1^+ becomes $\bar\sigma_1^+ = \sigma_1^+ + df$ while all other torsion components merely rescale. So σ_1^- does not really correspond to the class $\mathcal{W}_5$ but rather to "$3W_4 + 2W_5$". This choice for the one-forms was introduced by Martín Cabrera [25].

5. Locally conformally nearly Kähler 6-folds

For a 6-dimensional almost Hermitian manifold in the class $\mathcal{W}_1 + \mathcal{W}_4$ a (possibly local) choice of trivialization (ψ_+, ψ_-) allows us to write the structure equations (4.2), (4.4) as

$$\begin{aligned} d\omega &= 3\left(\sigma_0^+\psi_+ - \sigma_0^-\psi_-\right) + 2\sigma_1^+ \wedge \omega, \\ d\psi_+ &= -2\sigma_0^-\omega^2 + 3\sigma_1^+ \wedge \psi_+ - \sigma_1^- \wedge \psi_-, \\ d\psi_- &= -2\sigma_0^+\omega^2 + 3\sigma_1^+ \wedge \psi_- + \sigma_1^- \wedge \psi_+. \end{aligned} \tag{5.1}$$

Differentiating each equation yields

$$0 = 3(d\sigma_0^+ + \sigma_0^+\sigma_1^+ - \sigma_0^-\sigma_1^-)\psi_+ - 3(d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^-)\psi_- + 2d\sigma_1^+\omega, \tag{5.2}$$

$$0 = -2(d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^-)\omega^2 + 3d\sigma_1^+\psi_+ - d\sigma_1^-\psi_-, \tag{5.3}$$

$$0 = -2(d\sigma_0^+ + \sigma_0^+\sigma_1^+ - \sigma_0^-\sigma_1^-)\omega^2 + 3d\sigma_1^+\psi_- + d\sigma_1^-\psi_+. \tag{5.4}$$

These have the following immediate consequences. Equation (5.2) shows that $d\sigma_1^+$ is a $(2,0)+(0,2)$ form as a linear combination of one-forms contracted with ψ_+ and ψ_-. Using standard identities such as $J(\sigma \wedge \psi_+) = \sigma \wedge \psi_-$ for an arbitrary two-form σ and $J(\sigma \wedge \omega) = \sigma \wedge \omega$ for a $(1,1)$-form, as well as $*(\theta \wedge \psi_-) = \theta \lrcorner \psi_+ = (J\theta) \lrcorner \psi_-$ for a one-form θ, leads to the equivalent set of equations:

$$d\sigma_0^+ + \sigma_0^+\sigma_1^+ - \sigma_0^-\sigma_1^- = J(d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^-), \tag{5.5}$$

$$d\sigma_1^+ = 3(d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^-) \lrcorner \psi_+, \tag{5.6}$$

$$(d\sigma_1^-)^{2,0} = 7(d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^-) \lrcorner \psi_-. \tag{5.7}$$

Lemma 7. *Suppose (M^6, ω, J) is an almost Hermitian manifold in the class $\mathcal{W}_1 + \mathcal{W}_4$. Then the Lee form is closed if and only if (ω, J) is either* globally *conformal to a nearly Kähler structure (ω', J') on M or locally conformally equivalent to a Kähler structure.*

Proof. Suppose $d\sigma_1^+ = 0$. Locally, we pick a smooth trivialisation (ψ_+, ψ_-) of $[\![\Lambda^{3,0}]\!]$. Then, locally, equations (5.6) and (5.5) show that

$$\begin{aligned} d\sigma_0^+ + \sigma_0^+\sigma_1^+ - \sigma_0^-\sigma_1^- &= 0, \\ d\sigma_0^- + \sigma_0^-\sigma_1^+ + \sigma_0^+\sigma_1^- &= 0, \end{aligned}$$

whence

$$d\left((\sigma_0^+)^2 + (\sigma_0^-)^2\right) + ((\sigma_0^+)^2 + (\sigma_0^-)^2)(2\sigma_1^+) = 0.$$

However,

$$\phi := (\sigma_0^+)^2 + (\sigma_0^-)^2 = \tfrac{1}{9}\left\|d\omega^{3,0}\right\|^2$$

is a globally well-defined, smooth function, and $\alpha := 2\sigma_1^+$ is closed. So Lemma 3 applies and we conclude that $\left\|d\omega^{3,0}\right\|$ is either non-zero everywhere, or it vanishes at all points. □

Remark 8. Almost Hermitian manifolds in the class $\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4$ are characterized by $N_J^V = 0$, i.e. the Nijenhuis tensor is totally skew-symmetric. This amounts to the existence of a linear connection preserving the almost Hermitian structure with totally skew-symmetric torsion [13]. This is in particular true for almost Hermitian structures in the class $\mathcal{W}_1 + \mathcal{W}_4$. Another is the class of so-called quasi-integrable structures, see [5]. The properties of this in stark contrast to the situation for the $\mathcal{W}_1 + \mathcal{W}_4$ structures studied here. In fact, for quasi-integrable complex structures the Nijenhuis tensor either vanishes identically or the zero locus of N_J is of co-dimension at least 2 [5].

Theorem 9. *Let M be a 6-dimensional manifold with an almost Hermitian structure (ω, J) in the Gray-Hervella class $\mathcal{W}_1 + \mathcal{W}_4$. Then either (ω, J) is locally conformally equivalent to Kähler structure on M, or (ω, J) is a conformal transformation of a nearly Kähler structure.*

Proof. Write M as a disjoint union $M_0 \cup M_1$ where

$$M_0 := \left\{x \in M \colon (d\omega)^{3,0} = 0\right\}, \qquad M_1 := \left\{x \in M \colon (d\omega)^{3,0} \neq 0\right\}.$$

On the open submanifold M_1 there is a canonical choice of trivialization of $[\![\Lambda^{3,0}]\!]$ given by taking $\psi_+ = (d\omega)^{3,0}$. After a suitable transformation (as in section 4.2.2) we obtain the structure equations

$$\begin{aligned} d\tilde{\omega} &= 3\tilde{\psi}_+ + 2\tilde{\sigma}_1^+ \wedge \tilde{\omega}, \\ d\tilde{\psi}_+ &= 3\tilde{\sigma}_1^+ \wedge \tilde{\psi}_+ - \tilde{\sigma}_1^- \wedge \tilde{\psi}_-, \\ d\tilde{\psi}_- &= -2\tilde{\omega}^2 + 3\tilde{\sigma}_1^+ \wedge \tilde{\psi}_- + \tilde{\sigma}_1^- \wedge \tilde{\psi}_+. \end{aligned} \tag{5.8}$$

Equations (5.5), (5.6) and (5.7) then become

$$\begin{aligned} \tilde{\sigma}_1^+ &= J\tilde{\sigma}_1^-, \\ d\tilde{\sigma}_1^+ &= 3\tilde{\sigma}_1^- \lrcorner \tilde{\psi}_+, \\ (d\tilde{\sigma}_1^-)^{2,0} &= 7\tilde{\sigma}_1^- \lrcorner \tilde{\psi}_-. \end{aligned} \tag{5.9}$$

Using Lemma 5 with $\theta = \tilde{\sigma}_1^+$, $\theta' = \tilde{\sigma}_1^-$, and the identity $J_{(1)}(\sigma \lrcorner \psi_\pm) = (J\sigma) \lrcorner \psi_\pm = \mp\sigma \lrcorner \psi_\mp$ valid for all one-forms σ, we get

$$d\tilde{\sigma}_1^+ - J_{(1)}(d\tilde{\sigma}_1^-)^{2,0} = -2\tilde{\sigma}_1^+ \lrcorner \tilde{\psi}_- = -2\tilde{\sigma}_1^- \lrcorner \tilde{\psi}_+.$$

This is only compatible with the relations (5.9) if $\tilde{\sigma}_1^- \lrcorner \tilde{\psi}_+ = 0$. Therefore $\tilde{\sigma}_1^+ = \tilde{\sigma}_1^- = 0$ and the original one-forms $\sigma_1^\pm$ are, in fact, exact on M_1. Moreover, on the interior of M_0, $d\omega = 2\sigma_1^+ \wedge \omega$, so $d\sigma_1^+|_{\mathrm{int}(M_0)} = 0$ also holds.

So the set of points at which $d\sigma_1^+ \neq 0$, which clearly is open, is the common boundary of two open sets in M, at least one of which is non-empty. Therefore $d\sigma_1^+ = 0$ on all of M and Lemma 7 completes the proof. □

6. Proof of Theorem 1 and Theorem 2

Theorem 4 and Theorem 9 show that the Riemannian manifold (M, g) is globally conformal to an Einstein space of positive scalar curvature. Further, if the scalar curvature is constant then the Obata Theorem (see [28] or the more recent proof in [23]) tells us that the conformal transformation making the metric Einstein is trivial, or else (M, g) is the standard sphere. The two theorems also show that the conformal change takes the G structure to a nearly Kähler structure (in dimension 6) or nearly parallel structure (for dimension 7). On spheres such structures are unique up to isometry, see [12] □

7. Curvature classification

The Riemannian curvature tensor of a nearly Kähler 6-manifolds or a nearly parallel G_2-manifold is especially simple. In fact, viewing curvature tensors as bundle endomorphisms $R\colon \Lambda^2 \to \Lambda^2$, the curvature splits as

$$R^g = R^{\mathfrak{g}} + \frac{s_g}{2n(n-1)} Id_{\Lambda^2}. \tag{7.1}$$

where $R^{\mathfrak{g}}$, formally, is the curvature tensor of a space with holonomy algebra $\mathfrak{g}$ and n is the dimension of the underlying space, see [29]. This formula is reminiscent of the curvature formula for a Riemannian manifold with holonomy $Sp(n)\,Sp(1)$ of [30]. In the cases of concern, $\mathfrak{g}$ and n are equal to $\mathfrak{su}(3)$ and 6 for nearly Kähler and $\mathfrak{g}_2$ and 7 for nearly parallel G_2. In either situation the tensor $R^{\mathfrak{g}}$ takes values in a G-irreducible subspace of the space of algebraic Weyl tensors, i.e., algebraic curvature tensors with vanishing Ricci contraction. Standard identities [3] now make it possible to deduce the form of the curvature tensor for an almost Hermitian or G_2 space of type $\mathcal{W}_1+\mathcal{W}_4$, or $\mathcal{X}_1 + \mathcal{X}_4$, respectively. Further details may be found in [9].

Theorem 10. *(a) Suppose (M, ϕ) is a G_2 manifold of strict type $\mathcal{X}_1 + \mathcal{X}_4$ such that the associated metric g is complete and Einstein. Then (M, g) is isometric to either the sphere, the hyperbolic space or the euclidean space equipped with a constant curvature metric.*

(b) Suppose (M, ω, J) is an almost Hermitian 6-manifold of strict type $\mathcal{W}_1+\mathcal{W}_4$ such that the associated metric g is complete and Einstein. Then (M, g) is isometric to either the sphere, hyperbolic space or euclidean space equipped with a constant curvature metric.

Proof. Theorem 4 and Theorem 9 show that under the given assumptions the metrics in either case must be both Einstein and conformally Einstein. Up to isometry and homothety, there are only five Riemannian manifolds such that the metric is complete and Einstein and a conformal change of the metric is also Einstein, see the Main Theorem of [22]. Of these only the ones listed under item a and b have positive scalar curvature after the conformal change. □

Corollary 11. *Let M be a G_2 manifold or almost Hermitian 6-manifold of strict type $\mathcal{X}_1+\mathcal{X}_4$ or $\mathcal{W}_1+\mathcal{W}_4$, respectively with complete metric. Assume that the Riemannian curvature of M is of the form (7.1). Then M has constant sectional curvature and in particular $R^{\mathfrak{g}} = 0$.*

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