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THE ORDER OF PLURISUBHARMONICITY ON PSEUDOCONVEX DOMAINS WITH LIPSCHITZ BOUNDARIES

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Abstract. Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain with Lipschitz boundary. Diederich and Fornaess have shown that when the boundary of Ω is C^2, there exists a constant $0 < \eta < 1$ and a defining function ρ for Ω such that $-(-\rho)^\eta$ is a plurisubharmonic function on Ω. In this paper, we show that the result of Diederich and Fornaess still holds when the boundary is only Lipschitz.

1. Introduction

Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain. We denote the boundary of Ω by $\partial\Omega$ and say that $\partial\Omega$ is Lipschitz if it can be written locally as the graph of a Lipschitz function. Define the distance function δ for $\partial\Omega$ by $\delta(z) = \inf_{w\in\partial\Omega} |z - w|$. By Oka's Lemma, Ω is pseudoconvex if and only if there exists some open neighborhood U of $\partial\Omega$ such that $-\log\delta$ is plurisubharmonic on $U \cap \Omega$. By suitably modifying δ in the interior of Ω, we can obtain a plurisubharmonic exhaustion function for Ω.

In many applications, it is desirable to have a bounded plurisubharmonic exhaustion function for a given pseudoconvex domain (which $-\log\delta$ clearly fails to be). In [12], Kerzman and Rosay show that such functions exist locally on Lipschitz domains and globally on C^1 domains, but their functions fail to satisfy global estimates. Demailly [5] extends Kerzman and Rosay's result to obtain a global function on Lipschitz domains which is comparable to $\frac{1}{\log\delta}$. However, such functions remain quite singular at the boundary. In this paper, we wish to consider the possibility of Hölder continuous plurisubharmonic exhaustion functions on Lipschitz domains.

Following Cao, Shaw, and Wang [3], if PSH(Ω) is the space of plurisubharmonic function on Ω, we define the order of plurisubharmonicity for a domain Ω as follows:

$$\eta(\Omega) = \sup\left\{\eta \in [0,1] : \exists\lambda \in \mathrm{PSH}(\Omega) \text{ and } k > 1 \text{ such that } \frac{1}{k}\delta^\eta < -\lambda < k\delta^\eta\right\}.$$

In [7], Diederich and Fornaess show that whenever $\partial\Omega$ is C^2, $\eta(\Omega) > 0$. Furthermore, they use worm domains [6] to show that for any $0 < \eta < 1$, there exists a pseudoconvex domain with smooth boundary such that $\eta(\Omega) < \eta$. Hence, $\eta(\Omega) > 0$ is the strongest result which can be expected for generic pseudoconvex domains. On C^3 domains, Range [13] has a simplified proof which provides a concrete representation for λ in terms of any C^3 defining function ρ.

Our main result in this paper is to extend the result of Diederich and Fornaess to all Lipschitz pseudoconvex domains, as follows:

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Theorem 1.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain with Lipschitz boundary. Then for some $0 < \eta < 1$ there exists a strictly plurisubharmonic function λ on Ω and constants $k > 1$ and $c > 0$ such that $\frac{1}{k}\delta^\eta < -\lambda < k\delta^\eta$ and*

$$i\partial\overline{\partial}\lambda \geq ic\delta^\eta \partial\overline{\partial}\,|z|^2 \tag{1.1}$$

Remark 1.2. Equivalently, we can set $\rho = -(-\lambda)^{\frac{1}{\eta}}$ in Ω and conclude that Ω admits a defining function ρ such that $-(-\rho)^\eta$ is strictly plurisubharmonic in Ω and satisfies (1.1).

Several known results that use the Diederich and Fornaess result on C^2 domains can be immediately generalized to Lipschitz domains using this theorem. In [1], Berndtsson and Charpentier prove the following result for C^2 pseudoconvex domains, and observe that their result will also follow on any Lipschitz pseudoconvex domain satisfying the conclusions of Theorem 1.1.

Theorem 1.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain with Lipschitz boundary. Then for any $\frac{\eta(\Omega)}{2} > s > 0$, the Bergman projection P and the canonical solution operator $\overline{\partial}^* N$ for the $\overline{\partial}$ equation are bounded in the Sobolev space $W^s(\Omega)$.*

Here, the Bergman projection P denotes the orthogonal projection from $L^2(\Omega)$ to $\ker \overline{\partial} \cap L^2(\Omega)$, where $\overline{\partial}$ is the Cauchy-Riemann operator. For background on the L^2 theory for $\overline{\partial}$, see [10], [11], or [4].

In [3], Cao, Shaw, and Wang extend Berndtsson and Charpentier's result to obtain estimates for the $\overline{\partial}$-Neumann operator. Although they work in $\mathbb{CP}^n$ in this paper, their proof for this result also applies in $\mathbb{C}^n$ (see [9]).

Theorem 1.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain with Lipschitz boundary. Then for any $\frac{\eta(\Omega)}{2} > s > 0$, the $\overline{\partial}$-Neumann operator N is bounded in the Sobolev space $W^s_{(p,q)}(\Omega)$ for all $0 \leq p \leq n$ and $1 \leq q \leq n$.*

For background on the $\overline{\partial}$-Neumann problem, see [8], [2], or [4].

As in [12] and [5], the proof of Theorem 1.1 will begin by locally translating $-\log \delta$ in a direction transverse to the boundary, in order to obtain bounded plurisubharmonic functions satisfying certain estimates. We proceed to subdivide each neighborhood of the boundary into strips equidistant from the boundary so that we can refine our plurisubharmonic function on each strip. The key step of the proof is to show that the new functions on each strip can be patched together to obtain a global plurisubharmonic function.

2. Proof of the Main Theorem

Let $r > 0$ be some constant such that for any $p \in \partial\Omega$, $B(p, r)$ is a neighborhood on which $\partial\Omega$ can be written as the graph of a Lipschitz function. In other words, there exist local orthonormal coordinates $(z_p^1, z_p^2, \ldots, z_p^n) = (z_p', z_p^n)$ on $B(p, r)$ and a Lipschitz function φ_p such that

$$\Omega \cap B(p, r) = \left\{ z_p \in B(p, r) : \mathrm{Im} z_p^n < \varphi_p(z_p', \mathrm{Re} z_p^n) \right\}.$$

We define $\rho_p = \mathrm{Im} z_p^n - \varphi_p(z_p', \mathrm{Re} z_p^n)$. Note that for some $k_p > 1$, we have $\delta \leq -\rho_p \leq k_p\delta$ on $B(p, r) \cap \Omega$. Choose some finite indexing set $\mathcal{P} \subset \partial\Omega$ such that

$\{B(p,r/3)\}_{\mathcal{P}}$ is a finite cover of $\partial\Omega$, and let $k = \sup_{\mathcal{P}} k_p$. Let $\chi_p \in C_0^\infty(B(p,r/2))$ such that $\chi_p|_{B(p,r/3)} \equiv 1$ and $0 \leq \chi_p \leq 1$. Choose $t > 0$ such that $\chi_p + (t-1)|z|^2$ is plurisubharmonic for all $p \in \mathcal{P}$ (i.e. $i\partial\overline{\partial}(\chi_p + t|z|^2) \geq i\partial\overline{\partial}|z|^2$), and choose $d > 0$ such that $|z| \leq d$ on $\overline{\Omega}$.

Because we are planning to construct a plurisubharmonic function comparable to δ^η out of translations of $-\log\delta$, we need a way of comparing such functions. To that end, we define $f_\eta(x) = 1 + 2\eta\log\left(\frac{x+1}{2}\right) - x^\eta$. To estimate f_η, we compute $f'_\eta(x) = \frac{2\eta}{x+1} - \eta x^{\eta-1}$. Using convexity to bound x^η and $x^{\eta-1}$ by their tangent lines, we observe that $x^\eta \leq 1 + \eta(x-1)$ and $x^{\eta-1} \geq 1 + (\eta-1)(x-1)$ when $0 < \eta < 1$, so

$$f'_\eta(x) \geq \frac{2\eta}{x+1} - \frac{\eta}{x}(1 + \eta(x-1)) = \frac{\eta(x-1)}{x(x+1)}(1 - \eta(x+1)),$$
$$f'_\eta(x) \leq \frac{2\eta}{x+1} - \eta(1 + (\eta-1)(x-1)) = \frac{\eta(1-x)}{x+1}(1 - (1-\eta)(x+1)).$$

Since $f_\eta(1) = 0$, $f'_\eta(x) > 0$ when $1 < x < \frac{1-\eta}{\eta}$, and $f'_\eta(x) < 0$ when $\frac{\eta}{1-\eta} < x < 1$, we conclude that $f_\eta(x) \geq 0$ whenever $\frac{\eta}{1-\eta} < x < \frac{1-\eta}{\eta}$ (assuming now that $0 < \eta < \frac{1}{2}$).

Let $M = 2(2 + td^2)\log k^2$. Observe that

$$\lim_{\eta\to 0^+} \frac{f_\eta(x)}{\eta x^\eta} = 2\log\left(\frac{x+1}{2}\right) - \log x = \log\left(\frac{(x+1)^2}{4x}\right), \tag{2.1}$$

and

$$\lim_{x\to+\infty} \log\left(\frac{(x+1)^2}{4x}\right) = \lim_{x\to 0^+} \log\left(\frac{(x+1)^2}{4x}\right) = \infty,$$

so there exists some $a_0 > 1$ such that $\log\left(\frac{(x+1)^2}{4x}\right) > M$ whenever $x > a_0$ or $0 < x < \frac{1}{a_0}$. Choose $a > \max\left\{a_0, 2k^4 - 1\right\}$ (we will need $a > 2k^4 - 1$ in order to extend our function into the interior of Ω) and choose $\frac{1}{2} > \eta_0 > 0$ such that $\frac{1-\eta_0}{\eta_0} > a$. Then $f_\eta(x) \geq 0$ on $[\frac{1}{a}, a]$ for all $0 < \eta < \eta_0$. Choose such an $\eta > 0$ so that by (2.1) we have $f_\eta(a) > \eta a^\eta M$ and $f_\eta(1/a) > \eta a^{-\eta} M$.

Let $b > 0$ be such that $-\log\delta$ is plurisubharmonic in Ω when $0 < \delta < b$ and $\{z \in \Omega : 0 < \delta < b\} \subset \bigcup_{\mathcal{P}} B(p, r/3)$. Then for any $0 < \varepsilon < \frac{b}{a+1}$, we define $U_{\varepsilon,p} = \left\{z \in \Omega \cap B(p,r/2) : \frac{\varepsilon}{a} < \delta < \varepsilon a\right\}$. On $U_{\varepsilon,p}$, we set

$$\lambda_{\varepsilon,p}(z_p) = -\varepsilon^\eta\left(1 + 2\eta\log\left(\frac{k\delta(z'_p, z_p^n - i\varepsilon)}{2\varepsilon}\right) - 2\eta\log k^2(\chi_p - 1 + t(|z|^2 - d^2))\right). \tag{2.2}$$

Here, we emphasize that $|z|^2$ is defined in terms of global coordinates on $\mathbb{C}^n$, rather than the local coordinates on $B(p,r)$. Observe also that

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} \geq i2\eta\varepsilon^\eta \log k^2 \partial\overline{\partial}|z|^2 > i(2\eta a^{-\eta}\log k^2)\delta^\eta\partial\overline{\partial}|z|^2$$

on $U_{\varepsilon,p}$, since $-\log\delta(z'_p, z_p^n - i\varepsilon)$ is plurisubharmonic.

To compute estimates of $\lambda_{\varepsilon,p}$ that are independent of p, we will need to observe that

$$\delta(z_p', z_p^n - i\varepsilon) \leq -\rho_p(z_p) + \varepsilon \leq k(\delta(z) + \varepsilon), \tag{2.3}$$

$$\delta(z_p', z_p^n - i\varepsilon) \geq \frac{1}{k}(-\rho_p(z_p) + \varepsilon) \geq \frac{1}{k}(\delta(z) + \varepsilon). \tag{2.4}$$

Hence, when $\chi_p = 1$, we have

$$\lambda_{\varepsilon,p}(z) \geq -\varepsilon^\eta \left(1 + 2\eta \log\left(\frac{k^2(\delta(z)+\varepsilon)}{2\varepsilon}\right) - 2\eta t \log k^2(|z|^2 - d^2)\right),$$

and when $\chi_p = 0$, we have

$$\begin{aligned}\lambda_{\varepsilon,p}(z) &\leq -\varepsilon^\eta \left(1 + 2\eta \log\left(\frac{\delta(z)+\varepsilon}{2\varepsilon}\right) - 2\eta \log k^2(-1 + t(|z|^2 - d^2))\right) \\ &= -\varepsilon^\eta \left(1 + 2\eta \log\left(\frac{k^2(\delta(z)+\varepsilon)}{2\varepsilon}\right) - 2\eta t \log k^2(|z|^2 - d^2)\right).\end{aligned}$$

Since $\lambda_{\varepsilon,p_1} \geq \lambda_{\varepsilon,p_2}$ on $\overline{B(p_1, r/3)} \backslash B(p_2, r/2)$, the function

$$\lambda_\varepsilon(z) = \sup_{\{p \in \mathcal{P} : z \in B(p, r/2)\}} \lambda_{\varepsilon,p}(z)$$

is continuous on $U_\varepsilon = \left\{z \in \Omega : \frac{\varepsilon}{a} < \delta < \varepsilon a\right\}$ and

$$i\partial\overline{\partial}\lambda_\varepsilon(z) \geq i(2\eta a^{-\eta} \log k^2)\delta^\eta \partial\overline{\partial}|z|^2. \tag{2.5}$$

Note that this patching argument is a variation of the method used in [5], and that (2.5) follows from Lemma 2.10 in the same paper.

Now, observe that on U_ε, we have

$$\lambda_\varepsilon \leq -\varepsilon^\eta \left(1 + 2\eta \log\left(\frac{\delta+\varepsilon}{2\varepsilon}\right)\right) = -\varepsilon^\eta f_\eta\left(\frac{\delta}{\varepsilon}\right) - \delta^\eta \leq -\delta^\eta, \tag{2.6}$$

and

$$\begin{aligned}\lambda_\varepsilon &\geq -\varepsilon^\eta \left(1 + 2\eta \log\left(\frac{k^2(\delta+\varepsilon)}{2\varepsilon}\right) + 2\eta \log k^2(1 + td^2)\right) \\ &= -\varepsilon^\eta \left(f_\eta\left(\frac{\delta}{\varepsilon}\right) + 2\eta \log k^2(2 + td^2)\right) - \delta^\eta.\end{aligned} \tag{2.7}$$

Referring back to the defining inequalities for a and M, we see that when $\delta = \varepsilon a$, we have

$$\lambda_\varepsilon \leq -\varepsilon^\eta f_\eta(a) - \delta^\eta < -\delta^\eta \eta M - \delta^\eta.$$

Since $f_\eta(1) = 0$, we also have

$$\lambda_{\varepsilon a} = \lambda_\delta \geq -\delta^\eta \eta M - \delta^\eta.$$

Hence $\lambda_{\varepsilon a} > \lambda_\varepsilon$ when $\delta = \varepsilon a$, and similarly $\lambda_{\varepsilon/a} > \lambda_\varepsilon$ when $\delta = \varepsilon/a$. If we set $\varepsilon_n = a^{-n}\frac{b}{a+1}$, we can then define

$$\tilde{\lambda}(z) = \sup_{\{n \in \mathbb{N} : z \in U_{\varepsilon_n}\}} \lambda_{\varepsilon_n}(z).$$

As before, we have

$$i\partial\overline{\partial}\tilde{\lambda} \geq i(2\eta a^{-\eta} \log k^2)\delta^\eta \partial\overline{\partial}|z|^2,$$

while (2.6) and (2.7) give us

$$-\delta^\eta \geq \tilde{\lambda} \geq -\delta^\eta \left(a^\eta \left(\sup_{[1/a,a]} f_\eta + \eta M \right) + 1 \right).$$

To complete the proof, we need only find a strictly plurisubharmonic extension of $\tilde{\lambda}$ into the interior of Ω. Referring back to (2.2) and (2.4), we see that when $\delta = \varepsilon a$,

$$\lambda_\varepsilon \leq -\varepsilon^\eta \left(1 + 2\eta \log \left(\frac{a+1}{2} \right) - 2\eta t \log k^2 (|z|^2 - d^2) \right),$$

and when $\delta = \varepsilon$, we use (2.2) and (2.3) to show

$$\begin{aligned} \lambda_\varepsilon &\geq -\varepsilon^\eta \left(1 + 2\eta \log k^2 - 2\eta \log k^2 (-1 + t(|z|^2 - d^2)) \right) \\ &= -\varepsilon^\eta \left(1 + 4\eta \log k^2 - 2\eta t \log k^2 (|z|^2 - d^2) \right). \end{aligned}$$

Since $a > 2k^4 - 1$, we have $\eta \log \left(\frac{a+1}{2} \right) > 2\eta \log k^2$. Hence, we can define

$$\lambda_0 = -\varepsilon_1^\eta \left(1 + \eta \log \left(\frac{a+1}{2} \right) + 2\eta \log k^2 - 2\eta \log k^2 t (|z|^2 - d^2) \right)$$

satisfying $\lambda_0 < \lambda_{\varepsilon_1}$ when $\delta = \varepsilon_1$ and $\lambda_0 > \lambda_{\varepsilon_1}$ when $\delta = \varepsilon_1 a$. Clearly, λ_0 is also strictly plurisubharmonic. To conclude the proof, we define

$$\lambda = \begin{cases} \tilde{\lambda} & \delta \leq \varepsilon_1 \\ \sup \left\{ \tilde{\lambda}, \lambda_0 \right\} & \varepsilon_1 < \delta < \varepsilon_1 a \\ \lambda_0 & \delta \geq \varepsilon_1 a \end{cases}.$$

□

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