CURVES ON ALGEBRAIC MODELS OF SMOOTH MANIFOLDS

W. Kucharz

1. Introduction

Let X be a compact nonsingular real algebraic set (in $\mathbb{R}^n$ for some n). In real algebraic geometry one frequently encounters the subgroup

$$H_k^{\mathrm{alg}}(X, \mathbb{Z}/2)$$

of the homology group $H_k(X, \mathbb{Z}/2)$, generated by the homology classes represented by irreducible algebraic subsets of X of dimension k. The reader may refer to [4] for a short survey of properties and applications of the groups $H_k^{\mathrm{alg}}(-, \mathbb{Z}/2)$. Some homology classes are always in $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2)$. Namely, if $w_i(X)$ is the ith Stiefel-Whitney class of X, then the homology class Poincaré dual to the cup product $w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)$, where $i_1 + \cdots + i_r = \dim X - k$, is in $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2)$. In general, it is very difficult to compute $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2)$, even for $k = 1$. This paper deals with certain interesting, also hard to compute, subgroups of $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$.

By a *real algebraic curve* we mean a one-dimensional irreducible real algebraic set. If C is such a curve, then there is a unique (up to isomorphism) nonsingular projective complex algebraic curve V defined over $\mathbb{R}$, whose set of real points $V(\mathbb{R})$ is birationally equivalent to C. The genus of C is, by definition, the genus of V. In particular, C is rational (that is, birationally equivalent to $\mathbb{R}$) if and only if it is of genus 0. The curve C is said to be *dividing* if $V \backslash V(\mathbb{R})$ is disconnected.

Given a nonnegative integer g, denote by

$$A(X, g) \ (\text{resp. } B(X, g))$$

the subgroup of $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$ generated by the homology classes represented by algebraic curves in X of genus g (resp. of genus at most g). Obviously, $A(X, g)$ is a subgroup of $B(X, g)$. Moreover, $B(X, g) = H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$, provided g is sufficiently large (depending on X).

We are also interested in the subgroup

$$D(X)$$

of $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$ generated by the homology classes represented by dividing algebraic curves in X. Clearly, $A(X, 0)$ is a subgroup of $D(X)$.

Recall that every compact smooth (of class C^∞) manifold M is diffeomorphic to a nonsingular real algebraic set, called an *algebraic model* of M, cf. [11] or [3, Theorem 14.1.10] (and also [9] for a weaker but influential result). We precisely describe how the groups $A(X, g)$ and $B(X, g)$ vary as X runs through the class of all algebraic models of M.

Theorem 1.1. *Let M be a compact connected smooth manifold of dimension at least 2, and let g be a nonnegative integer. For any subgroup G of $H_1(M, \mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ with*

$$\varphi_*(G) = A(X, g) = B(X, g).$$

Here, as usual,

$$\varphi_* : H_*(M, \mathbb{Z}/2) \longrightarrow H_*(X, \mathbb{Z}/2)$$

denotes the homomorphism induced by φ. Note that in Theorem 1.1 one cannot take $G = 0$ and replace the groups $A(X, g)$ and $B(X, g)$ by $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$. Indeed, if X is any algebraic model of real projective n-space $\mathbb{P}^n(\mathbb{R})$, with n even, then $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2) = H_k(X, \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leq k \leq n$ since $w_1(X) \neq 0$. Moreover, it is not known whether there is an algebraic model X of the n-torus $\mathbb{R}^n/\mathbb{Z}^n$, with n odd greater than 1, for which $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2) = 0$.

We also have an analogous result to Theorem 1.1 dealing with the group $D(-)$.

Theorem 1.2. *Let M be a compact connected smooth manifold of dimension at least 2. Assume that M is orientable and the group $H_1(M, \mathbb{Z})$ has no 2-torsion. For any subgroup G of $H_1(M, \mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ with*

$$\varphi_*(G) = A(X, 0) = D(X).$$

We were not able to determine whether the assumption that M is orientable and $H_1(M, \mathbb{Z})$ has no 2-torsion is necessary. Theorem 1.2 implies that, in general, $D(X) \neq H_1^{\mathrm{alg}}(X, \mathbb{Z}/2)$. Indeed, there is an algebraic model X of $\mathbb{P}^2(\mathbb{C}) \times (\mathbb{R}/\mathbb{Z})$, where $\mathbb{P}^2(\mathbb{C})$ is the complex projective plane, with $D(X) = 0$. On the other hand, $w_4(X) \neq 0$ and hence $H_1^{\mathrm{alg}}(X, \mathbb{Z}/2) = H_1(X, \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Both Theorems 1.1 and 1.2 are proved in the next section.

2. Proofs

A compact nonsingular real algebraic set X is said to have *totally algebraic homology* if

$$H_k^{\mathrm{alg}}(X, \mathbb{Z}/2) = H_k(X, \mathbb{Z}/2)$$

for all $k \geq 0$. This notion will play an important role in our proofs.

Lemma 2.1. *Let g be a nonnegative integer. Any compact smooth manifold N has an algebraic model Y such that for every real algebraic curve D of genus at most g, every rational map from D into Y is constant.*

Proof. Let A be a compact connected nonsingular real algebraic curve of genus at least $g + 1$. Let $B = A \times \cdots \times A$ be the q-fold product, where $q \geq 2 \dim N + 1$, and let $e : N \to B$ be a smooth embedding. Endow the space $C^\infty(N, B)$ of smooth maps from N into B with the C^∞ topology and choose a neighborhood V of e in $C^\infty(N, B)$ consisting of smooth embeddings. Since A is connected, it follows that B has totally algebraic homology, and hence by [2, Theorem 3] or [1, Proposition 2.8], there exist an algebraic model Y of N, a smooth diffeomorphism $\psi : N \to Y$, and a regular map $f : Y \to B$ such that $f \circ \psi$ is in V. In particular, f is a smooth embedding, and consequently injective.

Let D be a real algebraic curve of genus at most g and let $\rho : D \dashrightarrow Y$ be a rational map. Since the genus of D is strictly less than the genus of A, the rational map $f \circ \rho : D \dashrightarrow B$ is constant in virtue of the Riemann-Hurwitz theorem. It follows that ρ is constant, f being injective. The proof is complete. $\square$

Denote by $\mathbb{G}_{r,s}(\mathbb{R})$ the Grassmannian of s-dimensional vector subspaces of $\mathbb{R}^r$. Recall that $\mathbb{G}_{r,s}(\mathbb{R})$ can be regarded in a natural way as an algebraic subset of $\mathbb{R}^m$ for some m, cf. [3, Theorem 3.4.4].

Lemma 2.2. *Let $C_1, \ldots, C_p$ be compact nonsingular algebraic curves in $\mathbb{R}^n$. Assume that*

(i) *$C_i \cap C_j \cap C_k = \emptyset$ whenever i, j, k are distinct.*
(ii) *for every point x in $C_i \cap C_j$, where $i \neq j$, the tangent spaces to C_i and C_j at x are distinct.*

Then every continuous map from $C = C_1 \cup \ldots \cup C_p$ into $\mathbb{G}_{r,s}(\mathbb{R})$ can be approximated by regular maps in the compact-open topology.

Proof.   Although a more direct proof is possible, the one below is probably the shortest. According to [3, Theorem 13.3.1], it suffices to prove that every topological real vector bundle ξ (of constant positive rank) on C is isomorphic to an algebraic vector bundle. Since ξ is isomorphic to the direct sum of a line bundle and a trivial bundle, we may assume that ξ itself is a line bundle. Let $h : C \to \mathbb{P}^d(\mathbb{R})$ be a classifying map for ξ, that is, h is continuous and ξ is isomorphic to $h^*\gamma_d$, where γ_d is the universal line bundle on $\mathbb{P}^d(\mathbb{R})$. Since γ_d is an algebraic bundle, it is enough to show that h can be approximated by regular maps. We proceed as follows. The union S of all the intersections $C_i \cap C_j$, with $i \neq j$, is a finite set. By [7, Lemma 2.1], for each i, there is a regular map $f_i : C_i \to \mathbb{P}^d(\mathbb{R})$ arbitrarily close to $h|C_i$ and equal to h on $C_i \cap S$. The map $f : C \to \mathbb{P}^d(\mathbb{R})$, defined by $f|C_i = f_i$ for $1 \leq i \leq p$, is continuous and close to h. Moreover, by [2, Lemma 3], f is actually a regular map (here (i) and (ii) are used, along with the fact that $\mathbb{P}^d(\mathbb{R})$ can be regarded as an algebraic subset of $\mathbb{R}^q$ for some q). The proof is complete. $\square$

It is convenient to introduce the following terminology and notation. By convention, submanifolds will always be closed subsets of the ambient manifold. For any compact smooth manifold M, we denote by $[M]$ its fundamental class in $H_m(M, \mathbb{Z}/2)$, where $m = \dim M$. If N is an n-dimensional smooth submanifold of M, we write $[N]_M$ for the homology class in $H_n(M, \mathbb{Z}/2)$ represented by N, that is, $[N]_M = i_*([N])$, where $i : N \hookrightarrow M$ is the inclusion map. By a smooth curve in M we mean a smooth submanifold of M of dimension 1.

Proof of Theorem 1.1.   We may assume that M is a smooth submanifold of $\mathbb{R}^n$, where $n \geq 2 \dim M + 1$. Let $C_1, \ldots, C_p$ be connected smooth curves in M, whose homology classes $[C_1]_M, \ldots, [C_p]_M$ generate G. We choose these curves in general position in M. Morover, we can select them in such a way that

(a_0) $e_*(H_1(C, \mathbb{Z}/2))$ is generated by $[C_1]_M, \ldots, [C_p]_M$,

where

$$C = C_1 \cup \ldots \cup C_p$$

and $e : C \hookrightarrow M$ is the inclusion map. Indeed, condition (a_0) can be easily arranged if $\dim M = 2$, while for $\dim M \geq 3$ is is automatically satisfied since then $C_1, \ldots, C_p$ are pairwise disjoint (being in general position).

There is a smooth diffeomorphism $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $\theta(C_i)$ is a nonsingular algebraic curve in $\mathbb{R}^n$ of genus g for $1 \leq i \leq p$. Hence we may assume that

(a_1) $\qquad$ C_i itself is a nonsingular algebraic curve in $\mathbb{R}^n$ of genus g for $1 \leq i \leq p$.

Set
$$\Gamma = \{v \in H^1(M, \mathbb{Z}/2) \mid \langle v, \alpha \rangle = 0 \text{ for all } \alpha \in G\},$$
where $\langle -, - \rangle$ is the scalar product (Kronecker index). Note that
$$G = \{\alpha \in H_1(M, \mathbb{Z}/2) \mid \langle v, \alpha \rangle = 0 \text{ for all } v \in \Gamma\}.$$

By Lemma 2.1, for any positive integer d, there is an algebraic model A of $\mathbb{P}^d(\mathbb{R})$ such that for every real algebraic curve D of genus at most g, every rational map from D into A is constant. If Γ is generated by q elements, $B = A \times \cdots \times A$ is the q-fold product, and d is sufficiently large (depending on M), then there is a smooth map
$$h : M \to B$$
with $\Gamma = h^*(H^1(B, \mathbb{Z}/2))$. Hence
$$G = \{\alpha \in H_1(M, \mathbb{Z}/2) \mid \langle h^*(b), \alpha \rangle = 0 \text{ for all } b \in H^1(B, \mathbb{Z}/2)\},$$
and since $\langle h^*(b), \alpha \rangle = \langle b, h_*(\alpha) \rangle$, we get

(a_2) $\qquad\qquad\qquad$ $G = \{\alpha \in H_1(M, \mathbb{Z}/2) \mid h_*(\alpha) = 0\}.$

Assume $d \geq 2$. Since each homology class $[C_i]_M$ is in G, we get $(h|C_i)_*([C_i]) = h_*([C_i]_M) = 0$, and hence in view of (a_0), the homomorphism
$$(h|C)_* : H_1(C, \mathbb{Z}/2) \to H_1(B, \mathbb{Z}/2)$$
is zero. This implies that the restriction map $h|C : C \to B$ is null homotopic since $H_1(B, \mathbb{Z}/2)$ can be identified with the fundamental group of B. Actually, the map h can be chosen in such a way that

(a_3) $\qquad\qquad\qquad$ $h : M \to B$ is constant on C.

Indeed, let K be a compact subset of M such that C is contained in the interior of K and is a deformation retract of K, while (M, K) is a polyhedral pair. Then $h|K : K \to B$ is null homotopic, and hence the homotopy extension theorem [10, p. 118, Corollary 5] implies the existence of a continuous map $h' : M \to B$ homotopic to h and constant on K. Thus there is a smooth map $h'' : M \to B$ homotopic to h' and equal to h' on C. Replacing h by h'', we may assume that (a_3) is satisfied.

Henceforth assume d is even. This implies that A, being an algebraic model of $\mathbb{P}^d(\mathbb{R})$, has totally algebraic homology (see Section 1). Consequently, by Künneth's theorem,

(a_4) $\qquad\qquad\qquad$ B has totally algebraic homology.

In view of (a_1), (a_3), (a_4), and Lemma 2.2, we can apply [2, Theorem 3] (cf. also [1, Proposition 2.8]) to M, C, $h : M \to B$. Therefore there exist a nonsingular algebraic subset X of $\mathbb{R}^n$, a smooth diffeomorphism $\varphi : M \to X$, and a regular map

$f : X \to B$ such that $C \subset X$, $\varphi(x) = x$ for all x in C, and $f \circ \varphi$ is homotopic to h. In particular, $\varphi_*([C_i]_M) = [C_i]_X$ for $1 \le i \le p$, which implies

$$\varphi_*(G) \subseteq A(X, g) \subseteq B(X, g).$$

Moreover, making use of (a_2), we get

$$\varphi_*(G) = \{\beta \in H_1(X, \mathbb{Z}/2) \mid f_*(\beta) = 0\}.$$

It remains to prove that if δ is a homology class in $B(X, g)$ represented by an algebraic curve D in X of genus at most g, then δ is in $\varphi_*(G)$, or equivalently, $f_*(\delta) = 0$. The last equality holds, the restriction map $f|D : D \to B$ being constant (see how B is constructed). The proof is complete. $\qquad\square$

Proof of Theorem 1.2. We may assume that M is a smooth submanifold of $\mathbb{R}^n$, where $n \ge 2 \dim M + 1$. Let $C_1, \ldots, C_p$ be connected smooth curves in M, whose homology classes $[C_1]_M, \ldots, [C_p]_M$ generate G. We choose these curves in general position in M, and so that

(b_0) $\qquad\qquad\qquad e_*(H_1(C, \mathbb{Z}))$ is generated by $o_{C_1}, \ldots, o_{C_p}$,

where

$$C = C_1 \cup \ldots \cup C_p,$$

$e : C \hookrightarrow M$ is the inclusion map, and o_{C_i} is the homology class in $H_1(M, \mathbb{Z})$ represented by C_i endowed with some orientation. Furthermore, as in the proof of Theorem 1.1, we may assume that

(b_1) $\qquad C_i$ is a nonsingular algebraic curve in $\mathbb{R}^n$ of genus 0 for $1 \le i \le p$.

Set

$$\Gamma = \{v \in H^1(M, \mathbb{Z}/2) \mid \langle v, \alpha \rangle = 0 \text{ for all } \alpha \in G\},$$

and let $v_1, \ldots, v_q$ be generators of Γ.

Since $H_1(M, \mathbb{Z})$ has no 2-torsion, the group $H^2(M, \mathbb{Z})$ has no 2-torsion, which in turn implies that the restriction modulo 2 homomorphism $H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{Z}/2)$ is surjective (the universal coefficient theorem). Hence, in view of the orientability of M, the homology class Poincaré dual to v_j can be represented by a codimension 1 orientable smooth submanifold N_j of M. If N_j is disconnected, then joining two connected components of N_j with a tube in M, without modifying the remaining components, we obtain an orientable smooth submanifold N'_j of M with $[N'_j]_M = [N_j]_M$ (orientability of N'_j is not a problem here since we do not preassign orientations to the connected components of N_j, and we join with a tube only two connected components). After performing such an operation finitely many times, we may assume that N_j is connected for $1 \le j \le q$. Moreover, we choose $N_1, \ldots, N_q$ so that $C_1, \ldots, C_p$, $N_1, \ldots, N_q$ are in general position in M.

Let E be a complex elliptic curve defined over $\mathbb{R}$, whose set of real points $A = E(\mathbb{R})$ is connected. In particular, A is diffeomorphic to the unit circle. According to the classical Pontryagin-Thom construction, there exist a smooth map $h_j : M \to A$ and a regular value $y_j \in A$ of h_j with $N_j = h_j^{-1}(y_j)$, cf. [8]. Consequently,

$$v_j = h_j^*(w),$$

where w is the unique generator of $H^1(A, \mathbb{Z}/2) \cong \mathbb{Z}/2$ (cf. [6, Proposition 2.15] if desired). We assert that each restriction map $h_j|C_i : C_i \to A$ is null homotopic for $1 \le i \le p$ and $1 \le j \le q$. Indeed, we have

$$\begin{aligned}
\langle w, (h_j|C_i)_*([C_i]) \rangle &= \langle w, h_{j_*}([C_i]_M) \rangle \\
&= \langle h_j^*(w), [C_i]_M \rangle \\
&= \langle v_j, [C_i]_M \rangle \\
&= 0,
\end{aligned}$$

where the last equality follows from the definition of Γ. Thus $(h_j|C_i)_*([C_i]) = 0$, which means that the topological degree modulo 2 of $h_j|C_i$ is 0. Since C_i is transverse to $N_j = h_j^{-1}(y_j)$ in M, the point y_j is a regular value of $h_j|C_i$, and hence the set

$$(h_j|C_i)^{-1}(y_j) = C_i \cap N_j$$

consists of an even number of points, say, m_{ij} of them. If orientations of M, C_i, N_j are fixed, the topological degree of $h_j|C_i$ is equal, up to sign, to the intersection number n_{ij} of C_i and N_j in M. Since C_i and N_j are connected, and m_{ij} is even, it follows that $n_{ij} = 0$. Hence $h_j|C_i$ is null homotopic, as asserted. Consequently, in view of (b_0), the homomorphism

$$(h_j|C)_* : H_1(C, \mathbb{Z}) \to H_1(A, \mathbb{Z})$$

is zero. This implies that the restriction map $h_j|C : C \to A$ is null homotopic, since $H_1(A, \mathbb{Z})$ can be identified with the fundamental group of A.

Let $B = A \times \cdots \times A$ be the q-fold product. The smooth map

$$h = (h_1, \ldots, h_q) : M \to B$$

satisfies

$$\Gamma = h^*(H^1(B, \mathbb{Z}/2)).$$

Moreover, the restriction map $h|C : C \to B$ is null homotopic. Now, arguing as in the proof of Theorem 1.1, we obtain

(b_2) $$G = \{\alpha \in H_1(M, \mathbb{Z}/2) \mid h_*(\alpha) = 0\}.$$

Also, modifying h without affecting (b_2), we may assume

(b_3) $$h : M \to B \text{ is constant on } C.$$

Since A is a compact connected nonsingular real algebraic curve,

(b_4) $$B \text{ has totally algebraic homology.}$$

The next step is again as in the proof of Theorem 1.1. In virtue of (b_1), (b_3), (b_4) and Lemma 2.2, there exist a nonsingular algebraic subset X of $\mathbb{R}^n$, a smooth diffeomorphism $\varphi : M \to X$, and a regular map $f : X \to B$ such that $C \subset X$, $\varphi(x) = x$ for all x in C, and $f \circ \varphi$ is homotopic to h. This in turn implies

$$\varphi_*(G) \subseteq A(X, 0) \subseteq D(X),$$

and combined with (b_2) yields

$$\varphi_*(G) = \{\beta \in H_1(X, \mathbb{Z}/2) \mid f_*(\beta) = 0\}.$$

It remains to prove that if δ is a homology class in $D(X)$ represented by a dividing algebraic curve D in X, then δ is in $\varphi_*(G)$, or equivalently, $f_*(\delta) = 0$. To this end,

choose a compact nonsingular real algebraic curve $\tilde{D}$ and a regular map $\pi : \tilde{D} \to D$ such that π is a birational isomorphism. In particular, $\tilde{D}$ is a dividing curve. Moreover, we have

$$f_*(\delta) = (f \circ e \circ \pi)_*([\tilde{D}]) = 0,$$

where $e : D \hookrightarrow X$ is the inclusion map, and the last equality is a consequence of [5, Theorem 1.1]. The proof is complete. $\qquad\square$

Acknowledgements

References

[1] S. Akbulut and H. King, *The topology of real algebraic sets with isolated singularities*, Ann. of Math. **113**(1981), 425-446.

[2] R. Benedetti and A. Tognoli, *Approximation theorems in real algebraic geometry*, Boll. Unione Mat. Ital., Suppl. **2**(1980), 198-211.

[3] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Math. und ihrer Grenzgeb. Folge 3, Vol. **36**, Berlin Heidelberg New York, Springer, 1998.

[4] J. Bochnak and W. Kucharz, *On homology classes represented by real algebraic varieties*, Banach Center Publications Vol. **44**, 21-35, Warsaw, 1998.

[5] J. Bochnak and W. Kucharz, *Complexification of real algebraic varieties and vanishing of homology classes*, Bull. London Math. Soc. **33**(2001), 32-40.

[6] A. Borel et A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89**(1961), 461-513.

[7] W. Kucharz, *Algebraic morphisms into rational real algebraic surfaces*, J. Algebraic Geometry **8**(1999), 569-579.

[8] J. Milnor, *Topology from the Differentiable Viewpoint*, University of Virginia Press, 1966.

[9] J. Nash, *Real algebraic manifolds*, Ann. of Math. **56**(1952), 405-421.

[10] E. Spanier, *Algebraic Topology*, New York Berlin Heidelberg, Springer.

[11] A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. (3) **27**(1973), 167-185.

Department of Mathematics and Statistics, University of New Mexico, MSC03 2150, Albuquerque, New Mexico 87131-0001, U.S.A.

E-mail address: kucharz@math.unm.edu