Math. Res. Lett. **15** (2008), *no.* 2, 221–238

A FAMILY OF COVERING PROPERTIES

Matteo Viale

Abstract. In the first part of this paper I present the main results of my Ph.D. thesis: several proofs of the singular cardinal hypothesis SCH are presented assuming either a strongly compact cardinal or the proper forcing axiom PFA. To this aim I introduce a family of covering properties which imply both SCH and the failure of various forms of square. In the second part of the paper I apply these covering properties and other similar techniques to investigate models of strongly compact cardinals or of strong forcing axioms like MM or PFA. In particular I show that if MM holds and all limit cardinals are strong limit, then any inner model W with the same cardinals has the same ordinals of cofinality at most $\aleph_1$.

In this paper I introduce a family of covering properties $\mathsf{CP}(\kappa, \lambda)$ indexed by pairs of regular cardinals $\lambda < \kappa$. In the first part I show that these covering properties imply the singular cardinal hypothesis SCH and capture the combinatorial content of many of the known proofs of the failure of square-like principles from forcing axioms or large cardinals. In the second part I show that a large class of these covering properties follows either from the existence of a strongly compact cardinal or from at least two combinatorial principles which hold under the proper forcing axiom and are mutually independent: the mapping reflection principle MRP introduced by Moore in [15] and the P-ideal dichotomy PID introduced in its full generality by Todorčević in [23] developing on preceding works by him and Abraham [1] on P-ideals of countable subsets of ω_1. This allows for a unified and simple proof of the failure of square and of the singular cardinal hypothesis assuming PFA. Finally in the last part of the paper I will apply these covering properties as well as other related ideas to investigate the "saturation properties"[1] of models of strong forcing axioms or of a strongly compact cardinal.

The paper is organized as follows: sections 1 and 2 introduce the combinatorial principles PID and MRP. Sections 3 and 4 introduce the covering properties $\mathsf{CP}(\kappa, \lambda)$ and outline some of their consequences among which SCH and the failure of square. Sections 5, 6 and 7 prove various instances of $\mathsf{CP}(\kappa, \lambda)$ assuming respectively the existence of a strongly compact cardinal, PID or[2] MRP. The last section study the rigidity of models of $\mathsf{CP}(\kappa, \lambda)$ for various κ and λ and prove that if MM holds and any limit cardinal is strong limit, then any inner model W with the same cardinals compute the same way the ordinals of cofinality at most $\aleph_1$. Caicedo and Veličković conjecture that any two models $W \subseteq V$ of MM with the same cardinals have the same ω_1-sequences of ordinals. I will show that set-forcing cannot be of any help

Received by the editors February 19, 2007.

[1] In a sense that will be made explicit in the last section of the paper.

[2] The shortest path to obtain a self-contained proof of the SCH starting from PID is to read sections 1, 3 and 6, from MRP is to read sections 2, 3 and 7.

in falsifying this conjecture. Moreover it seems plausible to conjecture that any two models of MM with the same cardinals have the same cofinalities. Results of a similar vein are obtained also for models of set theory with strongly compact cardinals.

The paper aims to be accessible and self-contained for any reader with a strong background in combinatorial set theory. While forcing axioms are a source of inspiration for the results that we present, all the technical arguments in this paper (except some of those in the last section) can be followed by a reader with no familiarity with the forcing techniques. When not otherwise explicitly stated [10] is the standard source for notation and definitions. For a regular cardinal θ, we use $H(\theta)$ to denote the structure $\langle H(\theta), \in, <\rangle$ whose domain is the collection of sets whose transitive closure is of size less than θ and where $<$ is a predicate for a fixed well ordering of $H(\theta)$. For cardinals $\kappa \geq \lambda$ we let $[\kappa]^\lambda$ be the family of subsets of κ of size λ. In a similar fashion we define $[\kappa]^{<\lambda}$, $[\kappa]^{\leq\lambda}$, $[X]^\lambda$, where X is an arbitrary set. If X is an uncountable set and θ a regular cardinal, $\mathcal{E} \subseteq [X]^\theta$ is unbounded if for every $Z \in [X]^\theta$, there is $Y \in \mathcal{E}$ containing Z. $\mathcal{E}$ is bounded otherwise. $\mathcal{E}$ is closed in $[X]^\omega$ if whenever $X = \bigcup_n X_n$ and $X_n \subseteq X_{n+1}$ are in $\mathcal{E}$ for all n, then also $X \in \mathcal{E}$. It is a well known fact that $\mathcal{C} \subseteq [X]^\omega$ is closed and unbounded (club) iff there is $f : [X]^{<\omega} \to X$ such that $\mathcal{C}$ contains the set of all $Y \in [X]^\omega$ such that $f[[Y]^{<\omega}] \subseteq Y$. $\mathcal{S} \subseteq [X]^\omega$ is stationary if it intersects all club subsets of $[X]^\omega$. The f-closure of X is the smallest Y containing X such that $f[[Y]^{<\omega}] \subseteq Y$. Given f as above, $\mathcal{E}_f$ is the club of $Z \in [X]^\omega$ such that Z is f-closed. If X is a set of ordinals, then $\overline{X}$ denotes the topological closure of X in the order topology. For regular cardinals $\lambda < \kappa$, $S^{\leq\lambda}_\kappa$ denotes the subset of κ of points of cofinality $\leq \lambda$. In a similar fashion we define S^λ_κ and $S^{<\lambda}_\kappa$. We say that a family $\mathcal{D}$ is covered by a family $\mathcal{E}$ if for every $X \in \mathcal{D}$ there is a $Y \in \mathcal{E}$ such that $X \subseteq Y$. We also recall the following definitions central to the arguments that follows:

The singular cardinal hypothesis SCH asserts that $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+ + 2^{\mathtt{cof}(\kappa)}$ for all infinite cardinals κ.

It is a celebrated result of Silver [21] that if SCH fails, then it first fails at a singular cardinal of countable cofinality. It is also known that the failure of SCH is a strong hypothesis as it entails the existence of models of ZFC with measurable cardinals [9].

Let κ be an infinite regular cardinal. The square principle $\square(\kappa)$ asserts the existence of a sequence $(C_\alpha : \alpha < \kappa)$ with the following properties:

(i) for every limit α, C_α is a closed unbounded subset of α,
(ii) if α is a limit point of C_β, $C_\alpha = C_\beta \cap \alpha$,
(iii) there is no club C in κ such that for all α there is $\beta \geq \alpha$ such that $C \cap \alpha = C_\beta \cap \alpha$,
(iv) $C_{\beta+1} = \{\beta\}$.

It is well known that the failure of $\square(\kappa)$ for all cardinals $\kappa \geq \aleph_1$ is a very strong large cardinal hypothesis. For example Schimmerling has shown that it entails the existence of models of set theory with Woodin cardinals [18].

Recall that λ is a strongly compact cardinal if for every $\kappa \geq \lambda$ there is a λ-complete, fine[3] ultrafilter on $[\kappa]^{<\lambda}$.

[3] $\mathcal{U}$ is a fine filter on $[\kappa]^{<\lambda}$ if $\hat{X} = \{Y \in [\kappa]^{<\lambda} : X \subseteq Y\} \in \mathcal{U}$ for every $X \in [\kappa]^{<\lambda}$.

1. The P-ideal dichotomy

Let Z be an uncountable set. $\mathcal{I} \subseteq [Z]^{\leq\omega}$ is a P-ideal if it is an ideal and for every countable family $\{X_n\}_n \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that for all n, $X_n \subseteq^* X$ (where $\subseteq^*$ is inclusion modulo finite).

Definition 1. *(Todorčević, [23])*
*The P-ideal dichotomy (*PID*) asserts that for every P-ideal $\mathcal{I}$ on $[Z]^{\leq\omega}$ for some fixed uncountable Z, one of the following holds:*

- *(i) There is Y uncountable subset of Z such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$.*
- *(ii) $Z = \bigcup_n A_n$ with the property that A_n is orthogonal to $\mathcal{I}$ (i.e. $X \cap Y$ is finite for all $X \in [A_n]^\omega$ and $Y \in \mathcal{I}$) for all n.*

PID is a principle which follows from PFA and which is strong enough to rule out many of the standard consequences of $V = L$. For example Abraham and Todorčević [1] have shown that under PID there are no Souslin trees while Todorčević has shown that PID implies the failure of $\square(\kappa)$ on all regular $\kappa > \aleph_1$ [23]. Due to this latter fact the consistency strength of this principle is considerable. Another interesting result by Todorčević is that PID implies that $\mathfrak{b} \leq \aleph_2$[4]. Nonetheless in [1] and [23] it is shown that this principle is consistent with CH. Other interesting applications of PID can be found in [26], [2], [23] and [1].

2. The mapping reflection principle

Almost all known applications of MM which do not follow from PFA are a consequence of some form of reflection for stationary sets. These types of reflection principles are a fundamental source in order to obtain proofs of all cardinal arithmetic result that follows from MM. In particular SCH and $\mathfrak{c} \leq \omega_2$ are a consequence of many of the known reflection principles which hold under MM. However up to a very recent time there was no such kind of principle which could be derived from PFA alone. This has been the main difficulty in the search for a proof that PFA implies $\mathfrak{c} = \omega_2$, a result which has been obtained by Todorčević and Veličković appealing to combinatorial arguments which are not dissimilar from the P-ideal dichotomy [25]. Later on this has also been the crucial obstacle in the search for a proof of SCH from PFA.

In 2003 Moore [15] found an interesting form of reflection which can be derived from PFA, the mapping reflection principle MRP. He has then used this principle to show that BPFA implies that $\mathfrak{c} = \aleph_2$ and that this principle is strong enough to entail the non-existence of square sequences. He has also shown in [17] that MRP could be a useful tool in the search of a proof of SCH from PFA. I first obtained my proof of this latter theorem elaborating from [17]. Many other interesting consequences of this reflection principle have been found by Moore and others. A complete presentation of this subject will be found in [4].

[4]In [23] it is shown that any gap in $P(\omega)/FIN$ is either an Hausdorff gap or a (κ, ω) gap with κ regular and uncountable. By another result of Todorčević (see [10] pp. 578 for a proof) if $\mathfrak{b} > \aleph_2$ there is an (ω_2, λ) gap in $P(\omega)/FIN$ for some regular uncountable λ. Thus PID is not compatible with $\mathfrak{b} > \aleph_2$.

Definition 2. *Let θ be a regular cardinal, let X be uncountable, and let $M \prec H(\theta)$ be countable such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is M-stationary if for all $\mathcal{E} \in M$ such that $\mathcal{E} \subseteq [X]^\omega$ is club, $\Sigma \cap \mathcal{E} \cap M \neq \emptyset$.*

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open if it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subseteq N$ is finite.

Definition 3. *Σ is an open stationary set mapping if there is an uncountable set X and a regular cardinal θ such that $[X]^\omega \in H(\theta)$, the domain of Σ is a club in $[H(\theta)]^\omega$ of countable elementary submodels M such that $X \in M$ and for all such M, $\Sigma(M) \subseteq [X]^\omega$ is open in the Ellentuck topology on $[X]^\omega$ and M-stationary.*

The mapping reflection principle (MRP) asserts that:

> If Σ is an open stationary set mapping, there is a continuous $\in$-chain $\vec{N} = (N_\xi : \xi < \omega_1)$ of elements in the domain of Σ such that for all limit ordinals $0 < \xi < \omega_1$ there is $\nu < \xi$ such that $N_\eta \cap X \in \Sigma(N_\xi)$ for all η such that $\nu < \eta < \xi$.

If $(N_\xi : \xi < \omega_1)$ satisfies the conclusion of MRP for Σ then it is said to be a reflecting sequence for Σ.

We list below some of the interesting consequences of MRP.

- (Moore [15]) PFA implies MRP.
- (Moore [15]) MRP implies that $\mathfrak{c} = \aleph_2$. As a simple outcome of his proof of the above theorem Moore obtains also that BPFA implies $\mathfrak{c} = \aleph_2$.
- (Moore [15]) Assume MRP. Then $\square(\kappa)$ fails for all regular $\kappa \geq \aleph_2$.

A folklore problem in combinatorial set theory for the last twenty years has been the consistency of the existence of a five element basis for the uncountable linear orders, i.e. the statement that there are five uncountable linear orders such that at least one of them embeds in any other uncountable linear order.

- (Moore [16]) Assume BPFA and MRP. Then there is a five element basis for the uncountable linear orders.

A considerable reduction of the large cardinal hypothesis needed for the consistency of the above conjecture has been obtained in [11]. A byproduct of their results yields to the following:

- (König, Larson, Moore, Veličković [11]) MRP implies that there are no Kurepa trees.

Other interesting consequences of MRP can be found in [5].

We remark that MRP and PID are mutually independent principles since PID is compatible with CH while, by a result of Myiamoto [14], MRP is compatible with the existence of Souslin trees.

3. A family of covering properties

In this section we introduce the main original concept of this paper. A careful analysis of Todorčević's proof that PID implies that $\square(\kappa)$ fails for all regular $\kappa > \aleph_1$ leads to the isolation of a family of covering properties $\mathsf{CP}(\kappa,\lambda)$ which on one side are strong enough to entail both SCH and the failure of square, and on the other side are weak enough to be a consequence of the existence of a strongly compact cardinal, of PID and of MRP.

Definition 4. *For regular cardinals $\lambda < \kappa$, $\mathcal{D} = (K(\alpha,\beta) : \alpha < \lambda, \beta \in \kappa)$ is a λ-covering matrix for κ if:*

(i): $\beta \subseteq \bigcup_{\alpha<\lambda} K(\alpha,\beta)$ for all β,
(ii): $K(\alpha,\beta)$ is strictly contained in $K(\eta,\beta)$ for all $\beta < \kappa$ and for all $\alpha < \eta < \lambda$,
(iii): for all $\gamma < \beta < \kappa$ and for all $\alpha < \lambda$, there is $\eta < \lambda$ such that $K(\alpha,\gamma) \subseteq K(\eta,\beta)$.
(iv): for all $X \in [\kappa]^{\leq\lambda}$, there is $\gamma_X < \kappa$ such that for all $\beta < \kappa$ and $\eta < \lambda$, there is α such that $K(\eta,\beta) \cap X \subseteq K(\alpha,\gamma_X)$

$\beta_{\mathcal{D}} \leq \kappa$ is the least β such that for all α and γ, $\mathtt{otp}(K(\alpha,\gamma)) < \beta$. $\mathcal{D}$ is trivial[5] *if $\beta_{\mathcal{D}} = \kappa$.*

We will mainly be interested in ω-covering matrices, which we will just call covering matrices. As we will see below square like principles are useful to construct several kinds of covering matrices. One successful strategy to negate square principles from large cardinals and forcing axioms is to use appropriate ultrafilters or specific forcing arguments to "diagonalize" through the covering matrix defined appealing to these square-like principles, as for example in the proofs of the failure of square from a strongly compact by Solovay [22] or from PID by Todorčevič [23][6]. The covering matrices induced by square-like principles that we will consider satisfy a stronger coherence property than the "local" property *(iv)*. This condition is replaced by the "global" property[7]:

> *(iv') For all $\gamma < \beta < \kappa$ and $\eta < \lambda$, there is α such that $K(\eta,\beta) \cap \gamma \subseteq K(\alpha,\gamma)$.*

The key point to introduce condition *(iv)* above in this weak form is that λ-covering matrices on κ can be defined in an elementary way in ZFC and the diagonalization argument which in the square-like cases leads to a contradiction, in the general case leads to a simple combinatorial argument to compute κ^λ. It is possible to prove the following:

[5]Essentially all the λ-covering matrices that will be of interest to us are non-trivial. We remark however that the ω-covering matrix induced by a $\square(\kappa)$-sequence (see theorem 18) is trivial according to this definition but nonetheless it is a mean to prove the failure of $\square(\kappa)$ assuming PFA or a strongly compact cardinal.

[6]See also the several arguments of this sort appearing in the sequel of this paper.

[7]For example the matrix produced by a square sequence using the ρ_2-function (see [24] or theorem 18 below) satisfies *(i),… (iii)* and *(iv')*, and this matrix can be used to show that CP implies the failure of square. Another interesting example of a covering matrix which is not defined appealing to lemma 6 below and which satisfies *(iv')* is the matrix used in the proof of theorem 27 in the last section.

Lemma 5. *Assume that there is a stationary set of points of uncountable cofinality in the approachability ideal*[8] $\mathcal{I}[\kappa]$*. Then there is an* ω*-covering matrix on* κ*. Moreover if* λ *is singular of countable cofinality, then there is* $\mathcal{D}$ ω*-covering matrix on* λ^+ *with* $\beta_{\mathcal{D}} = \lambda$*.*

Since the main original application of the existence of ω-covering matrices $\mathcal{D}$ that we have found is the proof of SCH from PFA, we will just prove a weaker form of the lemma:

Lemma 6. *Assume that* $\kappa \geq \mathfrak{c}$ *is a singular cardinal of countable cofinality. Then there is an* ω*-covering matrix* $\mathcal{C}$ *for* κ^+ *with* $\beta_{\mathcal{C}} = \kappa$*.*

Proof. The matrices we are going to define satisfy *(i),(ii)* and a stronger coherence property than what is required by *(iii)* and *(iv)* of the above definition. They will satisfy the following properties *(iii*)* and *(iv*)* from which *(iii)* and *(iv)* immediately follow:

- *(iii*):* For all $\alpha < \beta$ there is n such that $K(m, \alpha) \subseteq K(m, \beta)$ for all $m \geq n$.
- *(iv*):* For all $X \in [\kappa^+]^\omega$ there is $\gamma_X < \kappa$ such that for all $\beta \geq \gamma_X$ there is n such that $K(m, \beta) \cap X = K(m, \gamma_X) \cap X$ for all $m \geq n$.

Let $\phi_\eta : \kappa \to \eta$ be a surjection for all $0 < \eta < \kappa^+$. Fix also $\{\kappa_n : n < \omega\}$ increasing sequence of regular cardinals cofinal in κ with $\kappa_0 \geq \aleph_1$. Set

$$K(n, \beta) = \bigcup \{K(n, \gamma) \cup \{\gamma\} : \gamma \in \phi_\beta[\kappa_n]\}.$$

It is immediate to check that $\mathcal{D} = (K(n, \beta) : n \in \omega, \beta < \kappa^+)$ satisfies *(i)*, *(ii)* of definition 4, property *(iii*)* and that $\beta_{\mathcal{D}} = \kappa$. To prove *(iv*)*, let $X \in [\kappa^+]^\omega$ be arbitrary. Now since $\mathfrak{c} < \kappa^+$ and there are at most $\mathfrak{c}$ many subsets of X, there is an unbounded subset S of κ^+ and a fixed decomposition of X as the increasing union of sets X_n such that $X \cap K(n, \alpha) = X_n$ for all α in S and for all n. Now properties *(i), (ii), (iii*)* of the matrix guarantees that this property of S is enough to get *(iv*)* for X with $\gamma_X = \min(S)$. □

A similar argument can be used to prove the following:

Lemma 7. *Assume that* $\kappa > 2^\lambda$ *is singular of cofinality* λ*. Then there is a non-trivial* λ*-covering matrix* $\mathcal{C}$ *for* κ^+*.* □

We say that $\mathcal{D}$ is covered by $\mathcal{E}$ iff every $X \in \mathcal{D}$ is contained in some $Y \in \mathcal{E}$.

Definition 8. $\mathsf{CP}(\kappa, \lambda)$*:* κ *has the* λ*-covering property*[9] *if for every* $\mathcal{D}$*,* λ*-covering matrix for* κ *there is an unbounded subset* A *of* κ *such that* $[A]^\lambda$ *is covered by* $\mathcal{D}$*.* $\mathsf{CP}(\kappa)$ *abbreviates* $\mathsf{CP}(\kappa, \omega)$ *and* CP *is the statement that* $\mathsf{CP}(\kappa)$ *holds for all regular* $\kappa > \mathfrak{c}$*.*

Fact 9. *Assume* $\mathsf{CP}(\kappa^+)$ *for all singular* κ *of countable cofinality. Then* $\lambda^{\aleph_0} = \lambda$*, for every* $\lambda \geq 2^{\aleph_0}$ *of uncountable cofinality.*

---

[8]In [20] it is possible to find a definition of the approachability ideal $\mathcal{I}[\kappa]$. We avoid it in this paper since it is not relevant for the arguments we are presenting.

[9]Moore has first noticed that a covering property similar to the ω-covering property for a regular $\kappa > \mathfrak{c}$ followed from MRP reading a draft of [27].

Proof. By induction. The base case is trivial. If $\lambda = \kappa^+$ with $\mathtt{cof}(\kappa) > \omega$, then $\lambda^{\aleph_0} = \lambda \cdot \kappa^{\aleph_0} = \lambda \cdot \kappa = \lambda$, by the inductive hypothesis on κ. If λ is a limit cardinal and $\mathtt{cof}(\lambda) > \omega$ then $\lambda^{\aleph_0} = \sup\{\mu^{\aleph_0} : \mu < \lambda\}$, so the result also follows by the inductive hypothesis. Thus, the only interesting case is when $\lambda = \kappa^+$, with κ singular of countable cofinality. In this case we will show, using CP, that $(\kappa^+)^{\aleph_0} = \kappa^+$. To this aim let $\mathcal{D}$ be a covering matrix for κ^+ with $\beta_{\mathcal{D}} = \kappa$. Notice that by our inductive assumptions, since every $K(n,\beta)$ has order type less than κ, $|[K(n,\beta)]^\omega|$ has size less than κ. So $\bigcup\{[K(n,\beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\}$ has size κ^+. Use CP to find $A \subseteq \kappa^+$ unbounded in κ^+, such that $[A]^\omega$ is covered by $\mathcal{D}$. Then $[A]^\omega \subseteq \bigcup\{[K(n,\beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\}$, from which the conclusion follows. □

The following theorems motivate the introduction of these covering properties:

Theorem 10. *Assume λ is strongly compact. Then $\mathsf{CP}(\kappa,\theta)$ holds for all regular $\theta < \lambda$ and all regular $\kappa \geq \lambda$.*

Theorem 11. *Assume* PID*. Then* CP *holds.*

On the other hand MRP allows us to infer a slightly weaker conclusion than the one of the previous theorem.

Theorem 12. *Assume* MRP *and let $\mathcal{D}$ be a covering matrix for κ such that $K(n,\beta)$ is a closed set of ordinals for all $K(n,\beta)$. Then there is A unbounded in κ such that $[A]^\omega$ is covered by $\mathcal{D}$.*

In particular we obtain:

Corollary 13. PFA *implies* SCH*.*

Proof. PFA implies PID and PID implies CP. In particular PFA implies that $\kappa^\omega = \kappa$ for all regular $\kappa \geq \mathfrak{c}$. By Silver's theorem [21] the least singular $\kappa > 2^{\mathtt{cof}\kappa}$ such that $\kappa^{\mathtt{cof}\kappa} > \kappa^+$ has countable cofinality. Now assume PFA and let κ have countable cofinality. By fact 9, $\kappa^{\mathtt{cof}(\kappa)} \leq (\kappa^+)^{\aleph_0} = \kappa^+$. Thus assuming PFA there cannot be a singular cardinal of countable cofinality which violates SCH. Combining this fact with Silver's result we get that SCH holds under PFA. □

Before proving all the above theorems we analyze in more details the effects of CP and we give other interesting examples of λ-covering matrices.

4. Some other features of $\mathsf{CP}(\kappa,\lambda)$

First of all we investigate for what kind of pairs of regular cardinals κ and λ, $\mathsf{CP}(\kappa,\lambda)$ fails.

Lemma 14. $\mathsf{CP}(\kappa^+,\kappa)$ *fails for all regular $\kappa \geq \omega_1$.*

Proof. Fix a sequence $\mathcal{C} = \{C_\xi : \xi < \kappa^+\}$ such that for all limit α, C_α is a club subset of α of order type $\mathtt{cof}(\alpha)$. If $\xi = \alpha + 1$, $C_\xi = \{\alpha\}$. Define by induction on $\alpha \leq \beta < \kappa^+$,

$$\rho^*(\alpha,\beta) : [\kappa^+]^2 \to \kappa$$

as follows:

- $\rho^*(\alpha,\alpha) = 0$,

- $\rho^*(\alpha,\beta) = \max\{\mathtt{otp}(C_\beta \cap \alpha), \rho^*(\alpha, \min(C_\beta \setminus \alpha)),$
  $\sup\{\rho^*(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}\}$.

We will need the following properties of ρ^* which follows from the fact that κ is regular[10].

Lemma 15. *For all $\alpha \leq \beta \leq \gamma$:*

(a): $\rho^*(\alpha,\beta) \leq \max\{\rho^*(\alpha,\gamma), \rho^*(\beta,\gamma)\}$
(b): $\rho^*(\alpha,\gamma) \leq \max\{\rho^*(\alpha,\beta), \rho^*(\beta,\gamma)\}$

□

Lemma 16. *For all $\alpha < \kappa^+$ and $\nu < \kappa$:*

$$|\{\xi < \alpha : \rho^*(\xi,\alpha) < \nu\}| \leq |\nu| + \aleph_0.$$

□

Set $D(\alpha,\beta) = \{\xi < \beta : \rho^*(\xi,\beta) \leq \alpha\}$ for all $\alpha < \kappa$ and $\beta < \kappa^+$.

Fact 17. *The following holds:*

(i): $\mathtt{otp}(D(\alpha,\beta)) < \kappa$ *for all $\alpha < \kappa$ and $\beta < \kappa^+$,*
(ii): for all $\gamma < \beta < \kappa^+$,there is $\alpha_0 < \kappa$ such that $D(\alpha,\gamma) = D(\alpha,\beta) \cap \gamma$ for all $\alpha \geq \alpha_0$.

Proof. *(i)* follows from the second lemma on ρ^*. To prove *(ii)*, let α_0 be such that $\rho^*(\gamma,\beta) \leq \alpha_0$, $\alpha \geq \alpha_0$ and $\xi < \beta$ such that $\rho^*(\xi,\beta) \leq \alpha$. By *(a)* of lemma 15:

$$\rho^*(\xi,\gamma) \leq \max\{\rho^*(\xi,\beta), \rho^*(\gamma,\beta)\} \leq \max\{\alpha,\alpha_0\} = \alpha.$$

Conversely assume that $\rho^*(\xi,\gamma) \leq \alpha$, by *(b)* of the same lemma:

$$\rho^*(\xi,\beta) \leq \max\{\rho^*(\xi,\gamma), \rho^*(\gamma,\beta)\} \leq \max\{\alpha,\alpha_0\} = \alpha.$$

Thus $D(\alpha,\gamma) = D(\alpha,\beta) \cap \gamma$ for all $\alpha \geq \alpha_0$. □

This means that $\mathcal{D} = (D(\alpha,\beta) : \alpha < \kappa, \beta < \kappa^+)$ is a κ-covering matrix for κ^+ with $\beta_\mathcal{D} = \kappa$. Assuming $\mathsf{CP}(\kappa^+,\kappa)$ there would be A unbounded in κ^+ such that $[A]^\kappa$ is covered by $\mathcal{D}$. However there cannot be an unbounded subset A of κ^+ such that $[A]^\kappa$ is covered by $\mathcal{D}$, since any element of the matrix has order type less than κ. □

The function ρ^* defined above will be useful also for other applications of $\mathsf{CP}(\kappa,\lambda)$ in the final section (see theorem 27).
The following theorem follows closely Todorčević's proof that PID entails the failure of square and shows that CP is a very large cardinal property.

Theorem 18. *Assume $\kappa \geq \aleph_1$ is regular. Then $\mathsf{CP}(\kappa)$ implies that $\square(\kappa)$ fails.*

Proof. Todorčević has shown that assuming $\square(\kappa)$ it is possible to define a step function (see sections 6 and 8 of [24]):

$$\rho_2 : [\kappa]^2 \to \omega$$

with the following properties:

(i): For every A unbounded in κ, $\rho_2[[A]^2]$ is unbounded in ω,
(ii): for every $\alpha < \beta$ there is m such that $|\rho_2(\xi,\alpha) - \rho_2(\xi,\beta)| \leq m$ for all $\xi < \alpha$.

[10]For a proof see [24] lemmas 19.1 and 19.2

By *(ii)* it is immediate to check that $\mathcal{D} = \{K(n,\alpha) : n \in \omega \,\&\, \alpha < \kappa\}$ is a covering matrix for κ, where $K(n,\alpha) = \{\xi : \rho_2(\xi,\alpha) \leq n\}$. In fact it can be shown something stronger i.e. that for every $\alpha < \beta$ and n there is m such that $K(n,\alpha) \subseteq K(m,\beta)$ and $K(n,\beta) \cap \alpha \subseteq K(m,\alpha)$.
Using this coherence property of $\mathcal{D}$ one gets that whenever A is an unbounded subset of κ such that $[A]^\omega$ is covered by $\mathcal{D}$, then for all $\beta < \kappa$, $A \cap \beta \subseteq K(m_\beta, \beta)$ for some m_β. Thus one can refine any such A to an unbounded B such that for a fixed m, $B \cap \beta \subseteq K(m,\beta)$ for all $\beta \in B$. This contradicts property *(i)* of ρ_2. Assuming $\mathsf{CP}(\kappa)$ we would get that an A unbounded in κ and such that $[A]^\omega$ is covered by $\mathcal{D}$ exists. However we just remarked that this is impossible. □

Other interesting consequences of these covering properties follow from mild hypothesis on cardinal arithmetic and show that non-trivial θ-covering matrices provably exist only on successors of cardinals of cofinality θ:

Lemma 19. *Assume*[11] $\lambda^\theta = \lambda$. *Then* $\mathsf{CP}(\lambda^+,\theta)$ *implies that every* θ*-covering matrix* $\mathcal{D}$ *on* λ^+ *is trivial (i.e.* $\beta_{\mathcal{D}} = \lambda^+$*).*

We need the following fact:

Fact 20. *Let* θ *and* κ *be regular cardinals with* $2^\theta < \kappa$, $\mathcal{D} = \{K(\alpha,\beta) : \alpha \in \theta, \beta < \kappa\}$ *be a* θ*-covering matrix on* κ *and* A *be an unbounded subset of* κ. *The following are equivalent:*

(i) $[A]^\theta$ *is covered by* $\mathcal{D}$.
(ii) $[A]^\lambda$ *is covered by* $\mathcal{D}$ *for all* $\lambda < \kappa$ *such that* $\lambda^\theta < \kappa$.

Proof. (ii) implies *(i)* is evident. To prove the other direction, assume *(i)* and let $Z \subseteq A$ have size λ with $\lambda^\theta < \kappa$. We need to find $\alpha < \theta$ and $\beta < \kappa$ such that $Z \subseteq K(\alpha,\beta)$. For $X \in [Z]^\theta \subseteq [A]^\theta$ let by *(i)* $\alpha_X < \theta$, $\beta_X < \kappa$ be such that $X \subseteq K(\alpha_X,\beta_X)$. By our assumptions, $\lambda^\theta < \kappa$. For this reason $\beta = \sup_{X\in[Z]^\theta} \beta_X < \kappa$. Now by property *(iii)* of $\mathcal{D}$, we have that for all $X \in [Z]^\theta$, $X \subseteq K(\eta_X,\beta)$ for some η_X. Let $\mathcal{C}_\eta$ be the set of X such that $\eta_X = \eta$. Now notice that for at least one η, $\mathcal{C}_\eta$ must be unbounded in $[Z]^\theta$: if this is not the case let X_η witness that $\mathcal{C}_\eta$ is bounded, then $X = \bigcup_{\eta<\theta} X_\eta \in [Z]^\theta$ is not contained in any element of $\bigcup_{\eta<\theta} \mathcal{C}_\eta = [Z]^\theta$, contradiction. Thus $Z \subseteq K(\eta,\beta)$, since every $\alpha \in Z$ is in some $X \in \mathcal{C}_\eta$, as $\mathcal{C}_\eta$ is unbounded. This completes the proof of the fact. □

Now assume that the lemma fails and let $\mathcal{D}$ be a θ-covering matrix for λ^+ with $\beta_{\mathcal{D}} < \lambda^+$. By $\mathsf{CP}(\lambda^+,\theta)$ there should be an A unbounded in λ^+ such that $[A]^\theta$ is covered by $\mathcal{D}$. Appealing to fact 20, we can conclude in any case that $[A]^\lambda$ is covered by $\mathcal{D}$. Take β large enough in order that $\mathtt{otp}(A\cap\beta) \geq \beta_{\mathcal{D}}$. Since $A\cap\beta$ has size at most λ there are η,γ such that $A\cap\beta \subseteq K(\eta,\gamma)$. Thus $\beta_{\mathcal{D}} \leq \mathtt{otp}(A\cap\beta) \leq \mathtt{otp}K(\eta,\beta) < \beta_{\mathcal{D}}$, contradiction. □

The main difficulty towards a proof that PFA implies SCH has been the fact that all standard principles of reflection for stationary sets do not hold for PFA. In particular Baudoin [3] and Magidor (unpublished) have shown that PFA is compatible with the

[11] This assumption entails $\mathtt{cof}(\lambda) \neq \theta$ and follows from SCH for any $\lambda > 2^\theta$ with cofinality different from θ.

existence on any regular $\kappa \geq \aleph_2$ of a never reflecting stationary subset of S^ω_κ. However the following form of reflection holds under CP:

Fact 21. *Assume* CP *and let* $\mathcal{D}$ *be a covering matrix for a regular* $\kappa > \mathfrak{c}$ *with all* $K(n,\beta)$ *closed. Let* $\lambda < \kappa$ *be a regular cardinal and let* $(S_\eta : \eta < \lambda)$ *be an arbitrary family of stationary subsets of* $S^{\leq\lambda}_\kappa$*. Then there exist* n *and* β *such that* $S_\eta \cap K(n,\beta)$ *is non-empty for all* $\eta < \lambda$.

Proof. By CP and fact 20, there is X unbounded in κ such that $[X]^\lambda$ is covered by $\mathcal{D}$. Since $K(n,\beta)$ is closed for all n and β, we have that $[\overline{X} \cap S^{\leq\lambda}_\kappa]^\lambda$ is covered by $\mathcal{D}$. To see this, let Z be in this latter set and find $Y \subseteq X$ of size λ such that $Z \subseteq \overline{Y}$. Now find n and β such that $Y \subseteq K(n,\beta)$. Since $K(n,\beta)$ is closed, $Z \subseteq \overline{Y} \subseteq K(n,\beta)$.
Now pick $M \prec H(\Theta)$ with Θ large enough such that $|M| = \lambda \subseteq M$ and $\lambda, X, (S_\eta : \eta < \lambda) \in M$. Then $S_\eta \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ is non-empty for all η. By elementarity, M sees this and so $M \cap S_\eta \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ is non-empty for all η. However $M \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ has size λ so there are n and β such that $M \cap \overline{X} \cap S^{\leq\lambda}_\kappa \subseteq K(n,\beta)$. So $S_\eta \cap K(n,\beta)$ is non-empty for all η. □

5. Strongly compact cardinals and CP(κ,θ)

We turn to the proof of theorem 10. We will need the following trivial consequence of the existence of a strongly compact cardinal:

Lemma 22. *Assume* λ *is strongly compact. Then for every regular* $\kappa \geq \lambda$*, there is* $\mathcal{U}$*,* λ*-complete uniform ultrafilter on* κ *which concentrates on* $S^{<\lambda}_\kappa$.

Proof. Assume λ is strongly compact and $\kappa \geq \lambda$ is regular. By definition of λ there is a λ-complete, fine ultrafilter $\mathcal{W}$ on $[\kappa]^{<\lambda}$. Let $\phi(X) = \sup(X)$ for all $X \in [\kappa]^{<\lambda}$. Let $\mathcal{U}$ be the projection of $\mathcal{W}$ under ϕ, i.e.: $A \in \mathcal{U}$ if $\{X : \sup(X) \in A\} \in \mathcal{W}$. It is immediate to check that $\mathcal{U}$ is a λ-complete ultrafilter which concentrates on $S^{<\lambda}_\kappa$. □

Now let $\theta < \lambda$ and $\kappa \geq \lambda$ be regular cardinals and fix a θ-covering matrix $\mathcal{D} = (K(\alpha,\beta) : \alpha \in \theta, \beta \in \kappa)$ for κ. Let $A^\gamma_\alpha = \{\beta \geq \gamma : \gamma \in K(\alpha,\beta)\}$ and $A_\alpha = \{\gamma \in S^{<\lambda}_\kappa : A^\gamma_\alpha \in \mathcal{U}\}$. Since $\theta < \lambda$, by the λ-completeness of $\mathcal{U}$, for every $\gamma \in S^{<\lambda}_\kappa$, there is a least α such that $A^\gamma_\alpha \in \mathcal{U}$. Thus $\bigcup_{\alpha<\theta} A_\alpha = S^{<\lambda}_\kappa$. So there is $\alpha < \theta$ such that $A_\alpha \in \mathcal{U}$. In particular A_α is unbounded. Now let X be a subset of A_α of size θ. Then $A^\gamma_\alpha \in \mathcal{U}$ for all $\gamma \in X$. Since $|X| = \theta < \lambda$, $\bigcap_{\gamma\in X} A^\gamma_\alpha \in \mathcal{U}$ and thus is non-empty. Pick β in this latter set. Then $X \subseteq K(\alpha,\beta)$. Since X is an arbitrary subset of A_α of size θ, we conclude that $[A_\alpha]^\theta$ is covered by $\mathcal{D}$. This concludes the proof of theorem[12] 10. □

6. PID implies CP

We turn to the proof of theorem 11. As we will see below a model of PID retains enough properties of the supercompact cardinals from which it is obtained in order that a variation of the above argument can be run also in the context of ω-covering matrices. We break the proof of theorem 11 in two parts. Assume κ is regular and let $\mathcal{D} = (K(n,\alpha) : n \in \omega, \alpha < \kappa)$ be a covering matrix on κ. Let $\mathcal{I}$ be the family of $X \in [\kappa]^\omega$ such that $X \cap K(n,\alpha)$ is finite for all $\alpha < \kappa$ and for all $n < \omega$.

[12]Notice that for the proof of this theorem we just needed property *(i)* of a covering matrix. The proof of CP(κ,ω) assuming either PID or MRP will need properties *(i), (ii), (iii), (iv)*.

Claim 23. *$\mathcal{I}$ is a P-ideal.*

Proof. Let $\{X_n : n \in \omega\} \subseteq \mathcal{I}$. Let $Y = \bigcup_n X_n$. Let γ_Y witness *(v)* for $\mathcal{D}$ relative to Y. Since for every n, m, $X_n \cap K(m, \gamma_Y)$ is finite, let for every n and for every $j > 0$, $X(n, j)$ be the finite set

$$X_n \cap K(j, \gamma_Y) \setminus K(j-1, \gamma_Y)$$

and let:

$$X = \bigcup_n \bigcup_{j>n} X(n, j).$$

Notice that for every n, $X_n \setminus K(0, \gamma_Y) = \bigcup_{j>0} X(n, j)$ and $\bigcup_{j>n} X(n, j) \subseteq X$, so we have that $X_n \subseteq^* X$ for all n. Moreover $X \cap K(n, \gamma_Y) = \bigcup_{n \geq j > i} X(i, j)$, so it is finite. We claim that $X \in \mathcal{I}$. If not there would be some β and some l such that $X \cap K(l, \beta)$ is infinite. Now $X \cap K(l, \beta) \subseteq Y \cap K(l, \beta) \subseteq K(m, \gamma_Y)$ for some m. Thus we would get that $X \cap K(m, \gamma_Y)$ is infinite for some m contradicting the very definition of X. □

Now notice that if $Z \subseteq \kappa$ is any set of ordinals of size $\aleph_1$ and $\alpha = \sup(Z)$, there must be an n such that $Z \cap K(n, \alpha)$ is uncountable. This means that $\mathcal{I} \not\subseteq [Z]^\omega$, since any countable subset of $Z \cap K(n, \alpha)$ is not in $\mathcal{I}$. This forbids $\mathcal{I}$ to satisfy the first alternative of the P-ideal dichotomy. So the second possibility must be the case, i.e. we can split κ in countably many sets A_n such that $\kappa = \bigcup_n A_n$ and for each n, $[A_n]^\omega \cap \mathcal{I} = \emptyset$.

Claim 24. *For every n, $[A_n]^\omega$ is covered by $\mathcal{D}$.*

Proof. Assume that this is not the case and let $X \in [A_n]^\omega$ be such that $X \setminus K(l, \beta)$ is non-empty for all l, β. Now let X_0 be a countable subset of X such that $X_0 \cap K(l, \gamma_X)$ is finite for all l. Then exactly as in the proof of claim 23 we can see that $X_0 \in [A_n]^\omega \cap \mathcal{I}$. This contradicts the definition of A_n. □

This concludes the proof of theorem 11. □

7. MRP implies SCH

We prove theorem 12. Thus assume MRP and let $\mathcal{D}$ be a covering matrix on a regular $\kappa > \mathfrak{c}$ such that $K(n, \beta)$ is a closed set of ordinals for all n and β. Assume that for all A unbounded in κ, $[A]^\omega$ is not covered by $\mathcal{D}$. We will reach a contradiction. For each $\delta < \kappa$ of countable cofinality, fix C_δ cofinal in δ of order type ω. Let M be a countable elementary submodel of $H(\Theta)$ for some large enough regular Θ. Let $\delta_M = \sup(M \cap \kappa)$ and β_M be the ordinal $\gamma_{M \cap \kappa}$ provided by property *(v)* of $\mathcal{D}$ applied to $M \cap \kappa$. Set $\Sigma(M)$ to be the set of all countable $X \subseteq M \cap \kappa$ bounded in δ_M such that

$$\sup(X) \notin K(|C_{\delta_M} \cap \sup(X)|, \beta_M).$$

We will show that $\Sigma(M)$ is open and M-stationary. Assume this is the case and let $\{M_\eta : \eta < \omega_1\}$ be a reflecting sequence for Σ. Let $\delta_{M_\xi} = \delta_\xi$ and $\delta = \sup_{\omega_1} \delta_\xi$. Find $C \subseteq \omega_1$ club such that $\{\delta_\xi : \xi \in C\} \subseteq K(n, \delta)$ for some n (which is possible since the $K(n, \delta)$ are closed subsets of κ). Let α be a limit point of C. Let $M = M_\alpha$ and notice that by our choice of β_M for all m, there is l such that $K(m, \delta) \cap M \subseteq K(l, \beta_M)$. This means that for all $\eta \in C \cap \alpha$, $\delta_\eta \in K(n, \delta) \cap M \subseteq K(l, \beta_M)$ for some fixed l. Since α

is a limit point of C there is $\eta \in \alpha \cap C$ such that $|C_{\delta_M} \cap \delta_\eta| \geq l$ and $M_\eta \cap \kappa \in \Sigma(M)$. But this is impossible, since $M_\eta \cap \kappa \in \Sigma(M)$ means that $\delta_\eta \notin K(|C_{\delta_M} \cap \delta_\eta|, \beta_M)$, i.e. $\delta_\eta \notin K(l, \beta_M)$.

We now show that $\Sigma(M)$ is open and M-stationary:

Claim 25. $\Sigma(M)$ *is open.*

Proof. Assume $X \in \Sigma(M)$, we will find $\gamma \in X$ such that $[\{\gamma\}, X] \subseteq \Sigma(M)$. To this aim notice that $C_{\delta_M} \cap \sup(X)$ is a finite set. Let $n_0 = |C_{\delta_M} \cap \sup(X)|$ and $\gamma_0 = \max(C_{\delta_M} \cap \sup(X)) + 1$. Since $X \in \Sigma(M)$, $\sup(X) \notin K(n_0, \beta_M)$ and so, since $K(n_0, \beta_M)$ is closed, $\gamma_1 = \max(K(n_0, \beta_M) \cap \sup(X)) < \sup(X)$. Thus, let $\gamma \in X$ be greater or equal than $\max\{\gamma_1 + 1, \gamma_0\}$. If $Y \in [\{\gamma\}, X]$, then $\gamma_0 \leq \sup(Y) \leq \sup(X)$, so $|C_{\delta_M} \cap \sup(Y)| = |C_{\delta_M} \cap \sup(X)| = n_0$ and

$$\gamma_1 = \max(K(n_0, \beta_M) \cap \sup(X)) < \sup(Y) \leq \sup(X) < \min(K(n_0, \beta_M) \setminus \sup(X)).$$

Thus $\sup(Y) \notin K(|C_{\delta_M} \cap \sup(Y)|, \beta_M)$, i.e. $Y \in \Sigma(M)$. □

Claim 26. $\Sigma(M)$ *is* M*-stationary.*

Proof. Let $f : [\kappa]^{<\omega} \to \kappa$ in M. We need to find $X \in \Sigma(M)$ such that $f[[X]^{<\omega}] = X$. Let $N \prec H(\kappa^+)$ be a countable submodel in M such that $f \in N$ and let $C = \{\delta < \kappa : f[[\delta]^{<\omega}] = \delta\}$. Let also $n_0 = |C_{\delta_M} \cap \sup(N \cap \kappa)|$ and $\gamma_0 \in N$ be larger than $\max(C_{\delta_M} \cap \sup(N \cap \kappa))$. Then $(C \setminus \gamma_0) \in N$. We assumed that no A unbounded in κ is such that $[A]^\omega$ is covered by $\mathcal{D}$. So in particular by elementarity of N:

$$N \models [(C \setminus \gamma_0) \cap S^\omega_\kappa]^\omega \textit{is not covered by } \mathcal{D}$$

Thus there exists $X \in N$ countable subset of $(C \setminus \gamma_0) \cap S^\omega_\kappa$ such that for all n and β, $X \setminus K(n, \beta)$ is non-empty. Let $\gamma \in X \setminus K(n_0, \beta_M)$. Now find $Z \in N$ countable and cofinal in γ and let Y be the f-closure of Z. Then $Y \in N \subseteq M$. Now $\gamma \in C$ so $\sup(Y) = \sup(Z) = \gamma \notin K(n_0, \beta_M)$. Moreover $\gamma = \sup(Y) \in (C \setminus \gamma_0) \cap N$, so $\gamma_0 < \sup(Y) < \sup(N \cap \kappa)$, i.e. $|C_{\delta_M} \cap \sup(Y)| = |C_{\delta_M} \cap \sup(N \cap \kappa)| = n_0$. Thus:

$$\sup(Y) \notin K(|C_{\delta_M} \cap \sup(Y)|, \beta_M).$$

I.e. $Y \in \Sigma(M)$. □

This concludes the proof of theorem 12. □

8. "Saturation" properties of models of strong forcing axioms.

Since forcing axioms have been able to settle many of the classical problems of set theory, we can expect that the models of a forcing axiom are in some sense categorical. There are many ways in which one can give a precise formulation to this concept. For example, one can study what kind of forcing notions can preserve PFA or MM, or else if a model V of a forcing axiom can have an interesting inner model M of the same forcing axiom. There are many results in this area, some of them very recent. First of all there are results that shows that one has to demand a certain degree of resemblance between V and M. For example assuming large cardinals it is possible to use the stationary tower forcing introduced by Woodin[13] to produce two transitive models $M \subseteq V$ of PFA (or MM or whatever is not conflicting with large cardinal hypothesis)

[13] [13] gives a complete presentation of this subject.

with different ω-sequences of ordinals and an elementary embedding between them. However M and V do not compute the same way neither the ordinals of countable cofinality nor the cardinals. On the other hand, König and Yoshinobu [12, Theorem 6.1] showed that PFA is preserved by ω_2-closed forcing, while it is a folklore result that MM is preserved by ω_2-directed closed forcing. Notice however that all these forcing notions do not introduce new sets of size at most $\aleph_1$. In the other direction, in [25] Veličković used a result of Gitik to show that if MM holds and M is an inner model such that $\omega_2^M = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq M$ and Caicedo and Veličković [5] showed, using the mapping reflection principle MRP introduced by Moore in [17], that if $M \subseteq V$ are models of BPFA and $\omega_2^M = \omega_2$ then $\mathcal{P}(\omega_1) \subseteq M$. In any case all the results so far produced show that any two models $V \subseteq W$ of some strong forcing axiom and with the same cardinals have the same ω_1-sequences of ordinals. Thus it is tempting to conjecture that forcing axioms produce models of set theory which are "saturated" with respect to sets of size $\aleph_1$. One possible way to give a precise formulation to this idea may be the following:

Conjecture 1. *(Caicedo, Veličković) Assume $W \subseteq V$ are models of* MM *with the same cardinals. Then $[Ord]^{\leq\omega_1} \subseteq W$.*

We will show that one cannot prove the negation of the above conjecture by mean of forcing. Since no inner model theory is known for models of MM, PFA or of a supercompact cardinal, the results I will present brings to the conclusion that there are no suitable means to try to prove the negation of the conjecture.

Theorem 27. *Assume $\mathsf{CP}(\kappa^+, \theta)$. Let W be an an inner model such that κ is a regular cardinal of W and such that $(\kappa^+)^W = \kappa^+$. Then $\mathtt{cof}(\kappa) \neq \theta$.*

This shows that if λ is strongly compact, than one cannot change the cofinality of some regular $\kappa \geq \lambda$ to some $\theta < \lambda$ and preserve at the same time κ^+ and the strong-compactness of λ.

Corollary 28. *Assume* PFA *and let W be an inner model with the same cardinals. Then W computes correctly all the points of countable cofinality.*

Proof. PFA implies CP. Now apply the above theorem. □

In particular this gives another proof that Prikry forcing destroys PFA, since if g is a Prikry generic sequence on a measurable κ, $V[g]$ cannot model CP.
What about cofinality ω_1 assuming forcing axioms? I do not know whether $\mathsf{CP}(\kappa, \omega_1)$ holds under MM for regular cardinals[14] $\kappa > \aleph_2$. If we analyze the proofs of $\mathsf{CP}(\kappa, \omega)$ from PFA we see that the diagonalization arguments we used requires that the matrix has length ω and thus cannot generalize to ω_1. However in view of the preceding results it is natural to expect the following:

Theorem 29. *Assume* MM*. Let κ be a strong limit cardinal*[15] *and W be an inner model such that κ is regular in W and $\kappa^+ = (\kappa^+)^W$. Then $\mathtt{cof}(\kappa) > \omega_1$.*

---

[14] Notice that $\mathsf{CP}(\aleph_2, \aleph_1)$ fails by lemma 14.

[15] This cardinal arithmetic constraint can be certainly relaxed and I believe it to be unessential. I assumed this hypothesis to be sure that one can apply to this situation Dzamonja and Shelah's theorem 34 below.

Before I turn to the proofs, I'd like to add some more comments. First of all we can combine the above results to prove the following:

Proposition 30. *Assume $W \models \mathsf{MM}$ and all limit cardinals of W are strong limit. Moreover assume that W is a set-generic extension of V with the same ordinals of cofinality ω and ω_1 and such that $P(\omega_1)^W \subseteq V$. Then $[Ord]^{\leq\omega_1} \subseteq V$.*

Proof. W is a set-generic extension of V by some P-generic filter. Thus P is a set and satisfies the $|P|^+$-chain condition. Let $\kappa = |P|^+$. It is enough to show that $[\kappa]^{\leq\omega_1} \subseteq V$. Now by our assumption $S^{\leq\omega_1}_\kappa = (S^{\leq\omega_1}_\kappa)^W$. By the κ-chain condition we get that every stationary subset of $S^\omega_\kappa \in V$ remains stationary in W. Now fix a partition $\{S_\alpha : \alpha < \kappa\}$ in V of S^ω_κ in κ-many disjoint stationary subset of κ. Then this is still a partition in stationary sets of W. Given any $X \in [\kappa]^{\leq\omega_1}$ find by MM in W an ordinal $\delta < \kappa$ of cofinality ω_1 such that S_α reflects on δ iff $\alpha \in X$. Let $C \in V$ be a club subset of δ of order type ω_1 such that $S_\alpha \cap C$ is bounded in δ whenever $\alpha \notin X$. Then $X \in V$ since:

$$X = \{\eta : S_\eta \cap C \text{ is unbounded in } \delta\}.$$

□

The above results suggest that another interesting form of saturation of models of MM may hold. Gitik has shown [8] that assuming suitable large cardinals it is possible to produce a model of set theory W and a strongly inaccessible cardinal κ such that for all $\lambda < \kappa$ there is S_λ stationary subset of S^λ_κ such that $\mathsf{NS} \restriction S_\lambda$ is κ^+-saturated. Now if we force with $P(S_\lambda)/\mathsf{NS}$ we get a model V with the same cardinals, the same bounded subsets of κ and such that κ is singular of cofinality λ. However the ground model W is obtained by a cardinal preserving forcing which shoots Prikry sequences on a large number of cardinals below κ. Thus this approach cannot work to disprove the following conjecture:

Conjecture 2. *Assume $V \subseteq W$ are models of MM with the same cardinals. Then they have the same cofinalities.*

We now turn to proofs:
Proof of theorem 27: Assume the theorem is false. Then $\mathtt{cof}(\kappa) = \theta$. We need some preparation. Work in W. Fix a sequence $\mathcal{C} = \{C_\xi : \xi < \kappa^+\} \in W$ such that for all limit α, C_α is a club subset of α of order type $\mathtt{cof}(\alpha)$. If $\xi = \alpha + 1$, $C_\xi = \{\alpha\}$. Consider the function $\rho^* : [\kappa^+]^2 \to \kappa$ defined from such a sequence which was introduced in the proof of lemma 14.
Now work in V. Let $g : \theta \to \kappa$ be a cofinal sequence. Set $D(\alpha, \beta) = \{\xi < \alpha : \rho^*(\xi, \beta) \leq g(\alpha)\}$. The two lemmas 15 and 16 allows to prove the analogue of fact 17:

Fact 31. *The following holds:*

- *(i):* $\mathtt{otp}(D(\alpha, \beta)) < \kappa$ *for all* $\alpha < \theta$ *and* $\beta < \kappa^+$,
- *(ii): for all* $\gamma < \beta < \kappa^+$*,there is* $\alpha_0 < \theta$ *such that* $D(\alpha, \gamma) = D(\alpha, \beta) \cap \gamma$ *for all* $\alpha \geq \alpha_0$.

□

This means that $\mathcal{D} = (D(\alpha, \beta) : \alpha < \theta, \beta < \kappa^+)$ is a θ-covering matrix for κ^+ with $\beta_{\mathcal{D}} = \kappa$. By $\mathsf{CP}(\kappa^+, \theta)$ there is A unbounded in κ^+ such that $[A]^\theta$ is covered by $\mathcal{D}$.

Let β be large enough in order that $\mathtt{otp}(A \cap \beta) \geq \kappa$. Then for every $\alpha < \theta$ we can find $\xi_\alpha \in A \cap \beta \setminus K(\alpha, \beta)$ since $\mathtt{otp}(K(\alpha, \beta)) < \kappa \leq \mathtt{otp}(A \cap \beta)$. Now $X = \{\xi_\alpha : \xi < \theta\}$ is in $[A \cap \beta]^\theta$ and is not covered by any $D(\alpha, \beta)$. By *(ii)* of the above fact we can conclude that X is not covered by $D(\alpha, \gamma)$ for all $\alpha < \theta$ and $\gamma < \kappa$. This contradicts our assumption that $[A]^\theta$ is covered by $\mathcal{D}$. □

Proof of theorem 29: We proceed by contradiction. So assume the theorem is false. Let κ contradict the theorem, then by theorem 27, $\mathtt{cof}(\kappa) = \omega_1$ is the only possibility. Now notice that since SCH holds under MM, $2^\kappa = \kappa^+$ since κ is a singular strong limit cardinal. Then $2^\kappa = \kappa^+$ holds also in W. We will need this hypothesis in the sequel.

The idea is now to develop on a theorem of Cummings and Schimmerling where they assume that κ has countable cofinality and W is an inner model such that κ is regular and $\kappa^+ = (\kappa^+)^W$ to conclude that $\square_{\kappa,\omega}$ holds (section 4 of [6]). In analogy with their result, assuming that $\mathtt{cof}(\kappa) = \omega_1$ and the same hypothesis on W, we want to draw the conclusion that $\square_{\kappa,\omega_1}$ holds[16]. We will exploit however the proof of their theorem which appears in [24], since this proof uses the function ρ^* with which we are now familiar.

We thus recall the notion of a $\square_{\kappa,\aleph_1}$-square sequence.

Definition 32. *Let κ be an uncountable cardinal. $\{\mathcal{C}_\alpha : \alpha < \kappa^+\}$ is a $\square_{\kappa,\aleph_1}$-sequence if:*

- *(i): $\mathcal{C}_\alpha = (C_{\alpha,i} : i < \omega_1)$ is a sequence of club subsets of α of ordertype at most κ for all α,*
- *(ii): for all $\alpha < \beta < \kappa^+$ and $i < \omega_1$, if α is a limit point of $C_{\beta,i}$, there is a $j < \omega_1$ such that $C_{\beta,i} = C_{\beta,j} \cap \alpha$.*

It is a result of Magidor which develops on ideas of Todorčević, that PFA implies the failure of $\square_{\kappa,\aleph_1}$ (see [10] exercise 38.24 for a proof).

We also recall that Shelah [19] has shown that under MM, the strong Chang conjecture $\mathsf{SCC}(\lambda)$ holds on all regular $\lambda \geq \aleph_1$. Where $\mathsf{SCC}(\lambda)$ holds if for every club $\mathcal{D}$ in $[\lambda]^\omega$ there is a club $\mathcal{C} \subseteq \mathcal{D}$ such that for all $X \in \mathcal{C}$ and $\alpha < \lambda$ there is $Y \in \mathcal{C}$ containing X with $\alpha < \sup(Y)$ and such that $X \cap \omega_1 = Y \cap \omega_1$. Our aim is to prove the following[17]:

Theorem 33. *Assume* SCC *and let W be an inner model such that κ is strongly inaccessible in W, $\kappa^+ = (\kappa^+)^W = (2^\kappa)^W$, while $\mathtt{cof}(\kappa) = \omega_1$. Then $\square_{\kappa,\aleph_1}$ holds.*

Once this theorem is proved, we can combine it with Magidor's proof that PFA implies the failure of $\square_{\kappa,\aleph_1}$ for all $\kappa \geq \aleph_1$ to complete the proof of theorem 29.

To this aim I will now exploit the following weakening of a theorem of Džamonja and Shelah (theorem 2.0 of [7]):

Theorem 34. *Assume κ is singular of uncountable cofinality, W is an inner model such that κ is strongly inaccessible and $\kappa^+ = (\kappa^+)^W = (2^\kappa)^W$. Then there is E club*

---

[16] For those who are familiar with their proof, we remark that now we must define in a non-trivial way the square sequence on the points of cofinality ω_1 of W-cofinality κ, problem which they can ignore defining trivially the square sequence on these points. To overcome this difficulty we will use MM.

[17] The theorem below may need a reformulation of the conclusion in the case that SCC is incompatible with $\square_{\kappa,\aleph_1}$. This is plausible, since all the proof of $\mathsf{SCC}(\kappa^+)$ that I know assume as hypothesis principles which negates $\square_\kappa$.

subset of κ, such that $E \setminus C$ is bounded for all $C \in W$ club in κ and such that for all points $\xi \in E$ which are not limit, $\mathtt{cof}(\xi) > \omega$.

We are going to use this magic club $E \in V$, $\mathsf{SCC}(\kappa^+)$ in V and the function $\rho^* \in W$ to define a nicely cohering $\square_{\kappa,\aleph_1}$-sequence. We will be interested only in the non-limit points of E. Thus we assume that

$$E = \{\eta_\xi : \xi < \omega_1\}$$

is just a set of points of uncountable cofinality (and thus of uncountable W-cofinality) of type ω_1, cofinal in κ and such that $E \setminus C$ is bounded for all $C \in W$ club in κ.
Now work in W and set:

$$D_{\alpha,\eta} = \{\xi < \alpha : \rho^*(\xi, \alpha) < \eta\}$$

for all $\alpha \in S^{<\kappa}_{\kappa^+}$ and $\eta < \kappa$. Let

$$C_\alpha = \{\eta < \kappa : D_{\alpha,\eta} \textit{ is a countably closed subset of } \alpha \textit{ cofinal in } \alpha\}$$

It is easy to check in W that C_α is a club subset of $S^{>\omega}_\kappa$. Moreover using the coherence property of ρ^* (see lemma 16), it is immediate to check that for all $\alpha < \beta$, $C_\beta \setminus C_\alpha$ is bounded below $\rho^*(\alpha, \beta)$. Now work in V and for all $\alpha \in (S^{<\kappa}_{\kappa^+})^W$, let η be the least such that for some $\beta \geq \alpha$, $E \subseteq C_\beta \setminus \eta$ and α is a limit point of $D_{\beta,\min(E\setminus\eta)}$. Set $E_{\alpha,\xi} = \emptyset$ if $\eta_\xi < \eta$, $E_{\alpha,\xi} = \overline{D_{\beta,\eta_\xi}} \cap \alpha$ otherwise. We leave to the reader the proof of the following (see the proof of theorem 19.4 of [24]):

Lemma 35. *The following holds:*

- **(a):** *If $\alpha < \beta \in (S^{<\kappa}_{\kappa^+})^W$ and α is a limit point of $E_{\beta,\xi}$, then $E_{\beta,\nu} \cap \alpha = E_{\alpha,\nu}$ for all $\nu \geq \xi$.*
- **(b):** *For all $\beta \in (S^{<\kappa}_{\kappa^+})^W$ and for all $\xi < \omega_1$, D_{β,η_ξ} has size less than κ in W. So for all $\beta \in (S^{<\kappa}_{\kappa^+})^W$ and for all $\xi < \omega_1$, if $\alpha \in (S^{\kappa}_{\kappa^+})^W$, then α is not a limit point of $E_{\beta,\xi}$.*

$\square$

We thus almost have defined our $\square_{\kappa,\aleph_1}$-sequence, we need only to complete the definition of $E_{\alpha,\xi}$ for $\alpha \in (S^{\kappa}_{\kappa^+})^W$. To this aim we will use $\mathsf{SCC}(\kappa^+)$. There are two possibilities: either $S = (S^{\kappa}_{\kappa^+})^W \subseteq S^{\omega_1}_{\kappa^+}$ is non-stationary and in this case, if C is a club which avoids this set, $\{E_{\alpha,\xi} \cap C : \alpha \in C, \xi < \omega_1\}$ is a $\square_{\kappa,\aleph_1}$-sequence[18], or S is stationary. In this case we will find a club C such that for all $\alpha \in C \cap S$ and $\xi < \omega_1$, an $E_{\alpha,\xi}$ of order type at most κ can be defined in order that for any β limit point of $E_{\alpha,\xi}$, $(\mathtt{cof}(\beta))^W \neq \kappa$ and $E_{\beta,\xi} = E_{\alpha,\xi} \cap \beta$. If this is possible $\{E_{\alpha,\xi} \cap C : \alpha \in C, \xi < \omega_1\}$ will be our elected $\square_{\kappa,\aleph_1}$-sequence[19]. This is so because by (b) of the above claim and our construction of the sets $E_{\alpha,\xi}$ for $\alpha \in S$, α will not be a limit point of $E_{\gamma,\xi}$ for any $\gamma > \alpha$, so we just have to worry that the coherence properties are satisfied below α.
Fix $(M_\xi : \xi < \kappa^+)$ continuous $\in$-chain of models of size κ of some suitable $H(\theta)$ such that $\kappa \subseteq M_0$. Let C be the club of δ such that $M_\delta \cap \kappa^+ = \delta$. For any $\delta \in S \cap C$, we define $E_{\delta,\xi}$ as follows. By $\mathsf{SCC}(\kappa^+)$ in M_δ there is a suitable club $\mathcal{C} \in M_\delta$ of countable models X of $H(\kappa^{++})$ such that for every $X \in \mathcal{C}$ there is $Y \in \mathcal{C}$, with $X \subseteq Y$ and

[18] More precisely let π be the transitive collapse of C onto κ^+. Then $\{\pi(E_{\alpha,\xi} \cap C) : \alpha \in C, \xi < \omega_1\}$ is a $\square_{\kappa,\aleph_1}$-sequence.

[19] The previous footnote applies also to this situation.

$X \cap \omega_1 = Y \cap \omega_1$. Since $\kappa^\omega = \kappa$ it is easy to build by induction a sequence $(N_\xi : \xi < \omega_1)$ of elements of $\mathcal{C} \cap M_\delta$ such that for a fixed $\alpha < \omega_1$ and all $\xi < \eta$, $N_\xi \subseteq N_\eta$, $N_\xi \cap \omega_1 = \alpha$ and if $N = \bigcup_{\xi<\omega_1} N_\xi$, $\sup(N \cap \kappa^+) = \delta$. First of all fix $\{\delta_\xi : \xi < \omega_1\}$ club in δ of type ω_1. If η is limit and $(N_\xi : \xi < \eta)$ has been defined, there is a $\gamma < \delta$ such that $N_\eta = \bigcup_{\xi<\eta} N_\xi \subseteq M_\gamma$ since δ has cofinality ω_1 and $M_\delta = \bigcup_{\xi<\delta} M_\xi$. Notice that $N_\eta \in \mathcal{C}$ since it is the increasing union of elements in $\mathcal{C}$. Now since $|M_\gamma| = \kappa$ and $\kappa^\omega = \kappa$, $N_\eta \in [M_\gamma]^\omega \subseteq M_{\gamma+1} \subseteq M_\delta$. Thus we can put N_η on the top of our sequence. Now given $N_\eta \in M_\delta \cap \mathcal{C}$, we can apply $\mathsf{SCC}(\kappa^+)$ in M_δ to find $N_{\eta+1} \in M_\delta \cap \mathcal{C}$ such that $N_\eta \subseteq N_{\eta+1}$, $N_{\eta+1} \cap \omega_1 = N_\eta \cap \omega_1 = \alpha$ and $\sup(N_{\eta+1} \cap \kappa^+) \geq \delta_\eta$. After ω_1-many steps we get the desired sequence. Let $N = \bigcup_{\xi<\omega_1} N_\xi$. Now $N \cap \omega_1 = \alpha$, thus by elementarity of $N \prec H(\kappa^{++})$,

$$\sup(N \cap \kappa) = \sup(E \cap N) = \sup\{\eta_\xi : \xi < \alpha\} < \eta_\alpha.$$

So for all $\gamma < \beta \in N$, by elementarity of N there is $\xi < \alpha$ such that $\gamma \in E_{\beta,\xi} \subseteq E_{\beta,\alpha}$. Let C_0 be the club subset of limit points of $N \cap \delta \cap S^\omega_{\kappa^+}$. Notice that C_0 avoids S.

Claim 36. *For all $\gamma < \beta \in C_0$, $E_{\beta,i} \cap \gamma = E_{\gamma,i}$ for all $i \geq \alpha$.*

Proof. First of all notice that for all $\beta \in C_0$ and $\xi < \omega_1$, $E_{\beta,\xi}$ is defined since $\beta \notin S$. Now for any $\gamma, \beta \in C_0$, pick $\iota \in N \setminus S$ larger than γ and β. Since all of the points in $N \cap \iota$ are in $E_{\iota,\alpha} \subseteq E_{\iota,i}$, γ and β are limit points of $E_{\iota,i}$ and thus $E_{\gamma,i} = E_{\iota,i} \cap \gamma$ and $E_{\beta,i} = E_{\iota,i} \cap \beta$, from which the conclusion follows. □

Now we can set $E_{\delta,i} = \emptyset$ for all $i < \alpha$ and $E_{\delta,i} = \bigcup_{\gamma \in C_0} E_{\gamma,i}$, for all $i \geq \alpha$. Since $E_{\gamma,i}$ has order type less than κ for all $\gamma \in C$ and the union is coherent $\mathtt{otp}(E_{\delta,i}) \leq \kappa$. This complete the definition of the $\square_{\kappa,\aleph_1}$-sequence and the proof of theorems 29 and 33. □

Acknowledgements

This research has been partially supported by the Austrian Science Fund FWF project P19375-N18. I thank Boban Veličković, Andrés Caicedo and the referee for many useful suggestions and comments on the redaction of this paper.

References

[1] U. Abraham and S. Todorčević, *Partition Properties of ω_1 Compatible with CH*, Fundamenta Mathematicae **152** (1997) 165–180.

[2] B. Balcar, T. Jech, and T. Pazák, *Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann*, Bulletin of the London Mathematical Society **37** (2005) 885–898.

[3] R. E. Beaudoin, *The proper forcing axiom and stationary set reflection*, Pacific Journal of Mathematics **149(1)** (1991) 13–24.

[4] A. Caicedo and B. Veličković, *The Mapping Reflection Principle*. In preparation.

[5] ———, *Bounded proper forcing axiom and well orderings of the reals.*, Mathematical Research Letters (2006) 393–408.

[6] J. Cummings and E. Schimmerling, *Indexed squares*, Israel Journal of Mathematics **131** (2002) 61–99.

[7] M. Džamonja and S. Shelah, *On squares, outside guessing of clubs and $I_{<f}[\lambda]$*, Fundamenta Mathematicae **148** (1995) 165–198.

[8] M. Gitik, *Changing cofinalities and the non-stationary ideal*, israel journal of mathematics **56(3)** (1986) 280–314.

[9] ———, *The strength of the failure of SCH*, Annals of Pure and Applied Logics **51(3)** (1991) 215–240.
[10] T. Jech, Set theory, The Third Millennium Edition, Revised and Expanded, Springer (2002).
[11] B. König, P. Larson, J. T. Moore, and B. Veličković, *Bounding the consistency strength of a five element linear basis*. To appear in Israel Journal of Mathematics, 17 pages.
[12] B. König and Y. Yoshinobu, *Fragments of Martin's Maximum in generic extensions*, Mathematical Logic Quarterly **50(3)** (2004) 297–302.
[13] P. B. Larson, The Stationary Tower: Notes on a Course by W. Hugh Woodin, AMS (2004).
[14] T. Miyamoto, *a memo* (2006). Unpublished, 2 pages.
[15] J. T. Moore, *Set mapping reflection*, Journal of Mathematical Logic **5(1)** (2005) 87–97.
[16] ———, *A five element basis for the uncountable linear orders*, Annals of Mathematics **163(2)** (2006) 669–688.
[17] ———, *The Proper Forcing Axiom, Prikry forcing, and the Singular Cardinals Hypothesis*, Annals of Pure and Applied Logic **140(1-3)** (2006) 128–132.
[18] E. Schimmerling, *Combinatorial principles in the core model for one Woodin cardinal.*, Annals of Pure and Applied Logic **74(2)** (1995) 153–201.
[19] S. Shelah, Proper Forcing, Springer (1982).
[20] ———, Cardinal Arithmetic, Oxford University Press (1994).
[21] J. H. Silver, *On the singular cardinal problem*, Proceedings of the International Congress of Mathematicians, Vancouver, B.C., 1974 **1** (1975) 265–268.
[22] R. M. Solovay, *Strongly compact cardinals and the GCH*, Proc. Sympos. Pure Math., Vol XXV, Univ. of California, Berkeley, Calif. (1971) 365–372.
[23] S. Todorčević, *A dichotomy for P-ideals of countable sets*, Fundamenta Mathematicae **166(3)** (2000) 251–267.
[24] ———, *Coherent sequences*, in M. Foreman, A. Kanamori, and M. Magidor, editors, Handbook of Set Theory, North Holland (to appear).
[25] B. Veličković, *Forcing axioms and stationary sets*, Advances in Mathematics **94(2)** (1992) 256–284.
[26] ———, *CCC Forcing and Splitting Reals*, Israel Journal of Mathematics **147** (2005) 209–220.
[27] M. Viale, *The Proper Forcing Axiom and the Singular Cardinal Hypothesis*, Journal of Symbolic Logic **71(2)** (2006) 473–479.

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger strasse 25, A-1090 Wien, Austria

E-mail address: matteo@logic.univie.ac.at