GROTHENDIECK-RIEMANN-ROCH AND THE MODULI OF ENRIQUES SURFACES

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ABSTRACT. We give a short and "classical" proof of Borcherds' theorem that the moduli space of Enriques surfaces is quasi-affine. The use of the Borcherds' product is replaced in our proof by an application of the Grothendieck-Riemann-Roch theorem.

1

A (complex) Enriques surface is a projective smooth connected algebraic surface Y over $\mathbf{C}$ with $\mathrm{H}^1(Y, \mathcal{O}_Y) = \mathrm{H}^2(Y, \mathcal{O}_Y) = (0)$, $(\Omega_Y^2)^{\otimes 2} \simeq \mathcal{O}_Y$ but $\Omega_Y^2 \not\simeq \mathcal{O}_Y$ ([CD]). In this note we give a short proof of the fact that the coarse moduli space of complex Enriques surfaces is quasi-affine. This was first shown by Borcherds [B] using the denominator function of a generalized Kac-Moody superalgebra (the "Φ-function"). From this, it follows (see [BKPS]) that any complete family of complex Enriques surfaces is isotrivial. Our observation here is that the ingredient of Borcherds' theory of infinite products can be completely removed from the proof and can be replaced by a simple use of the Grothendieck-Riemann-Roch theorem.

2

We refer the reader to [CD] and [BPV] for background on Enriques and $K3$ surfaces. Let $L = E_8(-1) \oplus E_8(-1) \oplus H \oplus H \oplus H$ be the $K3$ lattice with involution $\rho : L \to L$ given by

$$\rho((x, y, z_1, z_2, z_3)) = (y, x, -z_1, z_3, z_2).$$

Here $H = \mathbf{Z} \cdot e \oplus \mathbf{Z} \cdot f$ is the hyperbolic plane lattice with $\langle e, f \rangle = 1$, and E_8 denotes the positive definite orthogonal lattice corresponding to the Dynkin diagram of type E_8. (The parentheses (m) mean that we modify the pairing by multiplying it by the integer m.) Let $L^+ = \{l \in L \mid \rho(l) = l\}$, $L^- = \{l \in L \mid \rho(l) = -l\}$. Then L^+ is isometric to $E_8(-2) \oplus H(2)$; the unimodular lattice $L^+(1/2)$ can be identified with the Enriques lattice and is isometric to $E_8(-1) \oplus H$. The lattice L^- has signature $(2, 10)$. We set

$$\Omega = \{\omega \in \mathbf{P}(L \otimes \mathbf{C}) \mid \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar\omega \rangle > 0\},$$
$$\Omega^- = \{\omega \in \mathbf{P}(L^- \otimes \mathbf{C}) \mid \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar\omega \rangle > 0\} \ .$$

for the 20-dimensional (resp. 10-dimensional) hermitian symmetric space for $O(3, 19)$ (resp. for $O(2, 10)$). (Ω^- has two 2 connected components.) Also let $O(L)^\rho = \{g \in O(L) \mid g \cdot \rho = \rho \cdot g\}$,

$$O(L)^- = \mathrm{restr}_{L^-} (O(L)^\rho)$$

where by O we denote the orthogonal group of automorphisms of the lattice that preserve the form. By work of Horikawa, $O(L)^- = O(L^-)$. For a vector d of norm $\langle d, d \rangle = -2$ in L^- let H_d be the divisor of points of Ω^- represented by vectors orthogonal to d and set $\Omega_0^- = \Omega^- - (\cup_d H_d)$. The group $O(L)^- = O(L^-)$ acts on Ω_0^-.

By the work of Namikawa and especially Horikawa ([H], [BPV])) on the Torelli theorem for Enriques surfaces, the (coarse) moduli variety D^0 of complex Enriques surfaces can be identified with the quotient

(2.1) $$D^0 = \Omega_0^- / O(L^-) \, .$$

(This has the structure of a quasi-projective variety over $\mathbf{C}$.) There is no "universal" family of Enriques surfaces over D^0 but this can be remedied by passing to a suitable finite cover:

Proposition 2.1. *There is a normal subgroup $\Gamma \subset O(L^-)$ of finite index such that:*
(i) The group Γ acts freely on Ω_0^-. The corresponding quotient

$$D_\Gamma^0 = \Omega_0^- / \Gamma$$

is a smooth variety which supports a smooth family of Enriques surfaces $f : Y \to D_\Gamma^0$.
(ii) There is a family of $K3$-surfaces $g : X \to D_\Gamma^0$ with a free involution $\iota : X \to X$ which respects g and is such that $Y = X/\langle \iota \rangle$. Denote by $p : X \to Y$ the covering morphism.
(iii) The line bundle of automorphic forms of weight 1 on Ω^- (with fiber over the point $\omega \in \Omega^- \subset \mathbf{P}(L^- \otimes \mathbf{C})$ given by the corresponding line in $L^- \otimes \mathbf{C}$) descends to the line bundle $\omega_\Gamma = g_*(\Omega_{X/D_\Gamma^0}^2)$ on D_Γ^0.

PROOF. The existence of such a subgroup Γ appears to be well-known to the experts; for the sake of completeness we sketch one possible construction here. Let us consider the vector $v = (0, 0, 0, e + f, e + f)$ in $L^+ \subset L$; we have $\langle v, v \rangle = 4$. Let us consider the groups $\Gamma_v = \{ g \in O(L) \mid g(v) = v \} < O(L)$,

$$\Gamma_v^\rho = \{ g \in O(L) \mid \rho \cdot g = g \cdot \rho, g(v) = v \} < O(L), \quad \Gamma_v^- = \mathrm{restr}_{L^-}(\Gamma_v^\rho) < O(L^-) \, .$$

(The structure of Γ_v^- is explained in [StI, Prop. 2.7].) For odd $n \geq 3$, we set $\Gamma_n = \{ g \in O(L) \mid g \equiv \mathrm{Id} \bmod nL \}$ and set $\Gamma_{v,n} := \Gamma_v \cap \Gamma_n$, $\Gamma_{v,n}^\rho := \Gamma_v^\rho \cap \Gamma_n$. Similarly, denote by $\Gamma_{v,n}^-$ the restriction $\mathrm{restr}_{L^-}(\Gamma_{v,n}^\rho)$; using [StI, Prop. 2.7] and the fact that n is odd we see that

$$\Gamma_{v,n}^- = \Gamma_v^- \cap \{ h \in O(L^-) \mid h \equiv \mathrm{Id} \bmod n \} \, .$$

This is a normal subgroup of finite index in Γ_v^-. By a well-known lemma of Serre, Γ_n and all our groups with a subscript n have only trivial torsion. The group $\Gamma_{v,n}$ acts on the homogeneous space $\Omega_v = \{ \omega \in \mathbf{P}((\mathbf{C} \cdot v)^\perp) \mid \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar{\omega} \rangle > 0 \}$ with trivial stabilizers. Using the period morphism and the Torelli theorem (e.g [BR], [Fr]), we can identify the smooth quasi-projective quotient $\Omega_v / \Gamma_{v,n}$ with the *fine* moduli space of polarized $K3$ surfaces with a polarization of degree 4 and n-level structure. Now notice that the subgroup $\Gamma_{v,n}^\rho$ preserves $\Omega^- \subset \Omega_v$ and its action factors through $\Gamma_{v,n}^\rho \to \Gamma_{v,n}^-$. We have a morphism

$$\Omega^- / \Gamma_{v,n}^- \to \Omega_v / \Gamma_{v,n} \, .$$

The group $\Gamma_{v,n}^-$ acts freely on Ω^-. The quotient $\Omega^-/\Gamma_{v,n}^-$ is also a smooth quasi-projective variety and supports a family $\tilde{g} : \tilde{X} \to \Omega^-/\Gamma_{v,n}^-$ of $K3$-surfaces (obtained by pulling back the universal family over $\Omega_v/\Gamma_{v,n}$). In addition, we see (using the Torelli theorem again) that the family supports an involution $\tilde{\iota} : \tilde{X} \to \tilde{X}$. Now restrict the family $\tilde{g}$ over the complement of the images of the divisors H_d, i.e over

$$D_{\Gamma_{v,n}^-}^0 = \left(\Omega^- - (\cup_d H_d)\right)/\Gamma_{v,n}^-.$$

Using results of Nikulin we see as in [Y, §1] that the involution $\tilde{\iota}$ is fixed-point-free over this complement. Denote by $g : X \to D_{\Gamma_{v,n}^-}^0$ the corresponding (restricted) family of $K3$'s with involution $\iota : X \to X$. The family $f : X/\langle \iota \rangle \to D_{\Gamma_{v,n}^-}^0$ is a family of Enriques surfaces with the desired properties. For any subgroup $\Gamma \subset \Gamma_{v,n}^-$ of finite index the families f and g pull back to families over D_Γ^0 with the same properties; this allows to make sure that Γ can be taken to be normal in $O(L^-)$. □

For simplicity of notation we will set $S = D^0$, $T = D_\Gamma^0$ and $\omega_\Gamma = \omega$.

The natural map $\pi : T \to S$ is a finite morphism of quasi-projective varieties and S is identified with the quotient T/G where $G := O(L^-)/\Gamma$. Now consider the line bundle $\omega^{\otimes 2}$ over T.

Lemma 2.2. *We have $f^*(\omega^{\otimes 2}) \simeq (\Omega_{Y/T}^2)^{\otimes 2}$ as line bundles on Y.*

PROOF. Since $f : Y \to T$ is a family of Enriques surfaces, the line bundle $(\Omega_{Y/T}^2)^{\otimes 2}$ is trivial along the fibers of f. Therefore, there is a line bundle δ on T such that $(\Omega_{Y/T}^2)^{\otimes 2} \simeq f^*(\delta)$ and we have to show that $\delta \simeq \omega^{\otimes 2}$. Notice that

$$g^*\delta \simeq p^*f^*\delta \simeq p^*((\Omega_{Y/T}^2)^{\otimes 2}) \simeq (\Omega_{X/T}^2)^{\otimes 2} \simeq g^*\omega^{\otimes 2} .$$

The result now follows by applying $\mathrm{R}^0 g_*$ to both sides of $g^*\delta \simeq g^*\omega^{\otimes 2}$ after using the projection formula and $\mathrm{R}^0 g_*(\mathcal{O}_X) \simeq \mathcal{O}_S$. □

Now let us apply the Grothendieck-Riemann-Roch theorem ([Fu, Theorem 15.2]) to the proper morphism $f : Y \to T$ and to the trivial line bundle $\mathcal{O}_Y$ on Y. This gives

$$(2.2) \qquad \mathrm{ch}(\mathrm{R}f_*\mathcal{O}_Y) = f_*(\mathrm{ch}(\mathcal{O}_Y) \cdot \mathrm{Td}(T_f)) = f_*(\mathrm{Td}(T_f))$$

in the Chow ring with rational coefficients $\mathrm{CH}^*(T)_\mathbf{Q} := \mathrm{CH}^*(T) \otimes_\mathbf{Z} \mathbf{Q}$. In this formula, ch is the Chern character and Td the Todd class. Also $\mathrm{R}f_*\mathcal{O}_Y$ denotes the (virtual) total cohomology bundle, and $T_f = (\Omega_{Y/T}^1)^\vee$ is the relative tangent bundle of the smooth morphism f. On the right hand side of the formula, $f_* : \mathrm{CH}^*(Y)_\mathbf{Q} \to \mathrm{CH}^{*-2}(T)_\mathbf{Q}$ is the push forward homomorphism on Chow groups. Set v_1 for the class of ω in $\mathrm{Pic}(T)$. By the above lemma, we have $f^*(v_1) = -2 \cdot c_1(T_f)$. For simplicity, let us denote by $c_i = c_i(T_f)$ the Chern classes of the relative tangent bundle. By reading off the degree 1 component of the Grothendieck-Riemann-Roch identity (2.2) we find

$$(2.3) \qquad c_1(\mathrm{R}f_*(\mathcal{O}_Y)) = \frac{1}{24} \cdot f_*(c_1 \cdot c_2)$$

in $\mathrm{Pic}(T) \otimes \mathbf{Q}$. Since for the family of Enriques surfaces $f : Y \to T$, we have $\mathrm{R}^0 f_*(\mathcal{O}_Y) = \mathcal{O}_T$, $\mathrm{R}^i f_*(\mathcal{O}_Y) = (0)$ if $i > 0$, we have $\mathrm{R}f_*(\mathcal{O}_Y) = \mathcal{O}_T$. Hence, the above gives

$$(2.4) \qquad f_*(c_1 \cdot c_2) = 0$$

in $\mathrm{Pic}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Hence, we obtain

$$0 = 2 \cdot f_*(c_1 \cdot c_2) = -f_*(f_*(v_1) \cdot c_2) = -v_1 \cdot f_*(c_2) \,.$$

By Noether's fomula $f_*(c_2) = 12$ in the Chow group of codimension 0 cycles $\mathrm{CH}^0(T)$ and so the class $v_1 = c_1(\omega)$ is trivial in $\mathrm{Pic}(T)/\text{torsion} \subset \mathrm{Pic}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Therefore, the line bundle ω over T is torsion.

Now notice that automorphic forms of weight n on D_Γ^0 are by construction sections of

$$(\pi_* \omega_\Gamma^{\otimes n})^G$$

over $S = D^0$ (i.e $G = O(L^-)/\Gamma$-invariant sections of $\omega^{\otimes n} = \omega_\Gamma^{\otimes n}$ over $T = D_\Gamma^0$). Since the line bundle ω is torsion and the group G is finite, for n sufficiently large and divisible, the line bundle $\omega_\Gamma^{\otimes n}$ has a nowhere vanishing G-invariant global section. On the other hand, by Baily-Borel ([BB]), for all sufficiently large and divisible n the line bundle of automorphic forms of weight n on D^0 is very ample. These two facts together imply that the moduli space $S = D^0$ is quasi-affine exactly as in [B]. As in loc. cit. it follows (using results of Sterk) that D^0 is in fact an affine variety with a copy of the affine line $\mathbf{C}$ removed.

Remark 2.3. The above argument applies to any family of (classical) Enriques surfaces $f : Y \to T$, even if T is a variety over a field of positive characteristic. The conclusion is that $\mathrm{R}^0 f_*((\Omega_{Y/T}^2)^{\otimes 2})$ is a torsion line bundle on T.

Acknowledgment:

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