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ALGEBRAIC CYCLES ON SEVERI-BRAUER SCHEMES OF PRIME DEGREE OVER A CURVE

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Abstract. Let k be a perfect field and let p be a prime number different from the characteristic of k. Let C be a smooth, projective and geometrically integral k-curve and let X be a Severi-Brauer C-scheme of relative dimension $p-1$. In this paper we show that $CH^d(X)_{\text{tors}}$ contains a subgroup isomorphic to $CH_0(X/C)$ for every d in the range $2 \leq d \leq p$. We deduce that, if k is a number field, the full Chow ring $CH^*(X)$ is a finitely generated abelian group.

1. Introduction.

Let k be a perfect field with algebraic closure $\overline{k}$ and let p be a prime number different from the characteristic of k. Let C be a smooth, projective and geometrically integral k-curve. In this paper we study a certain subgroup of $CH^d(X)_{\text{tors}}$ for a Severi-Brauer C-scheme $q\colon X \to C$ of relative dimension $p-1$ and any integer d such that $2 \leq d \leq p$. Let

$$CH_0(X/C) = \text{Ker}\Big[CH_0(X) \xrightarrow{q_*} CH_0(C)\Big]$$

and let $\pi^*\colon CH^d(X) \to CH^d\big(\overline{X}\big)$ be induced by the extension-of-scalars map $\overline{X} \to X$, where $\overline{X} = X \times_{\text{Spec}\, k} \text{Spec}\, \overline{k}$. Then the following holds.

Main Theorem. *For any d as above, there exists a canonical isomorphism*

$$\text{Ker}\Big[CH^d(X) \xrightarrow{\pi^*} CH^d\big(\overline{X}\big)\Big] \simeq CH_0(X/C).$$

Corollary. *Assume that k is a number field. Then the Chow ring $CH^*(X)$ is a finitely generated abelian group.*

The above corollary confirms a well-known conjecture of S.Bloch in a particular case. Previous work on Bloch's conjecture include [3], where $CH^2(X)$ is shown to be finitely generated for a certain class of varieties X, and [4], where the same result is obtained for $CH_0(X)$ when $X \to C$ is an arbitrary (i.e., not necessarily smooth over C) Severi-Brauer fibration of squarefree index.

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2. Preliminaries.

Let k be a perfect field, fix an algebraic closure $\overline{k}$ of k and let $\Gamma = \mathrm{Gal}\big(\overline{k}/k\big)$. Now let C be a smooth, projective and geometrically integral k-curve and let X be a Severi-Brauer scheme over C [6, §8] of dimension $m \geq 2$. There exists a proper and flat k-morphism $q\colon X \to C$ all of whose fibers are Severi-Brauer varieties of dimension $m-1$ over the appropriate residue field [loc.cit.]. We will write X_η for the generic fiber $X \times_C \operatorname{Spec} k(C)$ of q and A for the central simple $k(C)$-algebra associated to X_η. We define

$$CH_0(X/C) = \mathrm{Ker}\Big[CH_0(X) \xrightarrow{q_*} CH_0(C)\Big].$$

Now let C_0 be the set of closed points of C. The group of *divisorial norms* of X/C (cf. [8]) is the group

$$k(C)^*_{\mathrm{dn}} = \{f \in k(C)^* \colon \forall y \in C_0, \mathrm{ord}_y(f) \in (q_y)_*(CH_0(X_y))\}$$

where, for each $y \in C_0$, $q_y\colon X_y \to \operatorname{Spec} k(y)$ is the structural morphism of the fiber X_y. This group is closely related to $CH_0(X/C)$ (see [4, Proposition 3.1]). Indeed, there exists a canonical isomorphism

$$CH_0(X/C) \simeq k(C)^*_{\mathrm{dn}}/k^* \mathrm{Nrd}\, A^*.$$

Remark 2.1. Fix an integer d such that $1 \leq d \leq m$ and let

$$CH^d(X)' = \mathrm{Ker}\Big[CH^d(X) \xrightarrow{\pi^*} CH^d\big(\overline{X}\big)^{\Gamma}\Big],$$

where $\pi\colon \overline{X} \to X$ is the canonical map. A simple transfer argument shows that $CH^d(X)'$ is a subgroup of $CH^d(X)_{\mathrm{tors}}$. Now, since $\overline{X} \to \overline{C}$ has a section, $\overline{X}$ is a projective bundle over $\overline{C}$. Thus, by [5, Theorem 3.3(b), p.64], there exist isomorphisms

$$CH^d\big(\overline{X}\big) \simeq \begin{cases} \mathbb{Z} \oplus CH_0\big(\overline{C}\big) & \text{if } 1 \leq d \leq m-1 \\ CH_0\big(\overline{C}\big) & \text{if } \quad d = m. \end{cases}$$

Therefore, if $J_C(k)$ is finitely generated, where J_C is the Jacobian variety of C (e.g., k is a number field or $C = \mathbb{P}^1_k$), then $CH^d(X)$ is finitely generated if and only if $CH^d(X)'$ is finite.

3. The general method.

Let C be as above and let X be any smooth, projective and geometrically integral k-variety such that there exists a proper and flat morphism $q\colon X \to C$ whose generic fiber X_η is geometrically integral. We have an exact sequence [9]

$$H^{d-1}(X_\eta, \mathcal{K}_d) \xrightarrow{\delta} \bigoplus_{y \in C_0} CH^{d-1}(X_y) \to CH^d(X) \xrightarrow{j^*} CH^d(X_\eta) \to 0, \tag{1}$$

where $j\colon X_\eta \to X$ is the natural map and the map which we have labeled δ will play a role later when $k = \overline{k}$. A similar exact sequence exists over $\overline{k}$, and we have two natural exact commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} j^* & \longrightarrow & CH^d(X) & \longrightarrow & CH^d(X_\eta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\operatorname{Ker} \overline{\jmath}^*)^\Gamma & \longrightarrow & CH^d(\overline{X})^\Gamma & \longrightarrow & CH^d(\overline{X}_{\overline{\eta}})^\Gamma & & \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \dfrac{H^{d-1}(X_\eta, \mathcal{K}_d)}{j^* H^{d-1}(X, \mathcal{K}_d)} & \longrightarrow & \bigoplus_{y \in C_0} CH^{d-1}(X_y) & \twoheadrightarrow & \operatorname{Ker} j^* \\ & & \downarrow & & \downarrow {\scriptstyle \oplus_{\overline{y}\,|\,y} \pi^*_{\overline{y}}} & & \downarrow \\ 0 & \longrightarrow & \left(\dfrac{H^{d-1}(\overline{X}_{\overline{\eta}}, \mathcal{K}_d)}{\overline{\jmath}^* H^{d-1}(\overline{X}, \mathcal{K}_d)}\right)^\Gamma & \xrightarrow{\overline{\delta}} & \bigoplus_{\overline{y}\,|\,y} CH^{d-1}(\overline{X}_{\overline{y}})^{\Gamma_y} & \longrightarrow & (\operatorname{Ker} \overline{\jmath}^*)^\Gamma \end{array} \tag{2}$$

where, for each $y \in C_0$, we have fixed a closed point $\overline{y}$ of $\overline{C}$ lying above y and written $\Gamma_y = \operatorname{Gal}(\overline{k}/k(y))$. Set

$$CH^d(X_\eta)' = \operatorname{Ker}\Big[CH^d(X_\eta) \to CH^d(\overline{X}_{\overline{\eta}})^\Gamma \Big]$$

and, for each $y \in C_0$,

$$CH^{d-1}(X_y)' = \operatorname{Ker}\Big[CH^{d-1}(X_y) \xrightarrow{\pi^*_{\overline{y}}} CH^{d-1}(\overline{X}_{\overline{y}})^{\Gamma_y} \Big].$$

Now define

$$E(\overline{X}/\overline{C}) = \operatorname{Coker}\left[\frac{H^{d-1}(X_\eta, \mathcal{K}_d)}{j^* H^{d-1}(X, \mathcal{K}_d)} \longrightarrow \left(\frac{H^{d-1}(\overline{X}_{\overline{\eta}}, \mathcal{K}_d)}{\overline{\jmath}^* H^{d-1}(\overline{X}, \mathcal{K}_d)} \right)^\Gamma \right]. \tag{3}$$

Then, applying the snake lemma to the preceding diagrams, we obtain[1]

Proposition 3.1. *There exists a natural exact sequence*

$$\bigoplus_{y \in C_0} CH^{d-1}(X_y)' \to \operatorname{Ker}\big[CH^d(X)' \to CH^d(X_\eta)' \big]$$
$$\to \operatorname{Ker}\left[E(\overline{X}/\overline{C}) \to \bigoplus_{y \in C_0} \frac{CH^{d-1}(\overline{X}_{\overline{y}})^{\Gamma_y}}{\pi^*_{\overline{y}} CH^{d-1}(X_y)} \right] \to 0,$$

where $E(\overline{X}/\overline{C})$ is the group (3).

As regards the right-hand group in the exact sequence of the proposition, the following holds. Let

$$H^{d-1}(X_\eta, \mathcal{K}_d)' = \operatorname{Im}\Big[H^{d-1}(X_\eta, \mathcal{K}_d) \to H^{d-1}(\overline{X}_{\overline{\eta}}, \mathcal{K}_d)^\Gamma \Big]$$

[1] Proposition 3.1 was inspired by [1, Proposition 1.1].

and

$$\mathrm{Sal}_d(X/C) = \left\{ f \in H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big)^{\Gamma} : \forall y \in C_0,\ \overline{\delta}_{\overline{y}}(f) \in \pi_{\overline{y}}^{*} CH^{d-1}(X_y) \right\},$$

where $\overline{\delta}$ and $\pi_{\overline{y}}^{*}$ are the maps of diagram (2).

Proposition 3.2. *There exists a natural exact sequence*

$$\begin{aligned} 0 \to & \frac{\mathrm{Sal}_d(X/C)}{\big(\overline{\jmath}^{*} H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)\big)^{\Gamma} \cdot H^{d-1}(X_\eta, \mathcal{K}_d)'} \\ & \to \mathrm{Ker}\left[E\big(\overline{X}/\overline{C}\,\big) \to \bigoplus_{y \in C_0} \frac{CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y}}{\pi_{\overline{y}}^{*} CH^{d-1}(X_y)} \right] \\ & \to H^1\big(\Gamma, \overline{\jmath}^{*} H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)\big). \end{aligned}$$

Proof. This follows by applying the snake lemma to a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{A}^{\,\Gamma} & \longrightarrow & \overline{B}^{\,\Gamma} & \longrightarrow & \big(\overline{B}/\overline{A}\big)^{\Gamma} & \longrightarrow & H^1\big(\Gamma, \overline{A}\big) \end{array}$$

with $\overline{A} = \overline{\jmath}^{*} H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)$, $\overline{B} = H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big)$, etc. □

4. Proof of the main theorem.

Let C and A be as in Section 2, let p be a prime number different from the characteristic of k and let X be a Severi-Brauer scheme over C of relative dimension $p-1$.

Lemma 4.1. *There exists a Γ-isomorphism*

$$\overline{\jmath}^{*} H^{d-1}\big(\overline{X}, \mathcal{K}_d\big) \simeq \overline{k}^{*}.$$

Proof. Clearly, $\overline{\jmath}^{*} H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)$ is the kernel of the map

$$\overline{\delta} \colon H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big) \to \bigoplus_{\overline{y} \,|\, y} CH^{d-1}\big(\overline{X}_{\overline{y}}\big)$$

appearing in the exact sequence (1) over $\overline{k}$. Now $\overline{X}_{\overline{\eta}} \simeq \mathbb{P}^{p-1}_{\overline{\eta}}$ and $\overline{X}_{\overline{y}} \simeq \mathbb{P}^{p-1}_{\overline{k}}$ for every $\overline{y}$, whence we have Γ-isomorphisms

$$H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big) \simeq \overline{k}(C)^{*}$$

and

$$CH^{d-1}\big(\overline{X}_{\overline{y}}\big) \simeq \mathbb{Z}$$

for each $\overline{y}$. Under these isomorphisms, the map $\overline{\delta}$ above corresponds to the canonical map

$$\begin{aligned} \overline{k}(C)^{*} &\to \bigoplus_{\overline{y} \,|\, y} \mathbb{Z}, \\ f &\mapsto (\mathrm{ord}_{\overline{y}}(f))_{\overline{y} \,|\, y}, \end{aligned}$$

which yields the lemma. □

Theorem 4.2. *For every d such that $2 \leq d \leq p$, there exists a canonical isomorphism*

$$CH^d(X)' \simeq CH_0(X/C).$$

Proof. By Lemma 4.1, Hilbert's Theorem 90 and Proposition 3.2, there exists a natural isomorphism

$$\operatorname{Ker}\left[E(\overline{X}/\overline{C}) \to \bigoplus_{y \in C_0} \frac{CH^{d-1}(\overline{X}_{\overline{y}})^{\Gamma_y}}{\pi_{\overline{y}}^* CH^{d-1}(X_y)}\right] \simeq \frac{\operatorname{Sal}_d(X/C)}{k^* H^{d-1}(X_\eta, \mathcal{K}_d)'}.$$

On the other hand, by [7,(8.7.2)], $H^{d-1}(X_\eta, \mathcal{K}_d)' = \operatorname{Nrd} A^*$ for every d such that $2 \leq d \leq p$ and, for each $y \in C_0$,

$$\pi_{\overline{y}}^* CH^{d-1}(X_y) \simeq \pi_{\overline{y}}^* CH^{p-1}(X_y) \simeq (q_y)_* CH_0(X_y) \quad (= \mathbb{Z} \text{ or } p\mathbb{Z}).$$

The latter implies that $\operatorname{Sal}_d(X/C) = k(C)^*_{\mathrm{dn}}$, whence

$$\begin{aligned} \operatorname{Sal}_d(X/C)/k^* H^{d-1}(X_\eta, \mathcal{K}_d)' &\simeq k(C)^*_{\mathrm{dn}}/k^* \operatorname{Nrd} A^* \\ &\simeq CH_0(X/C). \end{aligned}$$

Finally, [loc.cit.] shows that the groups $CH^d(X_\eta)$ and $CH^{d-1}(X_y)$ ($y \in C_0$) are torsion free, whence $CH^d(X_\eta)'$ and $CH^{d-1}(X_y)'$ vanish. The theorem now follows from Proposition 3.1. □

Corollary 4.3. *Let d be such that $2 \leq d \leq p$. Then $CH^d(X)'$ is finite if*

(1) *k is a number field, or*
(2) *k is a field of finite type over $\mathbb{Q}$, $C = \mathbb{P}^1_k$ and X has a 0-cycle of degree one.*

Proof. Indeed, in these cases the group $CH_0(X/C)$ is finite [4]. □

Corollary 4.4. *In each of the cases listed in the previous corollary, the Chow ring $CH^*(X)$ is finitely generated as an abelian group.*

Proof. The above corollary and Remark 2.1 show that $CH^d(X)$ is finitely generated for any d such that $2 \leq d \leq p$. Since $CH^0(X)$ and $CH^1(X) = \operatorname{Pic}(X)$ are well-known to be finitely generated (see [2, §1]), the proof is complete. □

Remark 4.5. The referee has suggested the following alternative approach to this paper.

Since there is only p-torsion in the Chow groups and $\dim X = p$, it is not difficult to relate the E_2 and E_∞ terms in the Gersten-Quillen spectral sequence (see, e.g., [7, Proposition (8.6.2), p.320]). Hence if $K_0(X)$ is finitely generated, then the Chow groups of X are also finitely generated. Now let Λ be the Azumaya algebra over C corresponding to the Severi-Brauer scheme $X \to C$ (see [6]). Then, by a well-known theorem of Quillen, $K_0(X) \simeq K_0(C) \oplus K_0(\Lambda)^{p-1}$. Hence if $K_0(C)$ and $K_0(\Lambda)$ are finitely generated, then the Chow groups of X are also finitely generated. Now one can construct a commutative diagram with Swan's localization sequences for Λ and C and use it to relate the kernel of the restriction map from $K_0(\Lambda)$ to $K_0(C)$ (or $K_0(\overline{\Lambda})$) to the group $k(C)^*/k^* \operatorname{Nrd} A^*$. This gives more transparent proofs of the finiteness results and the introduction of the Azumaya algebra Λ provides a natural explanation for the appearance of the group $k(C)^*/k^* \operatorname{Nrd} A^*$.

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